Small perturbations of non-Hermitian matrices

Ofer Zeitouni
Based on joint works with Anirban Basak and Elliot Paquette

December 2019
An empirical fact

\[ J_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & 0
\end{pmatrix}, \quad P_N(z) = \det(zI - J_N) = z^N, \quad \text{roots}=0. \]
An empirical fact

\[ J_N = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & 1 \\
0 & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}, \quad P_N(z) = \det(zI - J_N) = z^N, \quad \text{roots}=0. \]

\( \hat{J}_N := U_N J_N U_N^* \) where \( U_N \) is random unitary matrix, Haar-distributed. Of course, \( \text{Spec}(\hat{J}_N) = \text{Spec}(J_N) \).
An empirical fact

\[ J_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}, \quad P_N(z) = \det(zI - J_N) = z^N, \quad \text{roots}=0. \]

\[ \hat{J}_N := U_N J_N U_N^*, \text{ where } U_N \text{ is random unitary matrix, Haar-distributed. Of course, } \text{Spec}(\hat{J}_N)=\text{Spec}(J_N). \]
An empirical fact

\[ J_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}, \quad P_N(z) = \det(zI - J_N) = z^N, \quad \text{roots}=0. \]

\( \hat{J}_N := U_N J_N U_N^* \) where \( U_N \) is random unitary matrix, Haar-distributed. Of course, \( \text{Spec}(\hat{J}_N) = \text{Spec}(J_N) \).

Goes back to Trefethen et al's - pseudo-spectrum.
A probability measure on $\mathbb{C}$ is characterized by its logarithmic potential

$$\mathcal{L}_\mu(z) = \int \log |z - x| \mu(dx).$$
A probability measure on $\mathbb{C}$ is characterized by its logarithmic potential

$$ \mathcal{L}_\mu(z) = \int \log |z - x|\mu(dx). $$

Further, $\mu_n \to \mu$ weakly if and only if $\mathcal{L}_{\mu_n}(z) \to \mathcal{L}_\mu(z)$, for Lebesgue almost every $z \in \mathbb{C}$. 
A probability measure on $\mathbb{C}$ is characterized by its logarithmic potential

$$\mathcal{L}_\mu(z) = \int \log |z - x| \mu(dx).$$

Further, $\mu_n \to \mu$ weakly if and only if $\mathcal{L}_{\mu_n}(z) \to \mathcal{L}_\mu(z)$, for Lebesgue almost every $z \in \mathbb{C}$.

For the empirical measure $L_N = N^{-1} \sum_{i=1}^{N} \lambda_i^A$ of eigenvalues of a matrix $A$, we have

$$\mathcal{L}_{L_N}(z) = \frac{1}{2} \log \det(z - A)(z - A)^*.$$
A probability measure on $\mathbb{C}$ is characterized by its logarithmic potential

$$\mathcal{L}_\mu(z) = \int \log |z - x| \mu(dx).$$

Further, $\mu_n \to \mu$ weakly if and only if $\mathcal{L}_{\mu_n}(z) \to \mathcal{L}_\mu(z)$, for Lebesgue almost every $z \in \mathbb{C}$.

For the empirical measure $L_N = N^{-1} \sum_{i=1}^{N} \lambda_i^A$ of eigenvalues of a matrix $A$, we have

$$\mathcal{L}_{L_N}(z) = \frac{1}{2} \log \det(z - A)(z - A)^*. $$

Thus, spectrum computations involves the determinant of a family of Hermitian matrices built from $A$!
Śniady’s theorem

Assume $A_N \to^* a$. 

Theorem (Śniady ’02)

$$
\lim_{t \to 0} \lim_{N \to \infty} L_{A_N(t)} = \nu_a.
$$

(Brown measure - given by log-potential of $a$)

In particular, some sequence of noise regularizes empirical measure to $\nu_a$.

Builds on regularization ideas of Haagerup.

Main ingredient of proof compares the singular values $\Sigma_{A_N(t)} = (\sigma_{A_N(1)}, \ldots, \sigma_{A_N(N)})$ of $A_N(t) = A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of $\Sigma_0$, $\Sigma_{A_N}$, for each coordinate-wise increasing $N-1 \text{tr} (f(\Sigma_{A_N(t)})) \geq N-1 \text{tr} (f(\Sigma_0(t)))$.

This gives required control of the determinant; Second part of theorem follows by diagonalization argument.

How can we take $t = t_N \to 0$?
\( A_N \rightarrow^* a \). Define \( A_N(t) = A_N + tN^{-1/2}G_N \).


\[ \text{Sniady’s theorem} \]

Assume \( A_N \to^* a \). Define \( A_N(t) = A_N + tN^{-1/2}G_N \).

**Theorem (Sniady ’02)**

\[ \lim_{t \to 0} \lim_{N \to \infty} L^A_N(t) = \nu_a. \quad (\text{Brown measure - given by log-potential of } a) \]
🚈 Noise Stability

**Śniady’s theorem**

Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

**Theorem (Śniady ’02)**

$$\lim_{t \to 0} \lim_{N \to \infty} L_{N}^{A_N(t)} = \nu_a. \quad (Brown \ measure\ - \ given \ by \ log-potential \ of \ a)$$

*In particular, some sequence of noise regularizes empirical measure to the Brown measure.*

Builds on regularization ideas of Haagerup.
**Śniady’s theorem**

Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

**Theorem (Śniady ’02)**

$$\lim_{t\to 0} \lim_{N\to \infty} L_N^{A_N(t)} = \nu_a \cdot (Brown \ measure \ - \ given \ by \ log\-potential \ of \ a)$$

In particular, some sequence of noise regularizes empirical measure to the Brown measure.

Builds on regularization ideas of Haagerup.

Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$;
**Śniady’s theorem**

Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

**Theorem (Śniady ’02)**

$$\lim_{t \to 0} \lim_{N \to \infty} L_N^{A_N(t)} = \nu_a. \text{ (Brown measure - given by log-potential of } a)$$

In particular, some sequence of noise regularizes empirical measure to the Brown measure.

Builds on regularization ideas of Haagerup.

Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of $\Sigma_0, \Sigma_A$, for $f$ coordinate-wise increasing,

$$N^{-1} \text{tr}(f(\Sigma_A(t))) \geq N^{-1} \text{tr}(f(\Sigma_0(t))).$$
Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

**Theorem (Śniady ’02)**

$$\lim_{t \to 0} \lim_{N \to \infty} L_N^{A_N(t)} = \nu_a. \text{ (Brown measure - given by log-potential of } a)$$

*In particular, some sequence of noise regularizes empirical measure to the Brown measure.*

Builds on regularization ideas of Haagerup.

Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of $\Sigma_0, \Sigma_A$, for $f$ coordinate-wise increasing,

$$N^{-1}\text{tr}(f(\Sigma_A(t))) \geq N^{-1}\text{tr}(f(\Sigma_0(t))).$$

This gives required control of the determinant; Second part of theorem follows by diagonalization argument.
**Śniady’s theorem**

Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

**Theorem (Śniady ’02)**

$$\lim_{t \to 0} \lim_{N \to \infty} L_N^{A_N(t)} = \nu_a. \quad (Brown \ measure \ - \ given \ by \ log-potential \ of \ a)$$

*In particular, some sequence of noise regularizes empirical measure to the Brown measure.*

Builds on regularization ideas of Haagerup.

Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of $\Sigma_0, \Sigma_A$, for $f$ coordinate-wise increasing,

$$N^{-1} \text{tr}(f(\Sigma_A(t))) \geq N^{-1} \text{tr}(f(\Sigma_0(t))).$$

This gives required control of the determinant; Second part of theorem follows by diagonalization argument.

**How can we take $t = t_N \to 0$?**
Regularization by noise

Consider the nilpotent $N$-by-$N$ matrix

$$J_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}$$
Regularization by noise

Consider the nilpotent $N$-by-$N$ matrix

$$J_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & 0 \\
\end{pmatrix}$$

Eigenvalues $\lambda_i = 0$, empirical measure $n^{-1} \sum \delta_{\lambda_i} = \delta_0$. 
Consider the nilpotent $N$-by-$N$ matrix

$$J_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Eigenvalues $\lambda_i = 0$, empirical measure $n^{-1} \sum \delta_{\lambda_i} = \delta_0$. 

Ofer Zeitouni
Regularization by noise II

Set $\gamma > 1/2$. 

Theorem (Guionnet-Wood-Z. '14) Set $A_N = \gamma N + N - \gamma G_N$, empirical measure of eigenvalues $L_{A_N}$. Then $L_{A_N}$ converges weakly to the uniform measure on the unit circle in the complex plane. Thus, $L_{J_N} = \delta_0$ but for a vanishing perturbation, $L_{A_N}$ has different limit. 

Earlier version - Davies-Hager '09 (Generalization to i.i.d. $G_N$: Wood '15.)
Regularization by noise II

Set $\gamma > 1/2$.

Theorem (Guionnet-Wood-Z. ’14)

Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues $L^A_N$. Then $L^A_N$ converges weakly to the uniform measure on the unit circle in the complex plane.
Regularization by noise II

Set $\gamma > 1/2$. 

Theorem (Guionnet-Wood-Z. ’14)

Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues $L^A_N$. Then $L^A_N$ converges weakly to the uniform measure on the unit circle in the complex plane.

Thus, $L^J_N = \delta_0$ but for a vanishing perturbation, $L^A_N$ has different limit. Earlier version - Davies-Hager ’09
Set $\gamma > 1/2$.

**Theorem (Guionnet-Wood-Z. ’14)**

Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues $L^A_N$. Then $L^A_N$ converges weakly to the uniform measure on the unit circle in the complex plane.

Thus, $L^J_N = \delta_0$ but for a vanishing perturbation, $L^A_N$ has different limit. Earlier version - Davies-Hager ’09 (Generalization to i.i.d. $G_N$: Wood ’15.)
Regularization by noise II

Set $\gamma > 1/2$.

**Theorem (Guionnet-Wood-Z. ’14)**

Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues $L_N^A$. Then $L_N^A$ converges weakly to the uniform measure on the unit circle in the complex plane.

Thus, $L_N^{J_N} = \delta_0$ but for a vanishing perturbation, $L_N^A$ has different limit.

Earlier version - Davies-Hager ’09
(Generalization to i.i.d. $G_N$: Wood ’15.)
What is going on?

\[ J_\delta^N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
& & \ddots & \vdots & \ddots & \vdots \\
0 & & \cdots & 0 & 1 \\
\delta_N & & \cdots & \cdots & 0 & 0
\end{pmatrix} \]

Characteristic polynomial:

\[ P_N(z) = \det(zI - J_\delta^N) = z^N \pm \delta^N. \]

Roots:

\[ \{ \delta_1^N \} \]

If \( \delta^N \to 0 \) polynomially slowly then \( L J_\delta^N \) converges to uniform on circle.

Why is this particular perturbation picked up?

General criterion - Guionnet, Wood, Z. Ofer Zeitouni

Small Perturbations
What is going on?

\[
J^\delta_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\delta_N & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]

Characteristic polynomial:

\[
P_N(z) = \det(zI - J^\delta_N) = z^N \pm \delta_N.
\]
What is going on?

\[ J^\delta_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 \\
\delta_N & \cdots & \cdots & \cdots & 0
\end{pmatrix} \]

Characteristic polynomial:

\[ P_N(z) = \det(zI - J^\delta_N) = z^N \pm \delta_N. \]

Roots: \[ \{ \delta^1_N e^{2\pi i/N} \}_{i=1}^N \].
What is going on?

\[
J_{\delta}^N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 \\
\delta_N & \cdots & \cdots & \cdots & 0 & 0
\end{pmatrix}
\]

Characteristic polynomial:

\[
P_N(z) = \det(zI - J_{\delta}^N) = z^N \pm \delta_N.
\]

Roots: \( \{\delta_{\delta}^{1/N} e^{2\pi i/N}\}_{i=1}^N \).

If \( \delta_N = 0 \) then \( L_{\delta}^N = \delta_0. \)
What is going on?

\[
J_N^\delta = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
& & & & & \\
& & & & & \\
0 & & & & & 0 & 1 \\
\delta_N & & & & & 0 & 0 \\
\end{pmatrix}
\]

Characteristic polynomial:

\[
P_N(z) = \det(zI - J_N^\delta) = z^N \pm \delta_N.
\]

Roots: \(\{\delta_N^{1/N} e^{2\pi i/N}\}_{i=1}^N\).

If \(\delta_N = 0\) then \(L_N^{J_N^\delta} = \delta_0\).

If \(\delta_N \to 0\) polynomially slowly then \(L_N^{J_N^\delta}\) converges to uniform on circle.
What is going on?

\[ J_{N}^{\delta} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \delta_{N} & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \]

Characteristic polynomial:

\[ P_{N}(z) = \det(zI - J_{N}^{\delta}) = z^{N} \pm \delta_{N}. \]

Roots: \( \{\delta_{N}^{1/N} e^{2\pi i/N}\}_{i=1}^{N} \).

If \( \delta_{N} = 0 \) then \( L_{N}^{J_{N}^{\delta}} = \delta_{0} \).

If \( \delta_{N} \to 0 \) polynomially slowly then \( L_{N}^{J_{N}^{\delta}} \) converges to uniform on circle.

Why is this particular perturbation picked up?
What is going on?

\[ J^\delta_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ \delta_N & \cdots & \cdots & \cdots & 0 \end{pmatrix} \]

Characteristic polynomial:

\[ P_N(z) = \det(zI - J^\delta_N) = z^N \pm \delta_N. \]

Roots: \( \{\delta^1_N e^{2\pi i/N}\}_{i=1}^N \).

If \( \delta_N = 0 \) then \( L_{N}^{J^\delta_N} = \delta_0 \).

If \( \delta_N \to 0 \) polynomially slowly then \( L_{N}^{J^\delta_N} \) converges to uniform on circle.

Why is this particular perturbation picked up?

General criterion - Guionnet, Wood, Z.
Noise Stability-Maximal Nilpotent

\( a \in \mathcal{A} \) is regular if for \( \psi \) smooth, compactly supported,

\[
\lim_{\epsilon \to 0} \int_{\mathbb{C}} \Delta \psi(z) \left( \int_0^\epsilon \log x \, d\nu^z_a(x) \right) \, dz = 0
\]

(\( \nu^z_a \) - spectral measure of \(|a - z|\)).
Noise Stability-Maximal Nilpotent

\( a \in \mathcal{A} \) is regular if for \( \psi \) smooth, compactly supported,

\[
\lim_{\epsilon \to 0} \int_C \Delta \psi(z) \left( \int_0^\epsilon \log x \, d\nu_z^z(x) \right) \, dz = 0
\]

(\( \nu_z^z \) - spectral measure of \( |a - z| \)).

Theorem (Guionnet-Wood-Z. ’14)

Assume: \( A_N \to^* a \), regular.
a ∈ A is **regular** if for ψ smooth, compactly supported,

\[
\lim_{\epsilon \to 0} \int_{\mathbb{C}} \Delta \psi(z) \left( \int_0^\epsilon \log x \, d\nu^z_a(x) \right) \, dz = 0
\]

(\nu^z_a - spectral measure of \(|a - z|\)).

**Theorem (Guionnet-Wood-Z. ’14)**

*Assume: \(A_N \to^* a, \text{ regular. } L^A_N \to \nu_a \text{ weakly.})*
Noise Stability-Maximal Nilpotent

\( a \in A \) is regular if for \( \psi \) smooth, compactly supported,

\[
\lim_{\epsilon \to 0} \int_C \Delta \psi(z) \left( \int_0^\epsilon \log x \, d\nu_{\bar{a}}^z(x) \right) \, dz = 0
\]

(\( \nu_{\bar{a}}^z \) - spectral measure of \(|a - z|\)).

**Theorem (Guionnet-Wood-Z. ’14)**

Assume: \( A_N \xrightarrow{\ast} a \), regular. \( L_N^A \to \nu_a \) weakly. \( \gamma > 1/2 \).
Noise Stability-Maximal Nilpotent

\( a \in \mathcal{A} \) is **regular** if for \( \psi \) smooth, compactly supported,

\[
\lim_{\epsilon \to 0} \int_{\mathbb{C}} \Delta \psi(z) \left( \int_0^\epsilon \log x \, d\nu_{z_\epsilon}(x) \right) \, dz = 0
\]

(\( \nu_{z_\epsilon} \) - spectral measure of \(|a - z|\)).

**Theorem (Guionnet-Wood-Z. ’14)**

Assume: \( A_N \to^* a, \text{ regular. } L_N^{A} \to \nu_a \text{ weakly. } \gamma > 1/2. \) Then,

\( L_N^{A_N+N^{-\gamma}G_N} \to \nu_a \text{ weakly, in probability.} \)
$a \in \mathcal{A}$ is regular if for $\psi$ smooth, compactly supported,

$$\lim_{\epsilon \to 0} \int_{\mathbb{C}} \Delta \psi(z) \left( \int_{0}^{\epsilon} \log x \, d\nu_{a}^{z}(x) \right) \, dz = 0$$

($\nu_{a}^{z}$ - spectral measure of $|a - z|$).

**Theorem (Guionnet-Wood-Z. ’14)**

Assume: $A_{N} \to^{*} a$, regular. $L_{N}^{A_{N}} \to \nu_{a}$ weakly. $\gamma > 1/2$. Then,

$L_{N}^{A_{N}+N^{-\gamma}G_{N}} \to \nu_{a}$ weakly, in probability.

The proof uses the regularity (of the limit) to truncate the singularity of the log... and depends crucially on convergence to $\nu_{a}$. 
$a \in \mathcal{A}$ is regular if for $\psi$ smooth, compactly supported,

$$\lim_{\epsilon \to 0} \int_{\mathbb{C}} \Delta \psi(z) \left( \int_0^\epsilon \log x \, d\nu_{a}^z(x) \right) \, dz = 0$$

($\nu_{a}^z$ - spectral measure of $|a - z|$).

**Theorem (Guionnet-Wood-Z. ’14)**

Assume: $A_N \to^* a$, regular. $L_N^A \to \nu_a$ weakly. $\gamma > 1/2$. Then,

$L_{N}^{A_{N} + N^{-\gamma} G_{N}} \to \nu_a$ weakly, in probability.

The proof uses the regularity (of the limit) to truncate the singularity of the $\log$... and depends crucially on convergence to $\nu_a$. But it is not useful in maximally nilpotent example, since $L_N^A = \delta_0 \not\to \nu_a = \delta_{S^1}$.
Theorem (Guionnet-Wood-Z. ’14)

Assume: $A_N \rightarrow^* a$, regular,
Theorem (Guionnet-Wood-Z. ’14)

Assume: $A_N \rightarrow^* a$, regular, $\|E_N\| \rightarrow 0$ polynomially. $L_{N}^{A_N+E_N} \rightarrow \nu_a$ weakly.

Then $L_{N}^{A_N+E_N} \rightarrow \nu_a$ weakly, in probability.

So it is enough to find a perturbation with correct limiting behavior!

Nilpotent example uses $a$ - unitary element (which is regular), $E_N$ is $(N,1)$ element.
Theorem (Guionnet-Wood-Z. ’14)

Assume: \( A_N \to^* a, \) \( \) regular, \( \|E_N\| \to 0 \) polynomially. \( L_N^{A_N+E_N} \to \nu_a \) weakly. Then \( L_N^{A_N+N^{-\gamma}G_N} \to \nu_a \) weakly, in probability.
Theorem (Guionnet-Wood-Z. ’14)

Assume: \( A_N \to^* a \), regular, \(|| E_N || \to 0\) polynomially. \( L_N^{A_N + E_N} \to \nu_a\) weakly. Then \( L_N^{A_N + N^{-\gamma} G_N} \to \nu_a\) weakly, in probability.

So it is enough to find a perturbation with correct limiting behavior!
Theorem (Guionnet-Wood-Z. ’14)

Assume: $A_N \rightarrow^* a$, regular, $\|E_N\| \rightarrow 0$ polynomially. $L_N^{A_N+E_N} \rightarrow \nu_a$ weakly. Then $L_N^{A_N+N^{-\gamma}G_N} \rightarrow \nu_a$ weakly, in probability.

So it is enough to find a perturbation with correct limiting behavior! Nilpotent example uses $a$- unitary element (which is regular), $E_N$ is $(N, 1)$ element.
Noise Stability-Nilpotent matrices

Maybe this always works?
Noise Stability-Nilpotent matrices

Maybe this always works? $J_b$ - maximally nilpotent of dimension $b$. 

\begin{align*}
J_b - \text{maximally nilpotent of dimension } b.
\end{align*}
Noise Stability

Noise Stability-Nilpotent matrices

Maybe this always works? $J_b$ - maximally nilpotent of dimension $b$.

$$J_{b,N} = \begin{bmatrix} J_b & & \\ & J_b & \\ & & J_b \end{bmatrix}$$

Theorem (Guionnet-Wood-Z '14)

If $b = a \log N$ and $\gamma$ is large enough, then the spectral radius of $J_{b,N} + N - \gamma G_N$ is uniformly strictly smaller than 1. In particular, $L_{J_{a \log N,N} + N - \gamma G_N} \not\to \delta_1$ even though $J_{a \log N,N}$ converges in $\ast$ moments to random unitary!
Noise Stability-Nilpotent matrices

Maybe this always works? $J_b$ - maximally nilpotent of dimension $b$.

$$J_{b,N} = \begin{bmatrix} J_b & & \\ & J_b & \\ & & \ddots & J_b \end{bmatrix}$$

**Theorem (Guionnet-Wood-Z ’14)**

If $b = a \log N$ and $\gamma$ is large enough, then the spectral radius of $J_{b,N} + N^{-\gamma} G_N$ is uniformly strictly smaller than 1. In particular,

$$L_{N}^{J_{a \log N, N} + N^{-\gamma} G_N} \not\rightarrow \delta_{S^1}$$

even though $J_{a \log N, N}$ converges in $\ast$ moments to random unitary!
A generalization: $B^i = B^i(N)$ - Jordan blocks, dimension $a_i(N) \log N$, eigenvalue $c_i(N)$. 

A generalization: $B^i = B^i(N)$ - Jordan blocks, dimension $a_i(N) \log N$, eigenvalue $c_i(N)$.

$$A_N = \begin{bmatrix} B^1 & & \\ & B^2 & \\ & & \ddots \\ & & & B^{\ell(N)} \end{bmatrix}.$$
Simulations inconclusive!

\[ \Re z \gamma = 1.0 \]

\[ \Re z \gamma = 0.8 \]
Noise Stability-Block Nilpotent IV

Simulations inconclusive!

More general models?

Figure: The eigenvalues of $J_N + J_N^2 + N^{-\gamma}G_N$, with $N = 4000$ and various $\gamma$. On left, actual matrix. On the right, $U_N(J_N + J_N^2)U_N^*$. 
More general models?

Figure: The eigenvalues of $D_N + J_N + N^{-\gamma} G_N$, with $N = 4000$ and various $\gamma$. Top: $D_N(i, i) = -1 + 2i/N$. Bottom: $D_N$ i.i.d. uniform on $[-2, 2]$. On left, actual matrix. On the right, $U_N(D_N + J_N)U_N^*$. 
More general models

Theorem (Basak, Paquette, Z. ’17)

\[ T_N = D_N + J_N, \quad M_N = T_N + N^{-\gamma} G_N, \quad \gamma > 1/2. \]

\( d_i \) iid uniform on \([-1, 1]\).
More general models

Theorem (Basak, Paquette, Z. ’17)

\[ T_N = D_N + J_N, \quad M_N = T_N + N^{-\gamma} G_N, \quad \gamma > 1/2. \]

\( d_i \) iid uniform on \([-1, 1]\).

Then \( L_N \to \mu, \mu \) explicit: log-potential of \( \mu \) at \( z \) is \((E \log |z - d_1|) \vee 0)\).
More general models

Theorem (Basak, Paquette, Z. ’17)

\[ T_N = \sum_{i=0}^{k} a_i J_N^i \] (Toeplitz, finite symbol, upper triangular). Then,

\[ L_N \to \text{Law of } \sum_{i=0}^{k} a_i U^i \]

where \( U \) is uniform on unit circle.
More general models

Theorem (Basak, Paquette, Z. '17)

\[ T_N = \sum_{i=0}^{k} a_i J_N^i \] (Toeplitz, finite symbol, upper triangular). Then,

\[ L_N \to \text{Law of } \sum_{i=0}^{k} a_i U^i \]

where \( U \) is uniform on unit circle.

Extends to twisted Toeplitz \( T_N(i, j) = a_i(j/N), i = 1, \ldots, k, a_i \) continuous:

\[ L_N \to \int_0^1 \text{Law of } \sum_{i=0}^{k} a_i(t) U^i \]
More general models

Theorem (Basak, Paquette, Z. ’17)

\[ T_N = \sum_{i=0}^{k} a_i J_N^i \] (Toeplitz, finite symbol, upper triangular). Then,

\[ L_N \rightarrow \text{Law of } \sum_{i=0}^{k} a_i U^i \]

where \( U \) is uniform on unit circle.

Extends to twisted Toeplitz \( T_N(i, j) = a_i(j/N) \), \( i = 1, \ldots, k \), \( a_i \) continuous:

\[ L_N \rightarrow \int_0^1 \text{Law of } \sum_{i=0}^{k} a_i(t) U^i \]

Confirms simulations and predictions (based on pseudo-spectrum) of Trefethen et als. Some two-diagonal Toeplitz cases studied by Sjöstrand and Vogel (2016)
Recall $T_N = M_N + N^{-\gamma} G_N$, $\gamma > 1/2$, $G_N$ complex Gaussian.
Recall $T_N = M_N + N^{-\gamma} G_N$, $\gamma > 1/2$, $G_N$ complex Gaussian

Write $zI - M_N = U \Sigma_N V^*$, $\Sigma_N$ - diagonal, singular values, arranged non-decreasing, and then

$$\Sigma = \Sigma_N = \begin{pmatrix} S_N & \ \ \\ B_N & \end{pmatrix}, \quad N^{-\gamma} G_N = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$ 

where $S_N$ has dimension $N^* \times N^*$.
Recall $T_N = M_N + N^{-\gamma} G_N$, $\gamma > 1/2$, $G_N$ complex Gaussian

Write $zI - M_N = U \Sigma_N V^*$, $\Sigma_N$ - diagonal, singular values, arranged non-decreasing, and then

$$\Sigma = \Sigma_N = \begin{pmatrix} S_N & \ & \ \\ & B_N & \ \\ & & \end{pmatrix}, \quad N^{-\gamma} G_N = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

where $S_N$ has dimension $N^* \times N^*$.

Define $N^*$ as

$$\sup\{i \geq 1 : \Sigma_{ii}(z) \leq \epsilon_N^{-1} N^{-\gamma} (N - i)^{1/2}\}, \quad \epsilon_N = N^{-\eta}$$
Recall $T_N = M_N + N^{-\gamma} G_N$, $\gamma > 1/2$, $G_N$ complex Gaussian

Write $z I - M_N = U \Sigma_N V^*$, $\Sigma_N$ - diagonal, singular values, arranged non-decreasing, and then

$$
\Sigma = \Sigma_N = \begin{pmatrix} S_N \\ B_N \end{pmatrix}, \quad N^{-\gamma} G_N = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.
$$

where $S_N$ has dimension $N^* \times N^*$.

Define $N^*$ as

$$
\sup\{i \geq 1 : \Sigma_{ii}(z) \leq \epsilon_N^{-1} N^{-\gamma}(N - i)^{1/2}\}, \quad \epsilon_N = N^{-\eta}
$$

Theorem (Basak-Paquette-Z. ’17 - Deterministic equivalence)

If $N^* = o(N/\log N)$ then

$$
\frac{1}{N} \log |\det T_N| - \frac{1}{N} \log |\det B_N| \to 0.
$$
Recall \( T_N = M_N + N^{-\gamma} G_N, \gamma > 1/2, \) \( G_N \) complex Gaussian

Write \( zI - M_N = U\Sigma_N V^*, \Sigma_N \) - diagonal, singular values, arranged non-decreasing, and then

\[
\Sigma = \Sigma_N = \begin{pmatrix} S_N & B_N \end{pmatrix}, \quad N^{-\gamma} G_N = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.
\]

where \( S_N \) has dimension \( N^* \times N^* \).

Define \( N^* \) as

\[
\sup\{i \geq 1 : \Sigma_{ii}(z) \leq \epsilon_N^{-1} N^{-\gamma}(N - i)^{1/2}\}, \quad \epsilon_N = N^{-\eta}
\]

Theorem (Basak-Paquette-Z. ’17 - Deterministic equivalence)

If \( N^* = o(N/\log N) \) then

\[
\frac{1}{N} \log |\det T_N| - \frac{1}{N} \log |\det B_N| \rightarrow 0.
\]

So only need to understand small singular values of \( M_N \).
Non triangular Toeplitz, non Gaussian noise

- $E \sum G_N(i,j)^2 = O(N^2)$
- There is $\beta = \beta(\alpha,\gamma)$ so that for any $M_N$ deterministic with $\|M_N\| = O(N^{-\alpha})$, $P(s_{\min}(M_N + N^{-\gamma}G_N) < N^{-\beta}) = o(1)$
Non triangular Toeplitz, non Gaussian noise

- $E \sum G_N(i,j)^2 = O(N^2)$
- There is $\beta = \beta(\alpha, \gamma)$ so that for any $M_N$ deterministic with $\|M_N\| = O(N^{-\alpha})$, $P(s_{\min}(M_N + N^{-\gamma}G_N) < N^{-\beta}) = o(1)$

**Theorem (Basak, Paquette, Z. ’18)**

$T_N = \sum_{i=-k_1}^{k_2} a_i J_N^i$ (Toeplitz, finite symbol, $J_N^{-1} := J_N^T$.) Then,

$$L^T_N \rightarrow \text{Law of } \sum_{i=-k_1}^{k_2} a_i U^i$$

where $U$ is uniform on unit circle.
Non triangular Toeplitz, non Gaussian noise

- \( E \sum G_N(i, j)^2 = O(N^2) \)
- There is \( \beta = \beta(\alpha, \gamma) \) so that for any \( M_N \) deterministic with \( \|M_N\| = O(N^{-\alpha}) \),
  \( P(s_{\min}(M_N + N^{-\gamma} G_N) < N^{-\beta}) = o(1) \)

**Theorem (Basak, Paquette, Z. ’18)**

\( T_N = \sum_{i=-k_1}^{k_2} a_i J_N^i \) (Toeplitz, finite symbol, \( J_N^{-1} := J_N^T \)). Then,

\[
L_N^{T_N + N^{-\gamma} G_N} \to \text{Law of} \sum_{i=-k_1}^{k_2} a_i U^i
\]

where \( U \) is uniform on unit circle.

Proof based on a two step approximation (related to GWZ14) - first find local (noisy) perturbation that gives required limit, then show that global noise does not destroy it.
Non triangular Toeplitz, non Gaussian noise

- $E \sum G_N(i,j)^2 = O(N^2)$
- There is $\beta = \beta(\alpha, \gamma)$ so that for any $M_N$ deterministic with $\|M_N\| = O(N^{-\alpha})$, $P(s_{\min}(M_N + N^{-\gamma} G_N) < N^{-\beta}) = o(1)$

**Theorem (Basak, Paquette, Z. ’18)**

$T_N = \sum_{i=-k_1}^{k_2} a_i J_N^i$ (Toeplitz, finite symbol, $J_N^{-1} := J_N^T$.) Then,

$$L_N^{T_N + N^{-\gamma} G_N} \rightarrow \text{Law of } \sum_{i=-k_1}^{k_2} a_i U^i$$

where $U$ is uniform on unit circle.

Proof based on a two step approximation (related to GWZ14) - first find local (noisy) perturbation that gives required limit, then show that global noise does not destroy it.

Related (different methods, Gaussian noise - Grushin problem) - Sjöstrand and Vogel ‘19.
Proof ingredients

Theorem (Replacement principle - after GWZ)

\( A_N \) - deterministic, bounded operator norm. \( \Delta_N \) and \( G_N \) - independent random matrices. Assume

(a) \( G_N \) and \( \Delta_N \) are independent. \( \| \Delta_N \| < N^{-\gamma_0} \) whp and \( G_N \) noise matrix as before.

(b) For Lebesgue a.e. \( z \in B_{\mathbb{C}}(0, R_0) \), the empirical distribution of the singular values of \( A_N - zI_N \) converges weakly to the law induced by \( |X - z| \), where \( X \sim \mu \) and \( \text{supp}\mu \subset B_{\mathbb{C}}(0, R_0/2) \).

(c) For Lebesgue a.e. every \( z \in B_{\mathbb{C}}(0, R_0) \),

\[
\mathcal{L}_{L_N^{A+\Delta}}(z) \to \mathcal{L}_{\mu}(z), \quad \text{as } N \to \infty, \text{ in probability.} \tag{1}
\]

Then, for any \( \gamma > \frac{1}{2} \), for Lebesgue a.e. every \( z \in B_{\mathbb{C}}(0, R_0) \),

\[
\mathcal{L}_{L_N^{A+\gamma G}}(z) \to \mathcal{L}_{\mu}(z), \quad \text{as } N \to \infty, \text{ in probability.} \tag{2}
\]
Proof ingredient II

Theorem

Let $T_N$ be any $N \times N$ banded Toeplitz matrix with a symbol $a$. Then, there exists a random matrix $\Delta_N$ with

$$P(\|\Delta_N\| \geq N^{-\gamma_0}) = o(1),$$

(3)

for some $\gamma_0 > 0$, so that $L_N^{T+\Delta}$ converges weakly, in probability, to $\mu_a$. 
Proof ingredient II

**Theorem**

Let $T_N$ be any $N \times N$ banded Toeplitz matrix with a symbol $a$. Then, there exists a random matrix $\Delta_N$ with

$$P(\|\Delta_N\| \geq N^{-\gamma_0}) = o(1),$$

(3)

for some $\gamma_0 > 0$, so that $L_N^{T+\Delta}$ converges weakly, in probability, to $\mu_a$.

This works for Toeplitz with banded symbol, but not for twisted Toeplitz! Main issue - Toeplitz determinant of un-perturbed matrix requires work, e.g. Widom’s theorem.
Outliers

\[ J_N + N^{-\gamma} G_N \]

\[ J_N + J_N^2 + N^{-\gamma} G_N \]

Outliers are random. What is the structure of outliers?

- \[ J_N + N^{-\gamma} G_N \]: Outliers are zeros of a limiting Gaussian field, all inside a disc.

- \[ J_N + J_N^2 + N^{-\gamma} G_N \]: Write \[ zI_N + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N) \].

- No outliers in \{ \[ z : |\lambda_i(z)| > 1 \}, \ i = 1, 2 \}.

- In \{ \[ z : 1 > |\lambda_1(z)| > |\lambda_2(z)| \} \}, outliers are roots of a Gaussian field, limit of terms involving a single Gaussian in expansion of characteristic polynomial.

- In \{ \[ z : |\lambda_1(z)| > |\lambda_2(z)| \} \}, outliers are roots of the limit of terms involving a product of two Gaussians in expansion of characteristic polynomial.
Outliers are random. What is structure of outliers?

\[ J_N + N^{-\gamma} G_N \]

\[ J_N + J_N^2 + N^{-\gamma} G_N \]

\[ \gamma = 0.75 \]
\[ \gamma = 1.75 \]
\[ \gamma = 4.00 \]
Outliers are random. What is structure of outliers?

- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.
Outliers are random. What is structure of outliers?

- \( J_N + N^{-\gamma} G_N \): outliers are zeros of a limiting Gaussian field, all inside disc.
- \( J_N + J_N^2 + N^{-\gamma} G_N \): Write \( zI + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N) \):
Outliers

Outliers are random. What is structure of outliers?

- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.

- $J_N + J_N^2 + N^{-\gamma} G_N$: Write $z I + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N)$:
  - No outliers in $\{ z : |\lambda_i(z)| > 1, i = 1, 2 \}$
Outliers are random. What is structure of outliers?
- $J_N + N^{-\gamma}G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.
- $J_N + J_N^2 + N^{-\gamma}G_N$: Write $zJ + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N)$:
  - No outliers in $\{z : |\lambda_i(z)| > 1, i = 1, 2\}$
  - In $\{z : |\lambda_1(z)| > 1 > |\lambda_2(z)|\}$, outliers are roots of a Gaussian field, limit of terms involving a single Gaussian in expansion of char. pol.
Outliers are random. What is structure of outliers?

- \( J_N + N^{-\gamma} G_N \): outliers are zeros of a limiting Gaussian field, all inside disc.
- \( J_N + J_N^2 + N^{-\gamma} G_N \): Write \( zI + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N) \):
  - No outliers in \( \{ z : |\lambda_i(z)| > 1, i = 1, 2 \} \)
  - In \( \{ z : |\lambda_1(z)| > 1 > |\lambda_2(z)| \} \), outliers are roots of a Gaussian field, limit of terms involving a single Gaussian in expansion of char. pol.
  - In \( \{ z : 1 > |\lambda_1(z)| > |\lambda_2(z)| \} \), outliers are roots of limit of terms involving a product of two Gaussians in expansion of char. pol.
Outliers

- Toeplitz, finite symbol $a(\lambda) = \sum_{i=-k_1}^{k_2} a_i \lambda^i$, set

$$z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|$$
Outliers

- Toeplitz, finite symbol $a(\lambda) = \sum_{i=-k_1}^{k_2} a_i \lambda^i$, set

$$z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|$$

Let $d_0 = d_0(z)$ be such that $|\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|$, and set $d = d(z) = k_1 - d_0$. Let $\mathcal{D}_k = \{z \in \mathbb{C} : d(z) = k\}$. 
• Toeplitz, finite symbol $a(\lambda) = \sum_{i=-k_1}^{k_2} a_i \lambda^i$, set

$$z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|$$

Let $d_0 = d_0(z)$ be such that $|\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|$, and set $d = d(z) = k_1 - d_0$. Let $D_k = \{ z \in \mathbb{C} : d(z) = k \}$. Let $T(a)$ denote the (infinite, band) Toeplitz operator of symbol $a(\lambda)$, with spectrum $D_\infty(a) = a(S^1) \cup D_0$.

Let $L_N$ be the empirical measure of eigenvalues of $T_N + N^{-\gamma} G_N$. Theorem (Basak-Z. '19 - No eigenvalues outside limiting support) Fix $\epsilon > 0$. Then,

$$P(L_N(D_\infty(a) \epsilon)) \rightarrow N \rightarrow \infty 1$$

This does not mean there are no outliers, as $a(S^1) \subset D_\infty(a)$. 
Conjectures and ongoing

Outliers

• Toeplitz, finite symbol $a(\lambda) = \sum_{i=-k_1}^{k_2} a_i \lambda^i$, set

$$z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|$$

Let $d_0 = d_0(z)$ be such that $|\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|$, and set $d = d(z) = k_1 - d_0$. Let $\mathcal{D}_k = \{ z \in \mathbb{C} : d(z) = k \}$. Let $T(a)$ denote the (infinite, band) Toeplitz operator of symbol $a(\lambda)$, with spectrum $D_\infty(a) (=a(S^1) \cup \mathcal{D}_0)$. Let $L_N$ be the empirical measure of eigenvalues of $T_N + N^{-\gamma} G_N$.

Theorem (Basak-Z. ’19 - No eigenvalues outside limiting support)

Fix $\epsilon > 0$. Then,

$$P(L_N(D_\infty(a)^\epsilon) = 0) \rightarrow_{N \rightarrow \infty} 1.$$
Outliers

- Toeplitz, finite symbol \( a(\lambda) = \sum_{i=-k_1}^{k_2} a_i \lambda^i \), set

\[
z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), \quad |\lambda_i| \geq |\lambda_{i+1}|
\]

Let \( d_0 = d_0(z) \) be such that \( |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}| \), and set \( d = d(z) = k_1 - d_0 \). Let \( D_k = \{ z \in \mathbb{C} : d(z) = k \} \).

Let \( T(a) \) denote the (infinite, band) Toeplitz operator of symbol \( a(\lambda) \), with spectrum \( D_\infty(a) (=a(S^1) \cup D_0) \).

Let \( L_N \) be the empirical measure of eigenvalues of \( T_N + N^{-\gamma}G_N \).

**Theorem (Basak-Z. ’19 - No eigenvalues outside limiting support)**

*Fix \( \epsilon > 0 \). Then,*

\[
P(L_N(D_\infty(a)\epsilon) = 0) \rightarrow_{N \rightarrow \infty} 1.
\]

*This does not mean there are no outliers, as \( a(S^1) \subset D_\infty(a) \).*
Outliers

\[ z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), \quad |\lambda_i| \geq |\lambda_{i+1}|, \]

\[ |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|, \quad d = d(z) = k_1 - d_0, \quad D_k = \{ z \in \mathbb{C} : d(z) = k \}. \]
Outliers

\[ z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), \quad |\lambda_i| \geq |\lambda_{i+1}|, \]

\[ |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|, \quad d = d(z) = k_1 - d_0, \quad D_k = \{ z \in \mathbb{C} : d(z) = k \}. \]

For \( k \neq 0 \), let \( A_{N,k} = \{ z \in D_k : z \) is an eigenvalue of \( T_N + N^{-\gamma}G_N \}. \)
Outliers

\[ z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), \ |\lambda_i| \geq |\lambda_{i+1}|, \]
\[ |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|, \ d = d(z) = k_1 - d_0, \ D_k = \{ z \in \mathbb{C} : d(z) = k \}. \]

For \( k \neq 0 \), let \( A_{N,k} = \{ z \in D_k : z \text{ is an eigenvalue of } T_N + N^{-\gamma} G_N \} \).

Theorem (Basak-Z ‘19 - Outlier fields)

For each \( k \neq 0 \), there exists a random set \( N_k \), finite on compact subsets of \( D_k \), so that \( A_{N,k} \) converges in distribution on compact subsets of \( D_k \) to \( N_k \).
Outliers

\[ z + \sum_{i=-k_1}^{k_2} a_i \lambda_i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|, \]

\[ |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|, d = d(z) = k_1 - d_0, D_k = \{z \in \mathbb{C} : d(z) = k\}. \]

For \( k \neq 0 \), let \( A_{N,k} = \{z \in D_k : z \text{ is an eigenvalue of } T_N + N^{-\gamma}G_N\} \).

Theorem (Basak-Z ‘19 - Outlier fields)

For each \( k \neq 0 \), there exists a random set \( N_k \), finite on compact subsets of \( D_k \), so that \( A_{N,k} \) converges in distribution on compact subsets of \( D_k \) to \( N_k \).

The random sets \( N_k \) are constructed as follows. There are random fields \( \xi_k^{(L)} \), polynomials in the \( \lambda_i(z) \), whose coefficients are specific minors of \( E_N \) of size \( |k| + k_2 \) (which minors appear admits a combinatorial description).
Outliers

\[ z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|, \]
\[ |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|, d = d(z) = k_1 - d_0, \mathcal{D}_k = \{z \in \mathbb{C} : d(z) = k\}. \]

For \( k \neq 0 \), let \( \mathcal{A}_{N,k} = \{z \in \mathcal{D}_k : z \text{ is an eigenvalue of } T_N + N^{-\gamma} G_N\}. \)

**Theorem (Basak-Z ‘19 - Outlier fields)**

*For each \( k \neq 0 \), there exists a random set \( \mathcal{N}_k \), finite on compact subsets of \( \mathcal{D}_k \), so that \( \mathcal{A}_{N,k} \) converges in distribution on compact subsets of \( \mathcal{D}_k \) to \( \mathcal{N}_k \).*

The random sets \( \mathcal{N}_k \) are constructed as follows. There are random fields \( \xi_k^{(L)} \), polynomials in the \( \lambda_i(z) \), whose coefficients are specific minors of \( E_N \) of size \( |k| + k_2 \) (which minors appear admits a combinatorial description).

The zero set of \( \xi_k^{(L)} \) is denoted \( \mathcal{N}_k^{(L)} \), and admits a distributional limit \( \mathcal{N}_k \) as \( L \to \infty \).
Outliers

\[ z + \sum_{i=-k_1}^{k_2} a_i \lambda^i = \lambda^{-k_2} \prod_{i=1}^{k_1+k_2} (\lambda_i(z) - \lambda), |\lambda_i| \geq |\lambda_{i+1}|, \]
\[ |\lambda_{d_0}| > 1 > |\lambda_{d_0+1}|, d = d(z) = k_1 - d_0, \mathcal{D}_k = \{ z \in \mathbb{C} : d(z) = k \}. \]

For \( k \neq 0 \), let \( \mathcal{A}_{N,k} = \{ z \in \mathcal{D}_k : z \text{ is an eigenvalue of } T_N + N^{-\gamma}G_N \} \).

Theorem (Basak-Z ‘19 - Outlier fields)

For each \( k \neq 0 \), there exists a random set \( \mathcal{N}_k \), finite on compact subsets of \( \mathcal{D}_k \), so that \( \mathcal{A}_{N,k} \) converges in distribution on compact subsets of \( \mathcal{D}_k \) to \( \mathcal{N}_k \).

The random sets \( \mathcal{N}_k \) are constructed as follows. There are random fields \( \xi_k^{(L)} \), polynomials in the \( \lambda_i(z) \), whose coefficients are specific minors of \( E_N \) of size \( |k| + k_2 \) (which minors appear admits a combinatorial description).

The zero set of \( \xi_k^{(L)} \) is denoted \( \mathcal{N}_k^{(L)} \), and admits a distributional limit \( \mathcal{N}_k \) as \( L \to \infty \).

Improves on counting estimates of Sjöstrand and Vogel (‘19).
In the particular case of $T_N = J_N$ with Gaussian complex noise:

- No outliers in compact subsets of $\{z : |z| > 1\}$.
- The outliers inside $\{z : |z| < 1\}$ have, asymptotically, the same law as zeros of the hyperbolic Gaussian analytic function, i.e.
  $$\sum_{i=0}^{\infty} a_i z^i$$
  with $a_i$ i.i.d. standard complex Gaussian.
- In particular, the outliers inside the unit disc form a determinantal process, and the first intensity is
  $$2\pi \left( 1 - |z|^2 \right)^{2/|z| < 1}$$
- Computation of intensity first performed by Sjostrand and Vogel (2018).
Outliers

In the particular case of $T_N = J_N$ with Gaussian complex noise:

- No outliers in compact subsets of $\{z : |z| > 1\}$.

Computation of intensity first performed by Sjostrand and Vogel (2018).
In the particular case of $T_N = J_N$ with Gaussian complex noise:

- No outliers in compact subsets of $\{z : |z| > 1\}$.
- The outliers inside $\{z : |z| < 1\}$ have, asymptotically, the same law as zeros of the hyperbolic Gaussian analytic function, i.e. $\sum_{i=0}^{\infty} a_i z^i$ with $a_i$ i.i.d. standard complex Gaussian.

In particular, the outliers inside the unit disc form a determinental process, and the first intensity is

$$\frac{2}{\pi (1 - |z|^2)^2} 1_{|z| < 1}$$
Outliers

In the particular case of $T_N = J_N$ with Gaussian complex noise:

- No outliers in compact subsets of $\{z : |z| > 1\}$.
- The outliers inside $\{z : |z| < 1\}$ have, asymptotically, the same law as zeros of the hyperbolic Gaussian analytic function, i.e. $\sum_{i=0}^{\infty} a_i z^i$ with $a_i$ i.i.d. standard complex Gaussian.

In particular, the outliers inside the unit disc form a determinental process, and the first intensity is

$$\frac{2}{\pi(1 - |z|^2)^2} 1_{|z| < 1}$$

Computation of intensity first performed by Sjostrand and Vogel (2018).
Conjectures and ongoing

Conjectures and open problems - Spectrum limits

• General twisted Toeplitz symbol: Expect mixture as in upper triangular case. Main obstacle: compute determinant of twisted Toeplitz with non-zero winding number.

• Toeplitz with infinite symbol - depends on rate of decay? Recent breakthrough of Sjöstrand-Vogel!
Conjectures and open problems - Spectrum limits

• General twisted Toeplitz symbol:
  Expect mixture as in upper triangular case.
Conjectures and open problems - Spectrum limits

• General twisted Toeplitz symbol:
  Expect mixture as in upper triangular case. Main obstacle: compute determinant of twisted Toeplitz with non-zero winding number.
• General twisted Toeplitz symbol :
  Expect mixture as in upper triangular case. Main obstacle: compute
determinant of twisted Toeplitz with non-zero winding number.
• Toeplitz with infinite symbol - depends on rate of decay?
Conjectures and open problems - Spectrum limits

- General twisted Toeplitz symbol:
  Expect mixture as in upper triangular case. Main obstacle: compute determinant of twisted Toeplitz with non-zero winding number.
- Toeplitz with infinite symbol - depends on rate of decay? Recent breakthrough of Sjöstrand-Vogel!