Sparse Random Block Matrices

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Outline

• the class of sparse random block matrices

• the ensemble in this work (the random spring system), a bit of physics

• evaluation of moments :
  – counting walks on trees
  – the limit \( d \to \infty, \quad \frac{Z}{d} \) fixed (non-crossing partitions)

• conclusions and generalizations.
the class of sparse random block matrices

Two prototypes of block matrices, $Nd \times Nd$

(with abuse of language) Adjacency $A$ and Laplacian $L$
$A = \begin{pmatrix}
0 & \alpha_{1,2}X_{1,2} & \alpha_{1,3}X_{1,3} & \cdots & \alpha_{1,N}X_{1,N} \\
\alpha_{2,1}X_{2,1} & 0 & \alpha_{2,3}X_{2,3} & \cdots & \alpha_{2,N}X_{2,N} \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
\alpha_{N,1}X_{N,1} & \alpha_{N,2}X_{N,2} & \alpha_{N,3}X_{N,3} & \cdots & 0
\end{pmatrix}$

$A_{i,j} = \alpha_{i,j}X_{i,j}$, edge weight in the graph, $i, j = 1, 2, \ldots, N$.

The set $\{\alpha_{i,j}\}, 1 \leq i < j \leq N$ is a set of $N(N-1)/2$ i.i.d. random variables, $\alpha_{j,i} = \alpha_{i,j}$. Erdős-Renyi probability law:

$$P(\alpha) = \left(\frac{Z}{N}\right) \delta(\alpha - 1) + \left(1 - \frac{Z}{N}\right) \delta(\alpha)$$

$Z = \langle \sum_{j=1}^{N} \alpha_{i,j} \rangle$ is the average connectivity, that is the average degree of the vertices of the graph.
The blocks $X_{i,j}$ are $d \times d$ real symmetric, rank one, random matrices 

$$X_{i,j} = X_{j,i} = (X_{i,j})^t = \hat{n}_{ij} \cdot \hat{n}_{ij}^t = |\hat{n}_{ij} > < \hat{n}_{ij}|$$

$\hat{n}_{ij}$ is a random vector of length one, 
with uniform probability on the sphere in $R^d$. 
The Laplacian block matrix

\[ L = \begin{pmatrix}
\sum_{j \neq 1} \alpha_{1,j} X_{1,j} & -\alpha_{1,2} X_{1,2} & -\alpha_{1,3} X_{1,3} & \ldots & -\alpha_{1,N} X_{1,N} \\
-\alpha_{2,1} X_{2,1} & \sum_{j \neq 2} \alpha_{2,j} X_{2,j} & -\alpha_{2,3} X_{2,3} & \ldots & -\alpha_{2,N} X_{2,N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\alpha_{N,1} X_{N,1} & -\alpha_{N,2} X_{N,2} & -\alpha_{N,3} X_{N,3} & \ldots & \sum_{j \neq N} \alpha_{N,j} X_{N,j}
\end{pmatrix} \]
Amorphous solids
The random spring model of the amorphous solid

\[
U = \frac{k}{2} \sum_{i,j} (r_{i,j} - r_{i,j}^0)^2, \quad r_{i,j} = \sqrt{\sum_{\alpha=1}^{d} (\mathbf{x}^{(i)}(\alpha) - \mathbf{x}^{(j)}(\alpha))^2}
\]

\[
H_{i,j;\alpha,\beta} = \left. \frac{\partial U}{\partial x^{(i)}(\alpha) \partial x^{(j)}(\beta)} \right|_{\text{eq}} = \frac{-k}{(r_{i,j})^2} \left( x^{(j)}(\alpha) - x^{(i)}(\alpha) \right) \left( x^{(j)}(\beta) - x^{(i)}(\beta) \right) |_{\text{eq}}
\]

The Hessian matrix \( H \) is a real symmetric block matrix \( Nd \times Nd \)

\[
H_{i,j} = -|\mathbf{n}_{i,j} > < \mathbf{n}_{i,j}| \quad \text{se} \quad i \neq j, \quad H_{i,i} = -\sum_j H_{i,j}
\]

The Hessian (or Stiffness) matrix is the Laplacian block matrix \( L \). Its eigenvalues are the squared eigenfrequencies of the amorphous solid.
Maxwell constraint counting

$N$ material points in $\mathbb{R}^d$ correspond to $Nd$ translational degrees of freedom

( let us ignore rotational degrees of freedom, Phillips and Thorpe 1985)

Let $Z$ be the average connectivity : Every material point interacts with $Z$ other points (on average)

Let’s suppose $E$ constraints $\sim$ two body interactions $\sim$ springs, $2E = ZN$

**isostatic system** : number of degrees of freedom equal number of constraints $Nd = \frac{ZN}{2}$ , or $Z = 2d$

If $Z < 2d$ the system is **floppy** or fluid

If $Z > 2d$ the system is **rigid**
The 3 figures in the top row are numerical simulations of the eigenvalues density of the Adjacency block matrix, for $Z/d = 2, 3, 4$

![Figure (a) Z = 2d](image1)
![Figure (b) Z = 3d](image2)
![Figure (c) Z = 4d](image3)

The 3 figures in the bottom row are numerical simulations of the eigenvalues density of the Laplacian block matrix, for $Z/d = 2, 3, 4$

For every $d$, the support of the spectral density originates at zero if $Z/d = 2$ (isostatic system, plenty of soft modes).
The moments of the spectral density

For any real $Z > 1$ and integer $d = 1, 2, ...$

$$\mu_k = \lim_{N \to \infty} \frac{1}{Nd} < \text{Tr} A^k >$$
\[
\mu_k = \lim_{N \to \infty} \frac{1}{Nd} < \text{Tr} \sum_{\text{paths}} \alpha_1 X_1 \alpha_2 X_2 \ldots \alpha_k X_k >
\]
\[
\mu_k = \lim_{N \to \infty} \frac{1}{N^d} \sum_{\text{paths}} \langle \alpha_1 \alpha_2 \ldots \alpha_k \rangle \langle \text{Tr } X_1 X_2 \ldots X_k \rangle,
\]

\[
\langle \alpha_1 \alpha_2 \ldots \alpha_k \rangle = \left( \frac{Z}{N} \right)^h
\]

\( h \) is the number of distinct edges of the path,

\[
\langle \text{Tr } X_1 X_2 \ldots X_k \rangle = \langle (\hat{a} \cdot \hat{b})(\hat{b} \cdot \hat{c}) \ldots (\hat{k} \cdot \hat{a}) \rangle
\]

here the integration is over \( h \) independent unit vectors.
On sparse graphs, for $N \to \infty$ only walks on trees contribute to the moments

\[ S_1 \cap S_2 \cap S_3 : \text{each edge is traveled an even number of times.} \]

\[ trA^{2n} = \sum a^{2n_1}b^{2n_2}c^{2n_3} \ldots \text{ where } \sum 2n_j = 2n. \]

$S_2$ : walks correspond to *non-crossing* partitions

$trA^{2n} = \sum \alpha_1\alpha_2\alpha_3...X_1(X_2)^2X_3X_4(X_5)^2(X_4)^3X_3X_1$

$S_3$ : Wigner walks, in each walk the edges are traveled exactly twice

$trA^{2n} = \sum a^2b^2c^2...$
The moments of spectral densities

The evaluation of the lowest order limiting moments suggest to express them as functions of $Z/d$ and $d$

$$\mu_k = \lim_{N \to \infty} \frac{1}{N^d} \langle \text{Tr} A^k \rangle \quad , \quad \mu_0 = 1 \quad , \quad \mu_{2k+1} = 0$$

$$\mu_2 = \frac{Z}{d} \quad , \quad \mu_4 = \frac{Z}{d} + 2 \left( \frac{Z}{d} \right)^2 \quad , \quad \mu_6 = \frac{Z}{d} + 6 \left( \frac{Z}{d} \right)^2 + 5 \left( \frac{Z}{d} \right)^3$$

$$\mu_8 = \frac{Z}{d} + \left( \frac{Z}{d} \right)^2 (12 + 2 c_2) + 28 \left( \frac{Z}{d} \right)^3 + 14 \left( \frac{Z}{d} \right)^4$$

$$\mu_{10} = \frac{Z}{d} + \left( \frac{Z}{d} \right)^2 (20 + 10 c_2) + \left( \frac{Z}{d} \right)^3 (90 + 20 c_2) + 120 \left( \frac{Z}{d} \right)^4 + 42 \left( \frac{Z}{d} \right)^5$$

$$\frac{c_m}{d} = \langle \left( \vec{a} \cdot \vec{b} \right)^{2m} \rangle = \frac{(2m - 1)!!}{2^m} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( m + \frac{d}{2} \right)} \quad ,$$

$$c_1 = 1 \quad , \quad c_2 = \frac{3}{(d+2)} \quad , \quad c_3 = \frac{5!!}{(d+2)(d+4)} \quad ,$$
Every $c_m = 1$ if $d = 1$, and $c_m = 0$ if $d = \infty$, $m \geq 2$.

The limiting moments of the Laplacian matrix are:

$$
\nu_k = \lim_{N \to \infty} \frac{1}{Nd} < \text{Tr} L^k >, \quad \nu_0 = 1
$$

$$
\nu_1 = \frac{Z}{d}, \quad \nu_2 = 2 \frac{Z}{d} + \left( \frac{Z}{d} \right)^2, \quad \nu_3 = 4 \frac{Z}{d} + 6 \left( \frac{Z}{d} \right)^2 + \left( \frac{Z}{d} \right)^3
$$

$$
\nu_4 = 8 \frac{Z}{d} + \left( \frac{Z}{d} \right)^2 (24 + c_2) + 12 \left( \frac{Z}{d} \right)^3 + \left( \frac{Z}{d} \right)^4
$$

$$
\nu_5 = 16 \frac{Z}{d} + \left( \frac{Z}{d} \right)^2 (80 + 10 c_2) + \left( \frac{Z}{d} \right)^3 (80 + 5 c_2) + 20 \left( \frac{Z}{d} \right)^4 + \left( \frac{Z}{d} \right)^5
$$

$$
\nu_6 = 32 \frac{Z}{d} + \left( \frac{Z}{d} \right)^2 (240 + 60 c_2 + c_3) + \left( \frac{Z}{d} \right)^3 \left( 400 + 72 c_2 + 4 c_2 \frac{1 + 2 c_2}{3} \right) +
$$

$$
+ \left( \frac{Z}{d} \right)^4 (200 + 15 c_2) + \left( \frac{Z}{d} \right)^5 30 + \left( \frac{Z}{d} \right)^6
$$
The Adjacency matrix ensemble has 3 interesting limiting regions of the parameters:

- Adjacency matrix of Erdos-Renyi graph (d=1)
- Sparse blocks Adjacency random matrix
  - \( d \to \infty \)
  - \( Z/d \) fixed
- Effective medium spectral distribution
- Semi-circle distribution
  - \( d \) fixed
  - \( Z/d \to \infty \)
The analogous 3 regions for the Laplacian matrix ensemble:

- Laplacian matrix of Erdos-Renyi graph
  - $d=1$

- Sparse blocks Laplacian random matrix
  - $d \to \infty$
  - $Z/d$ fixed
  - $d$ fixed
  - $Z/d \to \infty$

- Marchenko-Pastur spectral distribution
- Convolution of Gaussian and semi-circle
Walks and partitions

A walk of $2n$ steps with $h$ distinct edges is a partition of a sequence of $2n$ matrices in $h$ parts (or blocks).

Non-crossing partitions (Kreweras 1972, R. Simion 2000) play a role similar to planar graphs in the topological expansion.

Examples: two walks of 12 steps with 4 distinct edges

\[
X_1(X_2)^2(X_3)^2(X_2)^2(X_3)^2X_1(X_4)^2
\]

is a crossing partition because its 4 parts are
\[
\{1,10\} , \{2,3,6,7\} , \{4,5,8,9\} , \{11,12\}
\]
and the sequence 2,5,6,9 is such that \(\{2,6\} \in X_2\), \(\{5,9\} \in X_3\).
the sequence $X_1(X_2)^3(X_3)^4X_2(X_4)^2X_1$
is a **non-crossing** partition because its 4 parts are

\{1,12\} , \{2,3,4,9\} , \{5,6,7,8\} , \{10,11\}

there are no two parts with interlacing pairs.

We evaluate the ensemble averages

\[<\alpha_1\alpha_2\alpha_3\alpha_4>< TrX_1(X_2)^2(X_3)^2(X_2)^2(X_3)^2X_1(X_4)^2 > = \left(\frac{Z}{d}\right)^4 \frac{3}{d+2}\]

\[<\alpha_1\alpha_2\alpha_3\alpha_4>< TrX_1(X_2)^3(X_3)^4X_2(X_4)^2X_1 > = \left(\frac{Z}{d}\right)^4\]
The limit $d \to \infty$ with $Z/d$ fixed, steps of the proof

Definitions

$T_j^{(2n)} :=$ set of closed walks beginning and ending at vertex $j$, with $2n$ steps

$B_j^{(2n)} :=$ set of closed *primitive* walks beginning and ending at vertex $j$, with $2n$ steps

Generating functions

$$T_j(x) = 1 + \sum_{k=1}^{\infty} x^{2k} T_j^{(2k)} , \quad B_j(x) = \sum_{k=1}^{\infty} x^{2k} B_j^{(2k)}$$
Two basic equations: (1) the expansion of closed walks in primitive closed walks

\[ T_{jo}(x) = 1 + \sum_j B_{joj}(x) + \sum_{j,k} B_{joj}(x)B_{jok}(x) + \sum_{j,k,l} B_{joj}(x)B_{jok}(x)B_{jol}(x) + \ldots \]

(2) primitive closed walks are written in terms of first and last step plus generic closed walks (here written for walks on trees)

\[ B_{jok}(x) = \alpha_{jok}x^2X_{jok}T_k(x)X_{kjo} \]

A recursive algorithm that generates the walks on trees is obtained by inserting the second eq. in the first one.

In the \(d \to \infty\) limit, \(Z/d\) fixed, only non-crossing partitions survive, the sequences of closed walks are stochastically independent and one algebraic equation is obtained for the resolvent of Adjacency and Laplacian matrices.
For the resolvent $r(z)$ of the Adjacency matrix one obtains the cubic equation
\[
r^3(z) - \frac{1}{z}(1 - t)r^2(z) - r(z) + \frac{1}{z} = 0 \quad , \quad t = \frac{Z}{d}
\]
which is the effective medium approximation by Semerjian and Cugliandolo (2002).

For the resolvent $r(z)$ of the Laplacian matrix one obtains the quadratic equation
\[
2zr^2(z) - (2 - t + z)r(z) + 1 = 0 \quad , \quad t = \frac{Z}{d}
\]
which is the Marchenko Pastur equation (1967).
The generating function $f(z)$ of the moments of the two parameter families of densities $\pi_{s,t}$ (called “Free Bessel laws”, Banica et al. 2008)

$$f(z) = 1 + m_1z + m_2z^2 + .. , \quad f = 1 + zf^s(f+t-1)$$

The moments of EMA (Semerjian+Cugliandolo 2002) are the Fuss Narayama polynomials, $s = 2$

$$m_k = \sum_{b=1}^{k} \frac{1}{b} \binom{k-1}{b-1} \binom{2k}{b-1} t^b$$

Similar result for the moments of the Laplacian, $s = 1$, and a change of variables.
Conclusions and generalizations

• the determination of the spectral densities in the $d \to \infty$ limit, with $Z/d$ fixed was unexpected

• the hessian of this talk has a physics interest and the parameter $d$ is the space dimension

• other graph structures (all straightforward):
  – regular random graph
  – bipartite biregular graph (M. Pernici, in preparation)
  – complete graph, $\alpha \to 1$

• different ensembles for the blocks $X_{ij}$
  – $X_{ij} \in \text{GUE or GOE}$, analysis possible but not easy.