XV Brunel-Bielefeld Workshop
Random Matrix Theory and Applications
5-7 December 2019

Sparse Random Block Matrices

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G.M.C., Alessio Zaccone, Johannes Krausser, Rico Milkus,
Phys. Rev. E 97 (2018) 032113
Mario Pernici, G.M.C., Jour.Stat.Phys.(2019) 175, pag.384

Outline

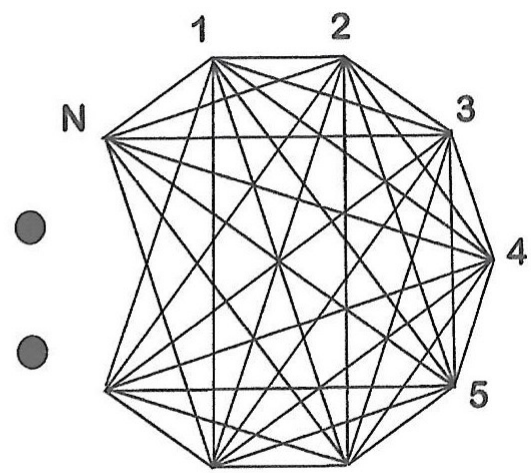
- the class of sparse random block matrices
- the ensemble in this work (the random spring system), a bit of physics
- evaluation of moments :
  - counting walks on trees
  - the limit $d \to \infty$, $\frac{Z}{d}$ fixed (non-crossing partitions)
- conclusions and generalizations.

the class of sparse random block matrices

Two prototypes of block matrices, $Nd \times Nd$

(with abuse of language) Adjacency A and Laplacian L

$$A = \begin{pmatrix} 0 & \alpha_{1,2}X_{1,2} & \alpha_{1,3}X_{1,3} & \dots & \alpha_{1,N}X_{1,N} \\ \alpha_{2,1}X_{2,1} & 0 & \alpha_{2,3}X_{2,3} & \dots & \alpha_{2,N}X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{N,1}X_{N,1} & \alpha_{N,2}X_{N,2} & \alpha_{N,3}X_{N,3} & \dots & 0 \end{pmatrix}$$



$A_{i,j} = \alpha_{i,j}X_{i,j}$, edge weight in the graph , $i, j = 1, 2, .., N$.

The set $\{\alpha_{i,j}\}$, $1 \leq i < j \leq N$ is a set of $N(N-1)/2$ i.i.d. random variables, $\alpha_{j,i} = \alpha_{i,j}$. Erdös-Renyi probability law :

$$P(\alpha) = \left(\frac{Z}{N}\right)\delta(\alpha - 1) + \left(1 - \frac{Z}{N}\right)\delta(\alpha)$$

$Z =< \sum_{j=1}^{N} \alpha_{i,j} >$ is the average connectivity, that is the average degree of the vertices of the graph.

The blocks $X_{i,j}$ are $d \times d$ real symmetric , rank one, random matrices

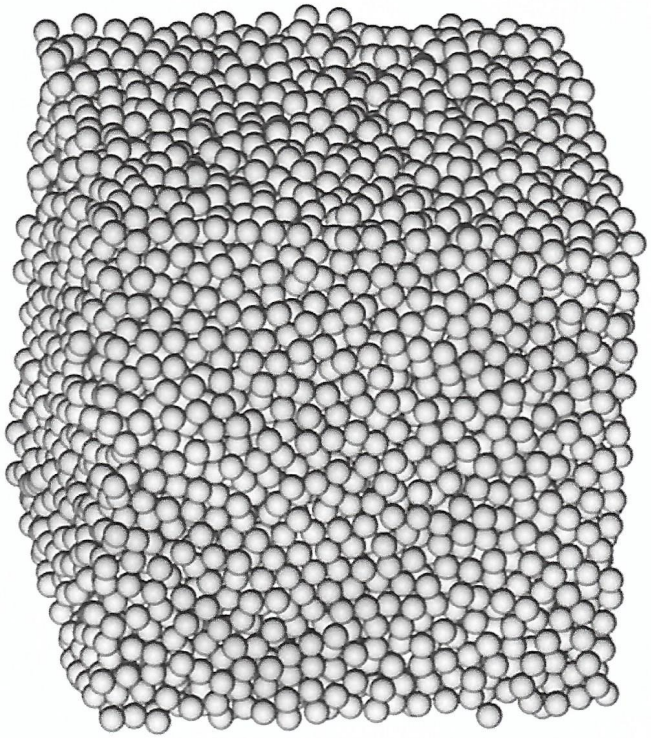
$$X_{i,j} = X_{j,i} = (X_{i,j})^t = \widehat{n}_{ij} \cdot \widehat{n}_{ij}^t = |\widehat{n}_{ij} >< \widehat{n}_{ij}|$$

$\widehat{n}_{ij}$ is a random vector of length one,
with uniform probability on the sphere in R^d.

The Laplacian block matrix

$$L = \begin{pmatrix} \sum_{j\neq 1} \alpha_{1,j} X_{1,j} & -\alpha_{1,2} X_{1,2} & -\alpha_{1,3} X_{1,3} & \dots & -\alpha_{1,N} X_{1,N} \\ -\alpha_{2,1} X_{2,1} & \sum_{j\neq 2} \alpha_{2,j} X_{2,j} & -\alpha_{2,3} X_{2,3} & \dots & -\alpha_{2,N} X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ -\alpha_{N,1} X_{N,1} & -\alpha_{N,2} X_{N,2} & -\alpha_{N,3} X_{N,3} & \dots & \sum_{j\neq N} \alpha_{N,j} X_{N,j} \end{pmatrix}$$

Amorphous solids



The random spring model of the amorphous solid

$$U = \frac{k}{2} \sum_{i,j} (r_{i,j} - r^0_{i,j})^2 \quad , \quad r_{i,j} = \sqrt{\sum_{\alpha=1}^{d} \left(\vec{x}^{(i)} - \vec{x}^{(j)} \right)^2_\alpha}$$

$$H_{i,j;\alpha,\beta} = \frac{\partial U}{\partial x^{(i)}_\alpha \partial x^{(j)}_\beta} \Big|_{\mathtt{eq}} = \frac{-k}{(r_{i,j})^2} \left(x^{(j)}_\alpha - x^{(i)}_\alpha \right) \left(x^{(j)}_\beta - x^{(i)}_\beta \right) \Big|_{\mathtt{eq}}$$

The Hessian matrix H is a real symmetric block matrix $Nd \times Nd$

$$H_{i,j} = -|\hat{n}_{i,j} >< \hat{n}_{i,j}| \quad \mathtt{se} \quad i \neq j \quad , \quad H_{i,i} = -\sum_j H_{i,j}$$

The Hessian (or Stiffness) matrix is the Laplacian block matrix L. Its eigenvalues are the squared eigenfrequencies of the amorphous solid.

Maxwell constraint counting

N material points in $\mathbf{R}^d$ correspond to $N\,d$ translational degrees of freedom

(let us ignore rotational degrees of freedom, Phillips and Thorpe 1985)

Let Z be the average connectivity : Every material point interacts with Z other points (on average)

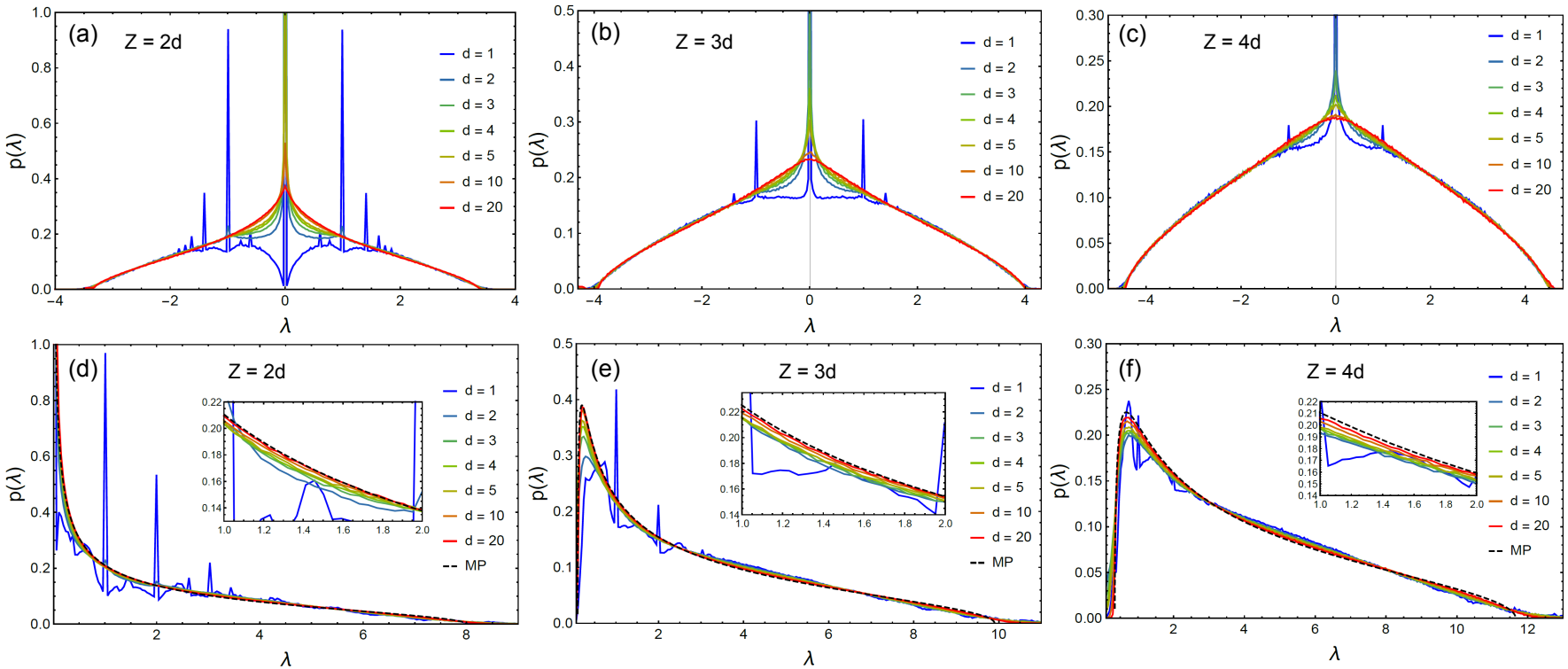
Let's suppose E constraints $\sim$ two body interactions $\sim$ springs, $2E = ZN$

isostatic system : number of degrees of freedom equal number of constraints $N\,d = \frac{ZN}{2}$, or $Z = 2d$

If $Z < 2d$ the system is **floppy** or fluid

If $Z > 2d$ the system is **rigid**

The 3 figures in the top row are numerical simulations of the eigenvalues density of the Adjacency block matrix, for $Z/d = 2, 3, 4$

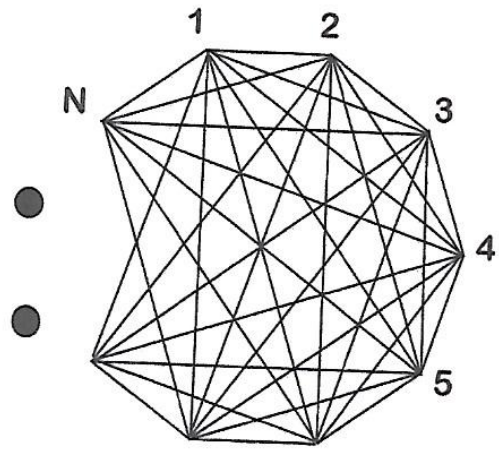


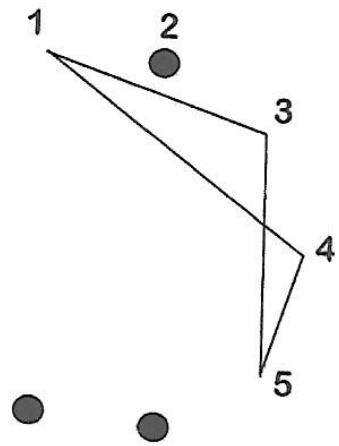
The 3 figures in the bottom row are numerical simulations of the eigenvalues density of the Laplacian block matrix, for $Z/d = 2, 3, 4$
For every d, the support of the spectral density originates at zero if $Z/d = 2$ (isostatic system , plenty of soft modes).

The moments of the spectral density

For any real $Z > 1$ and integer $d = 1, 2, ...$

$$\mu_k = \lim_{N \to \infty} \frac{1}{Nd} < \mathrm{Tr}\, A^k >$$





$$\mu_k = \lim_{N\to\infty} \frac{1}{Nd} < \mathrm{Tr} \sum_{\text{paths}} \alpha_1 X_1 \alpha_2 X_2 \ldots \alpha_k X_k >$$

$$\mu_k = \lim_{N\to\infty} \frac{1}{Nd} \sum_{\texttt{paths}} < \alpha_1\alpha_2..\alpha_k >< \mathrm{Tr}\, X_1 X_2 \ldots X_k > \qquad ,$$

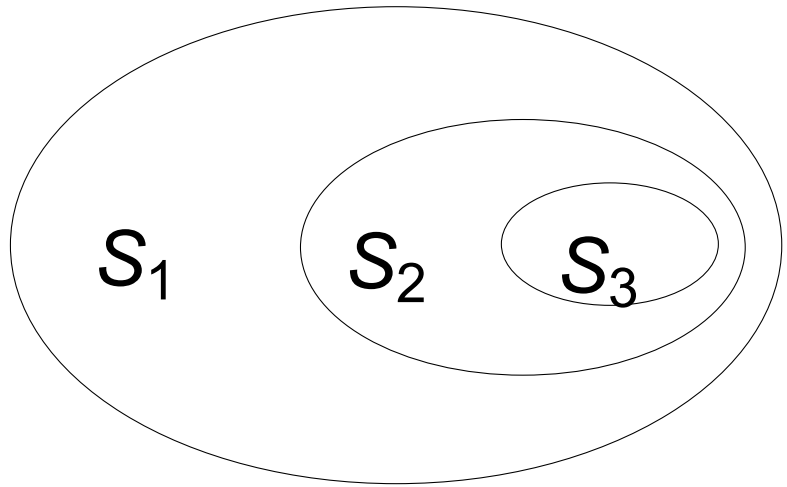
$$< \alpha_1\alpha_2..\alpha_k >= \left(\frac{Z}{N}\right)^h$$

h is the number of distinct edges of the path ,

$$< \mathrm{Tr}\, X_1 X_2 \ldots X_k >=< (\widehat{a}\cdot\widehat{b})(\widehat{b}\cdot\widehat{c})\ldots(\widehat{k}\cdot\widehat{a}) >$$

here the integration is over h independent unit vectors.

On sparse graphs, for $N \to \infty$ only walks on trees contribute to the moments



S_1 : each edge is traveled an even number of times.
$trA^{2n} = \sum a^{2n_1} b^{2n_2} c^{2n_3} ...$ where $\sum 2n_j = 2n$. M.Bauer, O.Golinelli (2001), Khorunzhy et al. (2000) and (2004)

S_2 : walks correspond to *non-crossing* partitions

$$trA^{2n} = \sum \alpha_1\alpha_2\alpha_3..X_1(X_2)^2X_3X_4(X_5)^2(X_4)^3X_3X_1$$

S_3 : Wigner walks , in each walk the edges are traveled exactly twice

$$trA^{2n} = \sum a^2b^2c^2...$$

The moments of spectral densities

The evaluation of the lowest order limiting moments suggest to express them as functions of Z/d and d

$$\mu_k = \lim_{N\to\infty} \frac{1}{N\,d} < \mathrm{Tr}\, A^k > \quad , \quad \mu_0 = 1 \quad , \quad \mu_{2k+1} = 0$$

$$\mu_2 = \frac{Z}{d} \quad , \quad \mu_4 = \frac{Z}{d} + 2\left(\frac{Z}{d}\right)^2 \quad , \quad \mu_6 = \frac{Z}{d} + 6\left(\frac{Z}{d}\right)^2 + 5\left(\frac{Z}{d}\right)^3$$

$$\mu_8 = \frac{Z}{d} + \left(\frac{Z}{d}\right)^2 (12 + 2\, c_2) + 28\left(\frac{Z}{d}\right)^3 + 14\left(\frac{Z}{d}\right)^4$$

$$\mu_{10} = \frac{Z}{d} + \left(\frac{Z}{d}\right)^2 (20 + 10\, c_2) + \left(\frac{Z}{d}\right)^3 (90 + 20\, c_2) + 120\left(\frac{Z}{d}\right)^4 + 42\left(\frac{Z}{d}\right)^5$$

$$\frac{c_m}{d} = < \left(\vec{a}\cdot\vec{b}\right)^{2m} > = \frac{(2m-1)!!}{2^m} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(m + \frac{d}{2}\right)} \quad ,$$

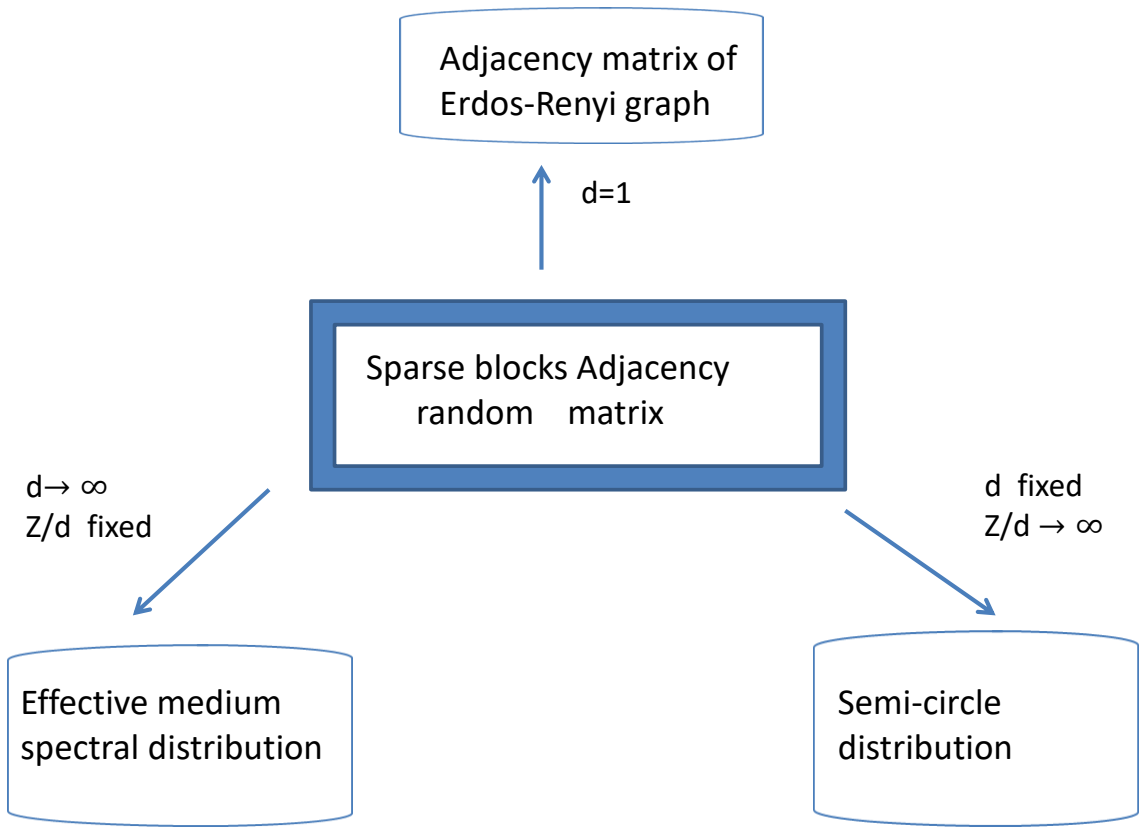
$$c_1 = 1 \quad , \quad c_2 = \frac{3}{(d+2)} \quad , \quad c_3 = \frac{5!!}{(d+2)(d+4)} \quad ,$$

Every $c_m = 1$ if $d = 1$, and $c_m = 0$ if $d = \infty$, $m \geq 2$.

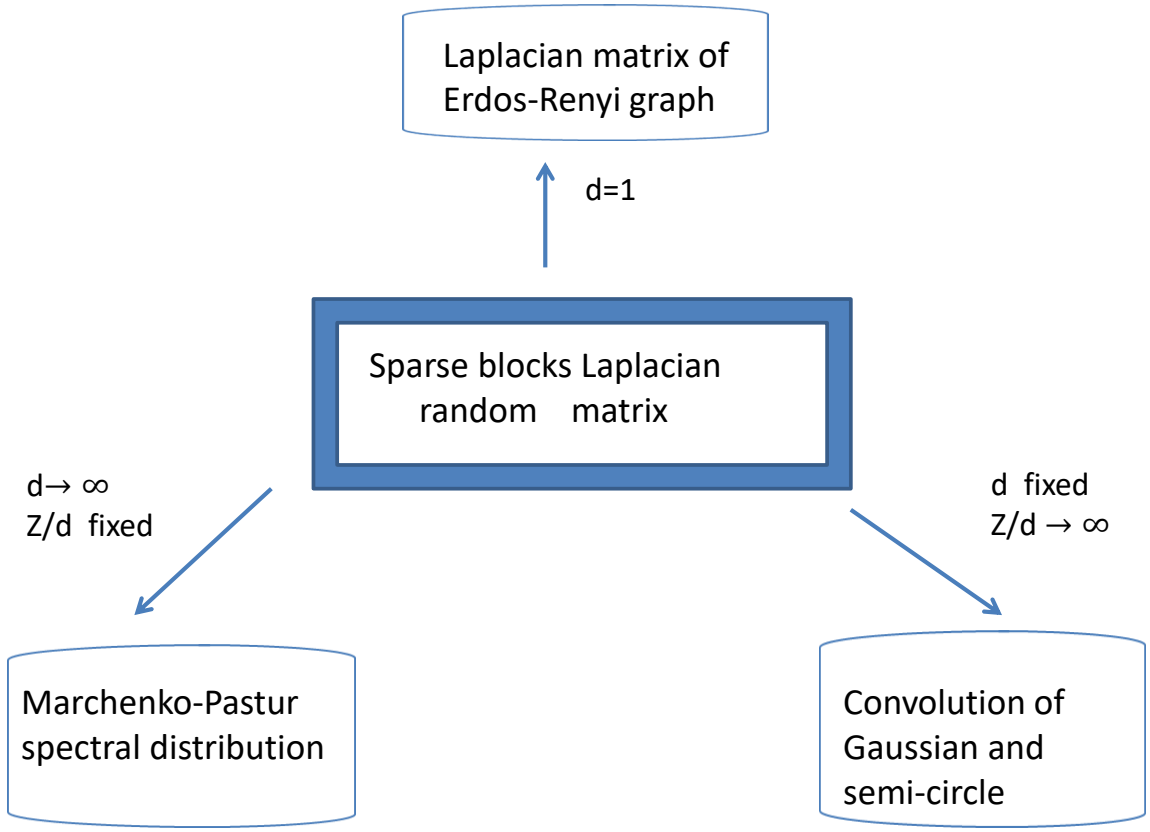
The limiting moments of the Laplacian matrix are :

$$
\begin{aligned}
\nu_k &= \lim_{N\to\infty} \frac{1}{N\,d} < \mathrm{Tr}\, L^k > \quad , \quad \nu_0 = 1 \\
\nu_1 &= \frac{Z}{d} \quad , \quad \nu_2 = 2\,\frac{Z}{d} + \left(\frac{Z}{d}\right)^2 \quad , \quad \nu_3 = 4\,\frac{Z}{d} + 6\,\left(\frac{Z}{d}\right)^2 + \left(\frac{Z}{d}\right)^3 \\
\nu_4 &= 8\,\frac{Z}{d} + \left(\frac{Z}{d}\right)^2 (24 + c_2) + 12\left(\frac{Z}{d}\right)^3 + \left(\frac{Z}{d}\right)^4 \\
\nu_5 &= 16\,\frac{Z}{d} + \left(\frac{Z}{d}\right)^2 (80 + 10\,c_2) + \left(\frac{Z}{d}\right)^3 (80 + 5\,c_2) + 20\,\left(\frac{Z}{d}\right)^4 + \left(\frac{Z}{d}\right)^5 \\
\nu_6 &= 32\,\frac{Z}{d} + \left(\frac{Z}{d}\right)^2 (240 + 60\,c_2 + c_3) + \left(\frac{Z}{d}\right)^3 \left(400 + 72\,c_2 + 4\,c_2 \frac{1 + 2\,c_2}{3}\right) + \\
&+ \left(\frac{Z}{d}\right)^4 (200 + 15\,c_2) + \left(\frac{Z}{d}\right)^5 30 + \left(\frac{Z}{d}\right)^6
\end{aligned}
$$

The Adjacency matrix ensemble has 3 interesting limiting regions of the parameters :



The analogous 3 regions for the Laplacian matrix ensemble :



Walks and partitions

A walk of $2n$ steps with h distinct edges is a partition of a sequence of $2n$ matrices in h parts (or blocks).

Non-crossing partitions (Kreweras 1972, R. Simion 2000) play a role similar to planar graphs in the topological expansion.

Examples: two walks of 12 steps with 4 distinct edges

$$X_1(X_2)^2(X_3)^2(X_2)^2(X_3)^2X_1(X_4)^2$$

is a **crossing** partition because its 4 parts are

$$\{1,10\} \quad , \quad \{2,3,6,7\} \quad , \quad \{4,5,8,9\} \quad , \quad \{11,12\}$$

and the sequence $2,5,6,9$ is such that $\{2,6\} \in X_2$, $\{5,9\} \in X_3$

the sequence $X_1(X_2)^3(X_3)^4X_2(X_4)^2X_1$
is a **non-crossing** partition because its 4 parts are

$$\{1,12\} \quad , \quad \{2,3,4,9\} \quad , \quad \{5,6,7,8\} \quad , \quad \{10,11\}$$

there are no two parts with interlacing pairs.
We evaluate the ensemble averages

$$< \alpha_1\alpha_2\alpha_3\alpha_4 >< TrX_1(X_2)^2(X_3)^2(X_2)^2(X_3)^2X_1(X_4)^2 >= \left(\frac{Z}{d}\right)^4 \frac{3}{d+2}$$

$$< \alpha_1\alpha_2\alpha_3\alpha_4 >< TrX_1(X_2)^3(X_3)^4X_2(X_4)^2X_1 >= \left(\frac{Z}{d}\right)^4$$

The limit $d \to \infty$ with Z/d fixed, steps of the proof

Definitions

$T_j^{(2n)}$:= set of closed walks beginning and ending at vertex j, with $2n$ steps

$B_j^{(2n)}$:= set of closed *primitive* walks beginning and ending at vertex j, with $2n$ steps

Generating functions

$$T_j(x) = 1 + \sum_{k=1}^{\infty} x^{2k} T_j^{(2k)} \quad , \quad B_j(x) = \sum_{k=1}^{\infty} x^{2k} B_j^{(2k)}$$

Two basic equations : (1) the expansion of closed walks in primitive closed walks

$$T_{j_0}(x) = 1 + \sum_j B_{j_0 j}(x) + \sum_{j,k} B_{j_0 j}(x) B_{j_0 k}(x) + \sum_{j,k,l} B_{j_0 j}(x) B_{j_0 k}(x) B_{j_o l}(x) + \dots$$

(2) primitive closed walks are written in terms of first and last step plus generic closed walks (here written for walks on trees)

$$B_{j_0 k}(x) = \alpha_{j_0 k} x^2 X_{j_0 k} T_k(x) X_{k j_0}$$

A recursive algorithm that generates the walks on trees is obtained by inserting the second eq. in the first one.

In the $d \to \infty$ limit, Z/d fixed, only *non-crossing* partitions survive, the sequences of closed walks are stochastically independent and one algebraic equation is obtained for the resolvent of Adjacency and Laplacian matrices.

For the resolvent $r(z)$ of the Adjacency matrix one obtains the cubic equation

$$r^3(z) - \frac{1}{z}(1-t)r^2(z) - r(z) + \frac{1}{z} = 0 \quad , \quad t = \frac{Z}{d}$$

which is the effective medium approximation by Semerjian and Cugliandolo (2002)

For the resolvent $r(z)$ of the Laplacian matrix one obtains the quadratic equation

$$2zr^2(z) - (2 - t + z)r(z) + 1 = 0 \quad , \quad t = \frac{Z}{d}$$

which is the Marchenko Pastur equation (1967).

Fuss Narayama polynomials

The generating function $f(z)$ of the moments of the two parameter families of densities $\pi_{s,t}$ (called "Free Bessel laws", Banica et al. 2008)

$$f(z) = 1 + m_1 z + m_2 z^2 + .. \quad , \quad f = 1 + z f^s (f + t - 1)$$

The moments of EMA (Semerjian+Cugliandolo 2002) are the Fuss Narayama polynomials, $s = 2$

$$m_k = \sum_{b=1}^{k} \frac{1}{b} \left(\begin{array}{c} k-1 \\ b-1 \end{array} \right) \left(\begin{array}{c} 2k \\ b-1 \end{array} \right) t^b$$

Similar result for the moments of the Laplacian, $s = 1$, and a change of variables.

Conclusions and generalizations

- the determination of the spectral densities in the $d \to \infty$ limit, with Z/d fixed was unexpected
- the hessian of this talk has a physics interest and the parameter d is the space dimension
- other graph structures (all straightforward):
  - regular random graph
  - bipartite biregular graph (M. Pernici, in preparation)
  - complete graph, $\alpha \to 1$
- different ensembles for the blocks X_{ij}
  - $X_{ij} \in$ GUE or GOE, analysis possible but not easy.