Non-linear evolution of CDM and massive neutrino perturbations

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Overview

- Neutrinos are massive $m_\nu = 0.05\text{eV} - 0.5\text{eV}$
- Not negligible $\Omega_\nu = 10^{-3} - 10^{-2}$
  - Contribute to large scale structure
- Density on small scales smaller then CDM, due to free streaming
- Need formalism to calculate higher order correlations

$$P \cdot \delta(\mathbf{k}_1 + \mathbf{k}_2) = \langle \delta(\mathbf{k}_1) \cdot \delta(\mathbf{k}_2) \rangle$$

$$\delta = \frac{\rho_{CDM} \cdot \delta_{CDM} + \rho_\nu \cdot \delta_\nu}{\rho_{CDM} + \rho_\nu}$$

$$B \cdot \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) = \langle \delta(\mathbf{k}_1) \cdot \delta(\mathbf{k}_2) \cdot \delta(\mathbf{k}_3) \rangle$$

CDM Perturbations

Structure formation of CDM well described by fluid equations:

$$\frac{\partial \delta}{\partial \tau} + \theta = -\int d^3k_1 d^3k_2\, \delta(\mathbf{k}-(\mathbf{k}_1+\mathbf{k}_2)) \cdot \alpha(\mathbf{k}_{1,}\mathbf{k}_2) \cdot \theta(\mathbf{k}_1) \cdot \delta(\mathbf{k}_2)$$

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H} \cdot \theta + \frac{3}{2}\mathcal{H}^2 \cdot \Omega_{CDM} \cdot \delta = -\int d^3k_1 d^3k_2\, \delta(\mathbf{k}-(\mathbf{k}_1+\mathbf{k}_2)) \cdot \beta(\mathbf{k}_{1,}\mathbf{k}_2) \cdot \theta(\mathbf{k}_1) \cdot \theta(\mathbf{k}_2)$$

Perturbation theory:

1. Find greens function of left side
2. Transform into integral equation
3. Solve iteratively

Generalize for CDM and massive neutrinos:

$$\frac{3}{2}\Omega_{CDM} \cdot \mathcal{H}^2 \cdot \delta \rightarrow \frac{3}{2}\mathcal{H}^2 \left(\Omega_{CDM} \cdot \delta + \Omega_\nu \cdot \delta_\nu\right)$$

Plus evolution equation for neutrinos

Apply same solution strategy, for coupled system

Neutrino Free Streaming

For Neutrinos stress not negligible

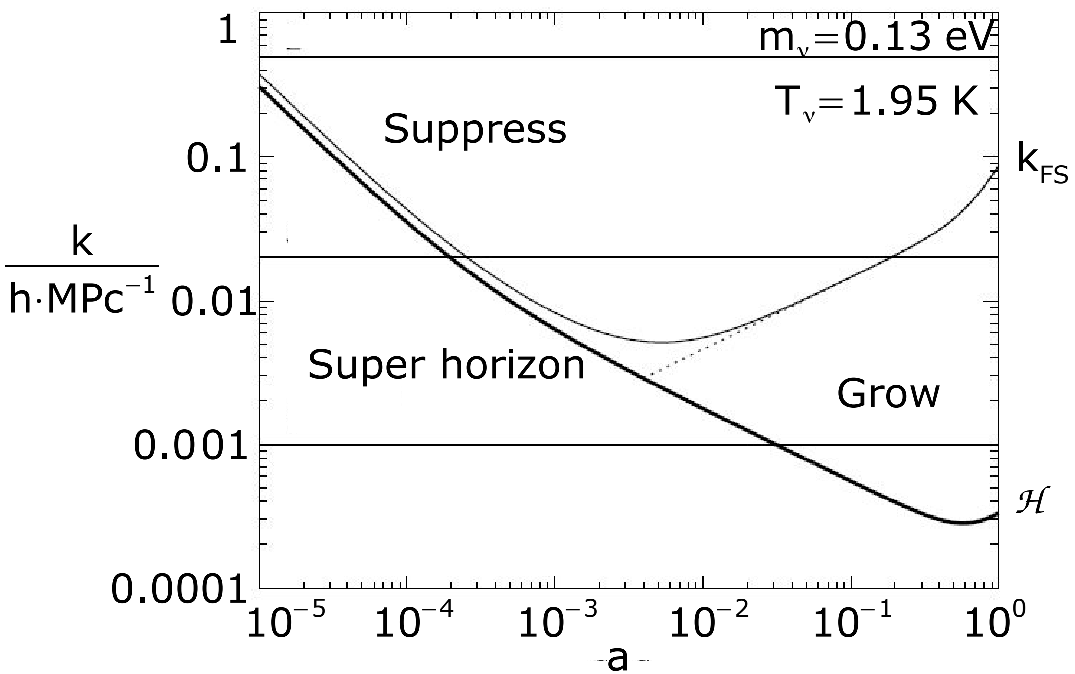
$$\frac{\partial\delta}{\partial\tau}+\theta=0$$

$$\frac{\partial\theta}{\partial\tau}+\mathcal{H}\cdot\theta+\left(\frac{3}{2}\mathcal{H}^2-k^2\cdot c_s^2\right)\delta=0$$

→ Gravitational potential effectively reduced

Free streaming scale:

$$k_{FS}=\sqrt{\frac{3}{2}\frac{\mathcal{H}}{c_s}}\approx\sqrt{\frac{3}{2}\frac{\mathcal{H}}{\sigma}}$$



$$c_s^2\approx\sigma^2=\frac{\int dq\, q^2\cdot\left(\frac{q}{\epsilon}\right)^2\cdot f^{(0)}(q)}{\int dq\, q^2\cdot f^{(0)}(q)}$$

Masatoshi Shoji & Eiichiro Komatsu(2010)
arxiv: 1003.0942

Boltzmann Equation

Without Knowledge of stress, have to start with full Boltzmann equation

$$p^A e^\mu_A \partial_\mu f - \Omega^a_{BC} p^B p^C \partial_{p^a} f = 0$$

Tetraed base: e^μ_A

Observer 4-velocity: $e^\mu_0 = u^\mu$

3-momentum direction: $n^a = e^a_\mu n^\mu$

$\Rightarrow p^\mu = E u^\mu + p n^\mu$

Spin connection: $\Omega^A_{BC} = e^A_\mu \left(e^\mu_{C,\lambda} + \Gamma^\mu_{\lambda\delta} e^\delta_C \right) e^\lambda_B$

Splitting into background and perturbation $f = f^{(0)} + \delta f$, $\mathbf{e} = \mathbf{e}^{(0)} + \boldsymbol{\delta} \mathbf{e}$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{(0)} + \boldsymbol{\delta\Omega}$ yields equation of the Form

$$D \delta f = S[\delta f]$$

linear differential-integral-operator (D)

non-linear differential-integral-operator (S)

This equation is equivalent to:

$$\delta f = \delta f_L + D^{-1} S[\delta f]$$

Linear Problem: $D \delta f_L = 0 \quad \delta f_L(\tau_i) = \delta f_i$

Operator inverse: $D \delta \tilde{f} = g \quad \delta \tilde{f}(\tau_i) = 0$

→ Solve iteratively

Boltzmann Hierarchy I

If D differential operator, then problem reduces to find greens function

Try to reduce Problem, in order to get differential equation

Expand the momentum direction dependence into spherical harmonics

$$f=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l \cdot \sqrt{\frac{4\pi}{2l+1}} \cdot f_{lm} \cdot Y_{lm}$$

$m=0$ scalar modes
$m=\pm 1$ vector modes
$m\pm 2$ tensor modes

$$f_{lm}=(-i)^l \cdot \sqrt{\frac{2l+1}{4\pi}} \cdot \langle Y_{lm}, f \rangle$$

$$\langle g, f \rangle := \int d\Omega\, g^* \cdot f$$

Boltzmann eqn. becomes an infinite hierarchy of eqns.

$$\frac{df_{lm}}{d\eta}(q,\mathbf{k},\eta)+\frac{k}{\mathcal{H}}\frac{q}{\epsilon} C_{lml'm'} f_{l'm'}(q,\mathbf{k},\eta)=S_{lm}$$

$q=a \cdot p$
$\epsilon=a \cdot E$
$\eta=\ln(a)$

Couples l with $l \pm 1 \times O(1)$ Non- and linear gravity

At linear order only can restrict to scalar modes therefore drop index m

Boltzmann Hierarchy II

Interested in late times $\longrightarrow$ massive neutrinos non relativistic:

$$\left(\frac{q}{\epsilon}\right)^2 \sim \frac{T_\nu}{m_\nu \cdot a} \ll 1$$

Influence of mode $l \pm n$ on l suppressed by $\left(\frac{k}{\mathcal{H}} \frac{q}{\epsilon}\right)^n$

Hierarchy can be truncated at $l = 2$ $\left(f_{lm} = 0 \quad l \geq 3\right)$

See: Masatoshi Shoji & Eiichiro Komatsu(2010)

But left with coupled integral-differential equation with dependence on 5(3) variables!

Boltzmann Hierarchy III

Only first few moments of physical interest

•Energy-momentum-tensor

•Density contrast

Define: $$f_{nl} = \frac{\int_0^\infty dq\; q^2 \cdot \epsilon \cdot \left(\frac{q}{\epsilon}\right)^{2n+l} \cdot f_l}{\int_0^\infty dq\; q^2 \cdot \epsilon \cdot f^{(0)}} \longrightarrow \begin{array}{l} f_{00} = \delta \\ k \cdot f_{01} \propto \theta \end{array}$$

For non-relativistic particles: $f_{nl} \gg f_{NL}$ for $N \geq n \quad L \geq l$

$$w_n = \frac{\int_0^\infty dq\; q^2 \cdot \epsilon \cdot \left(\frac{q}{\epsilon}\right)^{2n} \cdot f^{(0)}}{\int_0^\infty dq\; q^2 \cdot \epsilon \cdot f^{(0)}} = O\left(\left(\frac{T}{a \cdot m}\right)^n\right) \longrightarrow \text{Expansion parameter}$$

On sub horizon scales $\longrightarrow$ Newtonian limit

Boltzmann Hierarchy IV

$\kappa = 8\pi G$

$$f'_{nl} + (2n+l) f_{nl} + \frac{k}{\mathcal{H}} C_{nln'l'} f_{n'l'}$$

$$-\frac{\kappa}{2} \frac{a^2}{k \cdot \mathcal{H}} \delta_{1l} (3+2n) w_n (\rho_{CDM} \cdot \delta + \rho_\nu \cdot f_{00}) = S_{nl}$$

Boltzmann Hierarchy IV

$$\kappa = 8\pi G$$

$$f'_{nl} + (2n+l) f_{nl} + \frac{k}{\mathcal{H}} C_{nln'l'} f_{n'l'}$$

$$-\frac{\kappa}{2}\frac{a^2}{k\cdot\mathcal{H}}\delta_{1l}(3+2n) w_n (\rho_{CDM}\cdot\delta + \rho_\nu \cdot f_{00}) = S_{nl}$$

$$f_{nl} \propto a^{-2n+l}$$

Boltzmann Hierarchy IV

$\kappa = 8\pi G$

$$f'_{nl} + (2n+l) f_{nl} + \frac{k}{\mathcal{H}} C_{nln'l'} f_{n'l'}$$

$$-\frac{\kappa}{2} \frac{a^2}{k \cdot \mathcal{H}} \delta_{1l} (3+2n) w_n (\rho_{CDM} \cdot \delta + \rho_\nu \cdot f_{00}) = S_{nl}$$

Couples mode (n,l) with $(n,l+1)$ and $(n+1,l-1)$

Smaller, but pre factor large on small scales

→ Straightforward truncation not valid

Boltzmann Hierarchy IV

$\kappa = 8\pi G$

$$f'_{nl} + (2n+l) f_{nl} + \frac{k}{\mathcal{H}} C_{nln'l'} f_{n'l'}$$

$$-\boxed{\frac{\kappa}{2} \frac{a^2}{k \cdot \mathcal{H}} \delta_{1l} (3+2n) w_n} \left(\rho_{CDM} \cdot \delta + \rho_\nu \cdot f_{00}\right) = S_{nl}$$

Small for high n
Small on small scales

Boltzmann Hierarchy IV

$\kappa = 8\pi G$

$$f'_{nl} + (2n+l) f_{nl} + \frac{k}{\mathcal{H}} C_{nln'l'} f_{n'l'}$$

$$-\frac{\kappa}{2} \frac{a^2}{k \cdot \mathcal{H}} \delta_{1l} (3+2n) w_n (\rho_{CDM} \cdot \delta + \rho_\nu \cdot f_{00}) = S_{nl}$$

Full free streaming
Ignore self gravity for high moments

Gilbert's Equation I

Boltzmann equation can be written as an integral equation

$$f(\eta,\mathbf{k},\mathbf{q})=e^{-i\mathbf{k}\cdot\mathbf{l}(\eta,\eta_i)}\cdot f_i(\mathbf{k},\mathbf{q})+i\int_{\eta_i}^{\eta} d\eta' \frac{\mathbf{k}\cdot\mathbf{q}\cdot\epsilon}{a\cdot\mathcal{H}\cdot q} e^{-i\mathbf{k}\cdot\mathbf{l}(\eta,\eta')}\cdot\partial_q f^{(0)}\cdot\Phi+S[f]$$

Greens function without gravity

$$l(\eta,\eta_i)=\int_{\eta_i}^{\eta} d\eta' \frac{1}{\mathcal{H}}\frac{q}{\epsilon}$$

For one Moment at the Linear order this yields

$$f_{nlm}+(-i)^l\cdot\delta_{m0}\frac{\kappa}{2k}\int_{\eta_i}^{\eta} d\eta' \left(\rho_{CDM}\cdot\delta+\rho_\nu\cdot f_{000}\right)\Pi_{nl}(\eta,\eta')=I_{nlm}$$

$$I_{nlm}=\left(f_{nlm}\right)_{\Phi=0}$$

$$\Pi_{nl}(\eta,\eta')=\frac{a^2(\eta')}{\mathcal{H}(\eta')}\underbrace{\frac{1}{(a(\eta)\cdot m)^{2n+l-1}}\int_0^\infty dq \frac{f^{(0)}}{\tilde{f}_0^{(0)}} q^{(2n+l+2)}}_{\leqslant C_{nl}\cdot w_{n+\frac{l}{2}}(\eta')}\cdot F_{nl}(k\cdot l(\eta',\eta))$$

Gilbert's Equation II

Ignoring CDM this gives a closed equation for the density contrast

$$f_{00}+\frac{\kappa}{2k}\int_{\eta_i}^{\eta} d\eta'\,\rho\cdot f_{00}\cdot\Pi_{00}(\eta,\eta')=I_{000}$$

$$f_{00}=\frac{\int_0^\infty dq\, q^2\cdot\epsilon\int d\Omega\, f}{\int_0^\infty dq\, q^2\epsilon\int d\Omega\, f^{(0)}}$$

Has a solution as a power Series in $\frac{\kappa}{2k}$

$$f_{00}=I_{00}-\frac{\kappa}{2k}\int_{\eta_i}^{\eta} d\eta_1\,\rho\cdot I_{00}\cdot\Pi_{00}(\eta,\eta_1)$$

$$+\left(\frac{\kappa}{2k}\right)^2\int_{\eta_i}^{\eta} d\eta_1\,\rho\cdot\Pi_{00}(\eta,\eta_1)\cdot\int_{\eta_i}^{\eta_1} d\eta_2\,\rho\cdot I_{00}\cdot\Pi_{00}(\eta,\eta_2)+\dots$$

Only valid on small scales!

Combine this approach with moment equations

Closed equation for a set of (interesting) moments

Gilbert's Equation III

Replace stress and pressure by the integral equation

$$f'_{00} + \frac{k}{\mathcal{H}} \frac{1}{3} f_{01} = 0$$

$$f'_{01} + f_{01} - \frac{3\kappa}{2} \frac{w_0 \cdot a^2}{k \cdot \mathcal{H}} \rho \cdot f_{00}$$

$$+ \frac{\kappa}{\mathcal{H}^2} \int_{\eta_i}^{\eta} d\eta' \, \rho \cdot f_{00} \left(\frac{2}{7} \Pi_{02}(\eta, \eta') - \Pi_{10}(\eta, \eta') \right) = \frac{k}{\mathcal{H}} \left(I_{10} - \frac{2}{7} I_{02} \right)$$

Pressure: $$P \propto f_{10} = \frac{\int_0^\infty dq \, q^2 \cdot \epsilon \left(\frac{q}{\epsilon} \right)^2 \int d\Omega \, f}{\int_0^\infty dq \, q^2 \epsilon \int d\Omega \, f^{(0)}}$$

Scalar stress: $$\sigma \propto f_{02} = -\frac{\sqrt{5} \int_0^\infty dq \, q^2 \cdot \epsilon \left(\frac{q}{\epsilon} \right)^2 \int d\Omega \, Y_{20}^* f}{\int_0^\infty dq \, q^2 \epsilon \int d\Omega \, f^{(0)}}$$

Gilbert's Equation III

Replace stress and pressure by the integral equation

$$f'_{00}+\frac{k}{\mathcal{H}}\frac{1}{3}f_{01}=0$$

$$f'_{01}+f_{01}-\frac{3\kappa}{2}\frac{w_0\cdot a^2}{k\cdot\mathcal{H}}\rho\cdot f_{00}$$

→ Greens function: $\mathbf{g}(\eta,\eta')$

→ Resummation of terms: $\left(\frac{3\kappa\cdot w_0}{2k}\right)^n$

$$+\frac{\kappa}{\mathcal{H}^2}\int_{\eta_i}^{\eta} d\eta'\,\rho\cdot f_{00}\left(\frac{2}{7}\Pi_{02}(\eta,\eta')-\Pi_{10}(\eta,\eta')\right)=\frac{k}{\mathcal{H}}\left(I_{10}-\frac{2}{7}I_{02}\right)$$

$$\delta=f_{00}=\frac{\int_0^\infty dq\,q^2\cdot\epsilon\int d\Omega\,f}{\int_0^\infty dq\,q^2\epsilon\int d\Omega\,f^{(0)}}$$

$$\theta\propto k\cdot f_{01}=-ik\frac{\sqrt{3}\int_0^\infty dq\,q^2\cdot\epsilon\int d\Omega\,Y^*_{lm}f}{\int_0^\infty dq\,q^2\epsilon\int d\Omega\,f^{(0)}}$$

Gilbert's Equation III

Replace stress and pressure by the integral equation

$$f'_{00}+\frac{k}{\mathcal{H}}\frac{1}{3}f_{01}=0$$

→ Greens function: $\mathbf{g}(\eta,\eta')$

→ Resummation of terms: $\left(\frac{3\kappa\cdot w_0}{2k}\right)^n$

$$f'_{01}+f_{01}-\frac{3\kappa}{2}\frac{w_0\cdot a^2}{k\cdot\mathcal{H}}\rho\cdot f_{00}$$

$$+\frac{\kappa}{\mathcal{H}^2}\int_{\eta_i}^{\eta}d\eta'\,\rho\cdot f_{00}\left(\frac{2}{7}\Pi_{02}(\eta,\eta')-\Pi_{10}(\eta,\eta')\right)=\boxed{\frac{k}{\mathcal{H}}\left(I_{10}-\frac{2}{7}I_{02}\right)}$$

$$f=\mathbf{g}(\eta,\eta_i)\cdot\mathbf{f}_i+\int_{\eta_i}^{\eta}d\eta'\,\mathbf{g}(\eta,\eta')\cdot\begin{pmatrix}0\\1\end{pmatrix}\frac{k}{\mathcal{H}}\left(I_{10}-\frac{2}{7}I_{02}\right)$$

$$f=\begin{pmatrix}f_{00}\\f_{01}\end{pmatrix}$$

Gilbert's Equation III

Replace stress and pressure by the integral equation

$$f'_{00}+\frac{k}{\mathcal{H}}\frac{1}{3}f_{01}=0 \quad\longrightarrow\quad \text{Greens function: } \mathbf{g}(\eta,\eta')$$

$$\longrightarrow\quad \text{Resummation of terms: } \left(\frac{3\kappa\cdot w_0}{2k}\right)^n$$

$$f'_{01}+f_{01}-\frac{3\kappa}{2}\frac{w_0\cdot a^2}{k\cdot\mathcal{H}}\rho\cdot f_{00}$$

$$+\boxed{\frac{\kappa}{\mathcal{H}^2}\int_{\eta_i}^{\eta}d\eta'\,\rho\cdot f_{00}\left(\frac{2}{7}\Pi_{02}(\eta,\eta')-\Pi_{10}(\eta,\eta')\right)}=\frac{k}{\mathcal{H}}\left(I_{10}-\frac{2}{7}I_{02}\right)$$

$$f=\mathbf{g}(\eta,\eta_i)\cdot\mathbf{f}_i+\int_{\eta_i}^{\eta}d\eta'\,\mathbf{g}(\eta,\eta')\cdot\begin{pmatrix}0\\1\end{pmatrix}\frac{k}{\mathcal{H}}\left(I_{10}-\frac{2}{7}I_{02}\right)$$

$$+\int_{\eta_i}^{\eta}d\eta'\,\mathbf{g}(\eta,\eta')\cdot\begin{pmatrix}0\\1\end{pmatrix}\frac{\kappa}{\mathcal{H}^2}\int_{\eta_i}^{\eta'}d\tilde{\eta}\,\rho\cdot f_{00}\left(\frac{2}{7}\Pi_{02}(\eta,\tilde{\eta})-\Pi_{10}(\eta,\tilde{\eta})\right)$$

Valid on small and large scales!

Near the free streaming scale???

Extension to Higher Order

- Reduce to interesting moments
  - Like CDM
    - More quantities
    - Effective initial conditions
    - Small corrections to greens function
  - Large relative errors for "high" moments
- Take only density contrast, use formal solution to calculate full distribution function or some moments
  - Same relative error for all moments
  - Slightly more complicated
  - Physical less intuitive
    - Physical meaningful equations only intermediate step

Summary

- Boltzmann eqn. can be simplified by moment expansion
  - Momentum direction into spherical harmonics
  - Magnitude into powers of velocity
- High moments small compared to low
  - But straightforward truncation not valid
- Take free streaming full into account
- Influence of gravity on high modes leads to small corrections for low modes
- Solution of linear problem can be obtained
  - Use as starting point for a systematic treatment of higher orders