A non-linear approximation for cosmological inhomogeneities

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(K. Enqvist, S. Hotchkiss, GR) arXiv:1112.2995[astro-ph.CO]

(GR, W Valkenburg) arXiv:1203.2796[astro-ph.CO]

Bielefeld 4 May 2012

Inhomogeneities

The study of inhomogeneities through perturbation theory has been of crucial importance for the development of the Concordance model.

Perturbation theory is naturally limited when $\frac{\delta\rho}{\bar{\rho}} \sim 1$ ($\frac{\delta g_{\mu\nu}}{\bar{g}_{\mu\nu}} \sim 1$ in some gauges). In our Universe this happens at $k \sim 0.08h\mathrm{Mpc}^{-1}$ or $L \sim 79h^{-1}\mathrm{Mpc}$.

To explore non-linear structure formation employ

- Exact solutions of high symmetry
- Analytic approaches like the Zel'dovich approximation and its improvements
- Newtonian N-Body simulations

We would like to pursue an analytic approximation, of a different nature to perturbation theory that can follow the non-linear evolution for generic initial conditions: A gradient expansion

The Gradient Expansion

The gradient expansion builds a solution around a seed metric k_{ij} by including an increasing number of gradients. It is a priori valid when

$$\frac{1}{a}\partial_i\gamma_{jk} \ll \partial_t\gamma_{jk} \Leftrightarrow L \gg (aH)^{-1}$$

Extendible for $L > aH$?

$$\gamma_{ij} \sim \sum_n F_n(t)(\hat{R}^n_{ij} + \sum_r \hat{\nabla}^{2r}\hat{R}^{n-r})$$

A hat indicates that quantities are computed from k_{ij}.

For smooth enough initial conditions $L < aH$ can also be followed for a limited amount of time.

Hamilton Jacobi for ΛCDM

The gradient expansion can be applied directly to the Einstein equations. Here we apply Hamilton Jacobi theory

$$ds^2 = -dt^2 + \gamma_{ij}(t, \mathbf{x}) dx^i dx^j$$

hamiltonian density for the gravitational field

$$\mathcal{H} = \frac{2\kappa}{\sqrt{\gamma}} \pi_{ij} \pi^{kl} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) - \frac{\sqrt{\gamma}}{2\kappa}(R - 2\Lambda)$$

Hamilton Jacobi Equations

$$\frac{\partial \mathcal{S}}{\partial t} + \int d^3x \left\{ \frac{2\kappa}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl} \right) - \frac{\sqrt{\gamma}}{2\kappa}(R - 2\Lambda) \right\} = 0$$

$$\frac{\partial \gamma_{ij}}{\partial t} = \frac{2}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \left(2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right) , \quad \rho(t, \mathbf{x}) = \frac{\rho_0}{\sqrt{\gamma(t, \mathbf{x})}}$$

$$\mathcal{S} = \int d^3x \sqrt{\gamma}\, \mathcal{F}(t, \gamma_{ij})$$

Gradient Expansion for ΛCDM

Hamilton Jacobi Theory

Ansatz

$$\mathcal{F} = -2H(t) + J(t)R + L_1(t)R^2 + L_2(t)R^{ij}R_{ij} + \ldots$$

$$\frac{dH}{dt} + \frac{3}{2}H^2 - \frac{\Lambda}{2} = 0\,,$$

$$\frac{dJ}{dt} + JH - \frac{1}{2} = 0\,, \quad \frac{dL_1}{dt} - L_1 H - \frac{3}{4}J^2 = 0\,, \quad \frac{dL_2}{dt} - L_2 H + 2J^2 = 0$$

e.t.c.

$$H = \sqrt{\frac{\Lambda}{3}} \coth\left(\frac{\sqrt{3\Lambda}}{2}\ t\right)\,,$$

$$J(t) = e^{-\int_{t_i}^{t} H} \int_{t_i}^{t} \frac{1}{2}\, e^{\int_{t_i}^{x} H} dx = \frac{1}{\sqrt{3\Lambda}\left[\sinh \frac{\sqrt{3\Lambda}}{2}\ t\right]^{2/3}} \int_{\frac{\sqrt{3\Lambda}}{2}\ t_i}^{\frac{\sqrt{3\Lambda}}{2}\ t} (\sinh u)^{2/3}\, du\,,$$

$$L_1(t) = \frac{3}{4}L(t)\,, \quad L_2(t) = -2L(t)\,,$$

$$L(t) = e^{\int_{t_i}^{t} H} \int_{t_i}^{t} \left[e^{-\int_{t_i}^{x} H} J^2\right] dx = \frac{2}{\sqrt{3\Lambda}} \left[\sinh \frac{\sqrt{3\Lambda}}{2}\ t\right]^{2/3} \int_{\frac{\sqrt{3\Lambda}}{2}\ t_i}^{\frac{\sqrt{3\Lambda}}{2}\ t} (\sinh u)^{-2/3} J^2(u)\, du\,,$$

Gradient Expansion for ΛCDM

Hamilton Jacobi Theory

Knowing $\mathcal{S}$ we can compute γ_{ij}

$$\begin{aligned}\frac{\partial \gamma_{ij}}{\partial t} &= 2H\gamma_{ij} + J(R\gamma_{ij} - 4R_{ij}) \\ &+L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{|ij}\right) \\ &+L_2\Big(3\gamma_{ij}R^{kl}R_{kl} - 8R_{ik}R^k{}_j + 3\gamma_{ij}R^{|k}{}_{|k} \\ &+4R_i{}^k{}_{|jk} + 4R_j{}^k{}_{|ik} - 4R_{ij}{}^{|k}{}_{|k} - 4\gamma_{ij}R^{kl}{}_{|kl}\Big) + \mathcal{O}(6)\end{aligned}$$

Exact but Impossible to solve since:

- Contains an infinite number of terms
- R_{ij} is a function of γ_{ij}

Knowing $\mathcal{S}$ we can compute γ_{ij}

$$\begin{aligned}
\frac{\partial \gamma_{ij}}{\partial t} &= 2H\gamma_{ij} + J(R\gamma_{ij} - 4R_{ij}) \\
&+L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{|ij}\right) \\
&+L_2\Big(3\gamma_{ij}R^{kl}R_{kl} - 8R_{ik}R^k{}_j + 3\gamma_{ij}R^{|k}{}_{|k} \\
&+4R_i{}^k{}_{|jk} + 4R_j{}^k{}_{|ik} - 4R_{ij}{}^{|k}{}_{|k} - 4\gamma_{ij}R^{kl}{}_{|kl}\Big) + \mathcal{O}(6)
\end{aligned}$$

Solve iteratively starting at zero gradients

$$\frac{\partial \gamma_{ij}^{(0)}}{\partial t} = 2H\gamma_{ij}^{(0)} \Rightarrow \boxed{\gamma_{ij}^{(0)}(t) = A^2(t)k_{ij}}$$

$$A(t) = e^{\int_{t_i}^{t} H} = \left[\frac{\sinh \frac{\sqrt{3\Lambda}}{2}\, t}{\sinh \frac{\sqrt{3\Lambda}}{2}\, t_i}\right]^{2/3}$$

This is simply the separate universe approximation where each point evolves as a homogeneous FRW universe.

Knowing $\mathcal{S}$ we can compute γ_{ij}

$$\begin{aligned}\frac{\partial \gamma_{ij}}{\partial t} &= 2H\gamma_{ij} + J(R\gamma_{ij} - 4R_{ij}) \\ &+L_1\left(3\gamma_{ij}R^2 - 8RR_{ij} + 8R_{|ij}\right) \\ &+L_2\Big(3\gamma_{ij}R^{kl}R_{kl} - 8R_{ik}R^k{}_j + 3\gamma_{ij}R^{|k}{}_{|k} \\ &+4R_i{}^k{}_{|jk} + 4R_j{}^k{}_{|ik} - 4R_{ij}{}^{|k}{}_{|k} - 4\gamma_{ij}R^{kl}{}_{|kl}\Big) + \mathcal{O}(6)\end{aligned}$$

To second order in gradients we get

$$\frac{\partial \gamma^{(2)}_{ij}}{\partial t} = 2H\gamma^{(2)}_{ij} + J(\hat{R}k_{ij} - 4\hat{R}_{ij}) \Rightarrow \boxed{\gamma^{(2)}_{ij} = A^2(t)k_{ij} + \lambda(t)(\hat{R}k_{ij} - 4\hat{R}_{ij})}$$

where $\hat{R}_{ij} \equiv \hat{R}_{ij}(k_{lm})$

$$\lambda(t) = e^{2\int_{t_i}^{t}H}\int_{t_i}^{t}\left[e^{-2\int_{t_i}^{x}H}J\right]dx = \frac{2}{\sqrt{3\Lambda}}\left[\sinh\frac{\sqrt{3\Lambda}}{2}t\right]^{4/3}\int_{\frac{\sqrt{3\Lambda}}{2}t_i}^{\frac{\sqrt{3\Lambda}}{2}t}(\sinh u)^{-4/3}J(u)\,du\,.$$

$$ds^2 = -dt^2 + \gamma_{ij}(t, \mathbf{x}) dx^i dx^j$$

Up to fourth order

$$\begin{aligned}
\gamma_{ij}^{(4)} &= A^2(t) k_{ij} + \lambda(t)(\hat{R} k_{ij} - 4\hat{R}_{ij}) \\
&+ A^2(t) \int^t \frac{1}{A^2} \left(C_1 \hat{R}^2 k_{ij} + C_2 \hat{R}^{kl} \hat{R}_{kl} k_{ij} + C_3 \hat{R} \hat{R}_{ij} + C_4 \hat{R}_{ik} \hat{R}^k{}_j \right) \\
&+ A^2(t) \int^t \frac{1}{A^2} \left(D_1 \hat{R}^{|k}{}_{|k} k_{ij} + D_2 \hat{R}_{|ij} + D_3 \hat{R}_{ij}{}^{|k}{}_{|k} \right)
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= 8\frac{\lambda J}{A^2} - \frac{23}{4}\frac{L}{A^2}, \quad C_2 = -12\frac{\lambda J}{A^2} + 10\frac{L}{A^2} \\
C_3 &= -28\frac{\lambda J}{A^2} + 18\frac{L}{A^2}, \quad C_4 = 48\frac{\lambda J}{A^2} - 32\frac{L}{A^2}, \\
D_1 &= D_2 = 2\frac{\lambda J}{A^2} - 2\frac{L}{A^2}, \quad D_3 = -8\frac{\lambda J}{A^2} + 8\frac{L}{A^2}.
\end{aligned}$$

Range of validity

At late times $H \to \sqrt{\frac{\Lambda}{3}}$ and the gradient expansion becomes increasingly good.

For $\Lambda = 0$

$$\begin{aligned}\gamma_{ij} \simeq\ & \left(\frac{t}{t_\mathrm{i}}\right)^{4/3} k_{ij} + \frac{9}{20}\left(\frac{t}{t_i}\right)^2 t_i^2 \left[\hat{R} k_{ij} - 4\hat{R}_{ij}\right] \\ & + \frac{81}{350}\left(\frac{t}{t_i}\right)^{8/3} t_i^4 \left[\left(-4\hat{R}^{lm}\hat{R}_{lm} + \frac{5}{8}\hat{R}_{|k}{}^{|k} + \frac{89}{32}\hat{R}^2\right) k_{ij} \right. \\ & \left. -10\hat{R}\hat{R}_{ij} + 17\hat{R}^l{}_i\hat{R}_{lj} - \frac{5}{2}\hat{R}_{ij|k}{}^k + \frac{5}{8}\hat{R}_{|ij}\right],\end{aligned}$$

This should be valid for

$$t_{\rm con} \sim \mathcal{O}(\text{few})\frac{1}{t_i^2 \hat{R}^{3/2}} \quad \text{or} \quad t_{\rm con} \sim \mathcal{O}(\text{few})\frac{1}{t_i^2 (\nabla^2 \hat{R})^{3/4}}$$

Approximately the collapse time of regions with curvature $\hat{R}$

(Better behaved than a perturbative expansion?)

Range of validity

Our Universe

Assuming the standard inflationary initial conditions, the initial seed metric takes the form

$$k_{ij} = \left(\frac{t_{\rm i}}{t_0}\right)^{4/3} \delta_{ij} \left(1 + \frac{10}{3}\Phi(\mathbf{x})\right) ,$$

where $\Phi(\mathbf{x})$ is the primordial Newtonian potential. Then

$$\gamma_{ij} \simeq \left(\frac{t}{t_0}\right)^{4/3} \left[\delta_{ij} + 3\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij} + \left(\frac{t}{t_0}\right)^{4/3} t_0^4 \hat{B}_{ij}\right] + \mathcal{O}(6) ,$$

where

$$\hat{B}_{ij} = \frac{9}{28}\left[19\Phi_{,il}\Phi^{,l}{}_{,j} - 12\Phi_{,ij}\Phi^{,l}{}_{,l} + 3\delta_{ij}\left((\Phi^{,l}{}_{,l})^2 - \Phi_{,lm}\Phi^{,lm}\right)\right] .$$

Timescale for which the approximation for the metric is accurate

$$\boxed{\frac{t_{\rm con}}{t_0} \simeq 3.4 \times \frac{H_0^3}{(\nabla^2\Phi)^{3/2}}}$$

If used for the whole of t_0, the metric can include perturbations down to about $k = 0.3h\,\mathrm{Mpc}^{-1} > 0.08h\,\mathrm{Mpc}^{-1}$. Shorter scales can be accurately described but for shorter times, comparable to the collapse times of the over-dense regions.

The Density

The local density of matter can be obtained from

$$\rho(t,\mathbf{x}) = \frac{1}{6\pi G\, t^2} \frac{1}{\sqrt{\mathrm{Det}\left[\delta_{ij} + 3\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij} + \left(\frac{t}{t_0}\right)^{4/3} t_0^4 \hat{B}_{ij}\right]}} \, .$$

If this expression is expanded to linear order in Φ one recovers the result of linear perturbation theory for the density contrast in the synchronous gauge. Dropping the $\hat{B}_{ij}$

$$\rho(t,\mathbf{x}) = \frac{1}{6\pi G\, t^2} \frac{1}{\mathrm{Det}\left[\delta_{ij} + \frac{3}{2}\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij}\right]} \, ,$$

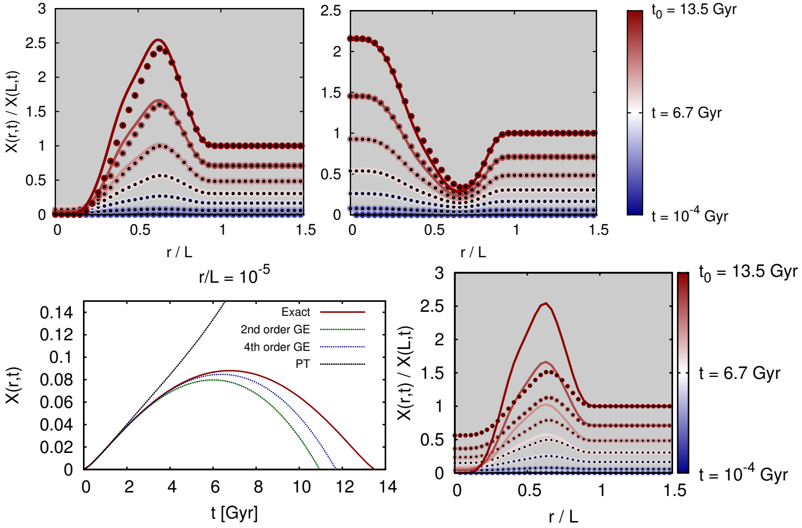
reproduces the well known result from the Zel'dovich approximation.

The gradient expansion in the comoving synchronous gauge can be thought of as providing a relativistic extension of the Zel'dovich approximation. In fact, the higher order terms are crucial for increased accuracy during the more advanced stages of the gravitational evolution. (Work in progress)

Accuracy?

Comparison with ΛLTB

$$ds^2 = dt^2 - \left(\partial_r Y(r,t)/\sqrt{1+2E(r)}\right)^2 dr^2 - Y^2(r,t)d\Omega^2$$



On Backreaction

What is the effect of cosmic inhomogeneities on the "average" dynamics of the universe?

Scalar equations for a CDM Universe (no vorticity)

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -4\pi G\rho - 2\sigma^2$$

$$\frac{1}{3}\Theta^2 = 8\pi G\rho - \frac{1}{2}{}^{(3)}R + \sigma^2$$

$$\dot{\rho} + \Theta\rho = 0$$

Averaged equations (Buchert)

$$3\frac{\ddot{a}_D}{a_D} = -4\pi G\langle\rho\rangle_D + Q_D$$

$$\left(\frac{\dot{a}_D}{a_D}\right)^2 = 8\pi G\langle\rho\rangle_D - \frac{1}{2}\langle{}^{(3)}R\rangle - \frac{1}{2}Q_D$$

$$\partial_t\langle\rho\rangle_D + 3\frac{\dot{a}_D}{a_D}\langle\rho\rangle_D = 0\,, \quad a_D^{-2}\partial_t\left(a_D^2\langle{}^{(3)}R\rangle\right) + a_D^{-6}\partial_t\left(a_D^6 Q_D\right) = 0$$

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$$Q_D = \frac{2}{3}\left(\langle\Theta^2\rangle_D - \langle\Theta\rangle_D^2\right) - 2\langle\sigma^2\rangle_D$$

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$$Q_D = \frac{1}{4}\langle\left(\gamma^{ij}\dot{\gamma}_{ij}\right)^2\rangle_D - \frac{1}{6}\langle\gamma^{ij}\dot{\gamma}_{ij}\rangle_D^2 - \frac{1}{4}\langle\gamma^{ki}\dot{\gamma}_{ij}\gamma^{jl}\dot{\gamma}_{lk}\rangle_D$$

On Backreaction

Numerical implementation

Use gradient expansion formulae on a grid

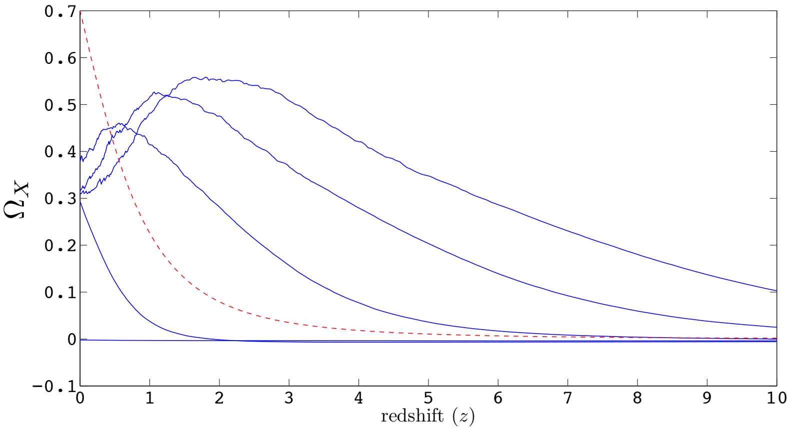
- Set up an initial gravitational potential $\Phi(\mathbf{x}) = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$
  - Gaussian random field with variance $\sigma_k^2 = \frac{9}{25}\frac{2\pi^2}{k^3}\frac{\Delta_{\mathcal{R}}^2}{V}\left(\frac{k}{k_{\rm piv}}\right)^{n_s-1} T(k)^2$
- At each point the metric is

$$\gamma_{ij} \simeq \left(\frac{t}{t_0}\right)^{4/3}\left[\delta_{ij} + 3\left(\frac{t}{t_0}\right)^{2/3} t_0^2 \Phi_{,ij} + \left(\frac{t}{t_0}\right)^{4/3} t_0^4 \hat{B}_{ij}\right]$$

- Explicitly calculate the average expansion of the grid $a_D(t)^3 \equiv \int_D d^3x \sqrt{\gamma(t,\mathbf{x})}$
- Crosscheck by calculating $Q_D = \frac{1}{4}\langle\left(\gamma^{ij}\dot{\gamma}_{ij}\right)^2\rangle_D - \frac{1}{6}\langle\gamma^{ij}\dot{\gamma}_{ij}\rangle_D^2 - \frac{1}{4}\langle\gamma^{ki}\dot{\gamma}_{ij}\gamma^{jl}\dot{\gamma}_{lk}\rangle_D$

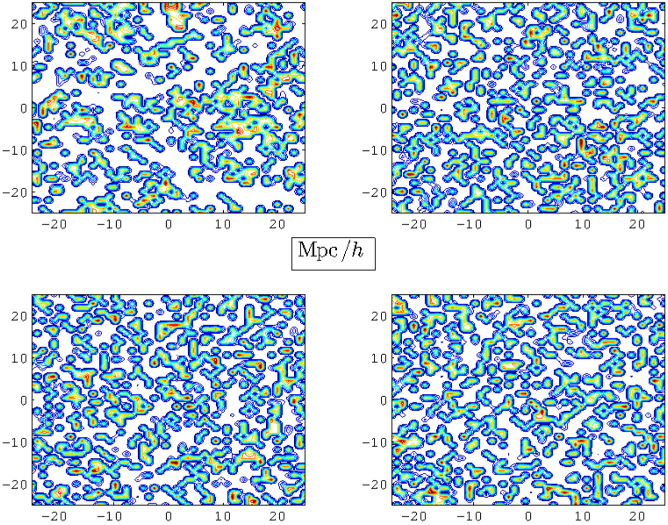
Some "fixes" required before this is implemented !

$$\Omega_X(t) = 1 - \Omega_m(t)$$



On Backreaction

Numerical implementation



A few percent effect?

- Better modeling needed to deal with the stabilization of structures and the behaviour of voids.
- A multi-scale computation
- Relation with other calculations (showing no effect)?
- Are the average quantities relevant at all?

Study light propagation!

Perhaps relevant for precision cosmology?

Discussion - future directions

The gradient expansion offers an analytic approximation that can describe non-linear gravitational evolution.

It can be thought of as a relativistic generalization of the Zel'dovich approximation and Langrangian perturbation theory.

It can be used with other choices of coordinates and more general fluids.

Can it be augmented to better model the formation of bound structures (Adhesion approximation)?

Light propagation through a clumpy spacetime using the gradient expansion metric.