Gaussian Multiplicative Chaos and random matrix theory

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1. Kahane’s Gaussian multiplicative chaos on the unit circle

2. The Circular $\beta$ Ensemble in random matrix theory

3. Orthogonal polynomials on the unit circle

4. Our main result
The logarithmically correlated field on the unit circle

- We consider a centered Gaussian field \( G_U \) with covariance

\[
\mathbb{E}[G_U(f)G_U(g)] = \int_U \int_U f(x)g(y)(2 \log(1/|x - y|))dxdy
\]

for smooth test functions \( f \) and \( g \) on \( U \).
The logarithmically correlated field on the unit circle

- We consider a centered Gaussian field $G_U$ with covariance

$$\mathbb{E}[G_U(f)G_U(g)] = \int_U \int_U f(x)g(y)(2 \log(1/|x-y|))dx\,dy$$

for smooth test functions $f$ and $g$ on $U$.

- We have formally the equality in law

$$G_U(e^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \mathcal{N}_k \frac{e^{ik\theta}}{\sqrt{k}}$$

where $(\mathcal{N}_1^C, \mathcal{N}_2^C, \ldots)$ to be a sequence of i.i.d standard complex Gaussians i.e:

$$\mathbb{P}(\mathcal{N}_i^C \in dx\,dy) = \frac{1}{\pi} e^{-x^2-y^2} dx\,dy,$$

so that:

$$\mathbb{E}\mathcal{N}_k^C = \mathbb{E}(\mathcal{N}_k^C)^2 = 0, \quad \mathbb{E}|\mathcal{N}_k^C|^2 = 1.$$
Harmonic extension to the unit disc

We can consider the harmonic extension $G$ of $G_U$ to the open unit disc $\mathbb{D}$ (not to be confused with the Gaussian free field on $\mathbb{D}$):

$$G(re^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \frac{N_k}{\sqrt{k}} r^k e^{ik\theta} = P_r \ast G_U(e^{i\theta}),$$

where $P_r$ is the Poisson kernel. Notice that $G$ is a random holomorphic function on $\mathbb{D}$, whereas $G_U$ is only a distribution (in the Sobolev space $H^{-\varepsilon}$ for all $\varepsilon > 0$).
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$$G(re^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \frac{\mathcal{N}_k^C}{\sqrt{k}} r^k e^{ik\theta} = P_r * G_U(e^{i\theta}) ,$$

where $P_r$ is the Poisson kernel. Notice that $G$ is a random holomorphic function on $\mathbb{D}$, whereas $G_U$ is only a distribution (in the Sobolev space $H^{-\varepsilon}$ for all $\varepsilon > 0$).
The covariance structure of $G$ is given by

$$\mathbb{E}[G(re^{i\theta})G(r'e^{i\theta'})] = -2 \log |1 - rr'e^{i(\theta-\theta')}|,$$

in particular, we have

$$\mathbb{E}[(G(re^{i\theta}))^2] = -2 \log (1 - r^2).$$
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$$

We can then define the following random measure:

$$
GMC_\gamma(d\theta) := e^{\gamma G(re^{i\theta})} - \frac{\gamma^2}{2} \text{Var}[G(re^{i\theta})] \frac{d\theta}{2\pi} = e^{\gamma G(re^{i\theta})}(1 - r^2)^{\gamma^2} \frac{d\theta}{2\pi},
$$

which naturally appears in the work by Kahane and in the study of Liouville conformal field theory (work by Rhodes-Vargas, Miller-Sheffield and their students: I am not an expert of this topic!).
Definition of the subcritical Gaussian multiplicative chaos

For $\gamma \in (0, 1)$ (subcritical regime), the Gaussian multiplicative chaos (GMC) is obtained by taking the limit of $GMC_\gamma r$ when $r$ goes to 1 from below, we can be formally written as

$$GMC_\gamma (d\theta) := e^{\gamma G_U(e^{i\theta}) - \frac{\gamma^2}{2} \text{Var}[G_U(e^{i\theta})]} \frac{d\theta}{2\pi}.$$ 

This definition is not rigorous since the field $G_U$ is not defined at points of the unit circle. However, the following holds:
Definition of the subcritical Gaussian multiplicative chaos

- For $\gamma \in (0, 1)$ (subcritical regime), the Gaussian multiplicative chaos (GMC) is obtained by taking the limit of $\text{GMC}_r^\gamma$ when $r$ goes to 1 from below, we can be formally written as

$$\text{GMC}_r^\gamma(d\theta) := e^{\gamma \text{G}_U(e^{i\theta}) - \frac{\gamma^2}{2} \text{Var}[\text{G}_U(e^{i\theta})]} \frac{d\theta}{2\pi}. $$

This definition is not rigorous since the field $\text{G}_U$ is not defined at points of the unit circle. However, the following holds:

**Theorem (Kahane, Rhodes-Vargas, Berestycki)**

*For $\gamma \in (0, 1)$, there exists a random measure $\text{GMC}^\gamma$ on $\mathbb{U}$ such that for all smooth functions $f$ from $\mathbb{U}$ to $\mathbb{R}$, the following convergence holds in $L^1$

$$\text{GMC}_r^\gamma(f) \xrightarrow{r \to 1} \text{GMC}^\gamma(f):$$

where $M(f)$ denotes the integral of $f$ with respect to the measure $M$. The limiting measure $\text{GMC}^\gamma$ is called Kahane’s Gaussian multiplicative chaos (GMC).*
For $\gamma \geq 1$, the construction above gives the zero measure when $r$ goes to 1. In the critical case $\gamma = 1$, a non-trivial version of the Gaussian multiplicative chaos can be constructed in different ways (articles by Duplantier-Rhodes-Sheffield-Vargas, Junnila-Saksmann, Aru-Powell-Sepulveda). One can show that all these constructions agree, and give the limit in distribution of $(1 - \gamma)^{-1}GMC^\gamma$ when $\gamma$ goes to 1 from below. We will denote $GMC^1$ for this limit.
The critical and the supercritical phases

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- In the supercritical phase $\gamma > 1$ (papers by Barral-Jin-Rhodes-Vargas, Madaule-Rhodes-Vargas), several constructions are possible, which are not expected to agree with each other. These constructions give measures which are almost surely atomic.
1. Kahane’s Gaussian multiplicative chaos on the unit circle

2. The Circular $\beta$ Ensemble in random matrix theory

3. Orthogonal polynomials on the unit circle

4. Our main result
For $\beta > 0$, consider the Circular $\beta$ Ensemble ($C\beta E$), defined as the following distribution of $n$ points on the circle:

$$(C\beta E_n) \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < l \leq n} |e^{i\theta_k} - e^{i\theta_l}|^\beta d\theta.$$
The model

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- For $\beta = 2$, this distribution corresponds to the joint law of the eigenvalues of a uniformly distributed unitary matrix of order $n$, i.e. the Circular Unitary Ensemble. Other random matrix models for general $\beta$ have been constructed, for example by Killip and Nenciu.
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(C\beta E_n) \quad \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < l \leq n} |e^{i\theta_k} - e^{i\theta_l}|^\beta \, d\theta.
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- One can then consider the characteristic polynomial of a random unitary matrix $U_n$ whose spectrum $\{e^{i\theta_j}, 1 \leq j \leq n\}$ corresponds to the $C\beta E_n$:

$$
X_n(z) := \det (\text{id} - zU_n^*) = \prod_{1 \leq j \leq n} (1 - ze^{-i\theta_j})
$$
If the spectrum of a random matrix $U_n$ follows the $C\beta E_n$, then the traces of fixed powers of $U_n$ converge to independent complex Gaussian variables when $n$ goes to infinity:

$$\left(\text{tr} \left( U_n^k \right) \right)_{1 \leq k \leq K} \xrightarrow{n \to \infty} \left( \sqrt{\frac{2k}{\beta}} N^C_k \right)_{1 \leq k \leq K}$$

for all integers $K \geq 1$. This result has been proven by Diaconis and Shahshahani for $\beta = 2$ and by Jiang-Matsumoto for general $\beta > 0$. 
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An shorter proof of the result by Jiang-Matsumoto can be obtained by combining results on symmetric functions given in Macdonald’s book.
Convergence in law of the characteristic polynomial

If one takes the continuous version of the logarithm of the characteristic polynomial, one gets

\[ \log X_n(z) = -\sum_{k \geq 1} \frac{tr(U_n^k)}{k} z^k, \]

for \( z \in \mathbb{D} \). Because of the convergence in law given above, it is natural to expect the following:
Convergence in law of the characteristic polynomial

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for \( z \in \mathbb{D} \). Because of the convergence in law given above, it is natural to expect the following:

**Proposition (Hughes-Keating-O’Connell for \( \beta = 2 \), Chhaibi-N. for \( \beta > 0 \))**

*We have the convergence in law to the log-correlated field:*

\[
(\log(|X_n(z)|^2))_{z \in \mathbb{D}} \overset{n \to \infty}{\longrightarrow} \left( \sqrt{\frac{2}{\beta}} G(z) \right)_{z \in \mathbb{D}}
\]

uniformly in \( z \in K \) when \( K \) is a compact subset of \( \mathbb{D} \). Moreover, a similar convergence holds for \( z \in U \), with \( G \) replaced by \( G_U \), in the Sobolev space \( H^{-\varepsilon}(U) \).
From the convergence above, it is natural to construct a random probability measure from the characteristic polynomial and to compare it to the GMC. For $\beta = 2$, the following result holds:
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**Proposition (Nikula, Saksman and Webb (2018))**

For every $\delta \in [0, 1)$, consider $X_n(z) = \det(I_n - zU_n^*)$ to be the characteristic polynomial of the CUE, i.e. the $C_\beta E$ when $\beta = 2$. Then, the random measure

$$\frac{|X_n(e^{i\theta})|^{2\delta}}{\mathbb{E} |X_n(e^{i\theta})|^{2\delta} 2\pi} \, d\theta$$

converges as $n \to \infty$ for topology of weak convergence, (in law), to GMC$^\delta$. 

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It is not known if the result above can be generalized to $\beta \neq 2$. The method used by Nikula, Saksman and Webb is specific to the case $\beta = 2$, since it uses the determinantal structure of the CUE and some techniques related to Riemann-Hilbert problems.
It is not known if the result above can be generalized to $\beta \neq 2$. The method used by Nikula, Saksman and Webb is specific to the case $\beta = 2$, since it uses the determinantal structure of the CUE and some techniques related to Riemann-Hilbert problems.

In our recent paper with Chhaibi, we construct a modification of the characteristic polynomial of the $C\beta E$ which couples all the dimensions $n$ together on a single probability space, in such a way that a strong convergence to the GMC occurs when $n \to \infty$. 
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In our recent paper with Chhaibi, we construct a modification of the characteristic polynomial of the $C_{\beta}E$ which couples all the dimensions $n$ together on a single probability space, in such a way that a strong convergence to the GMC occurs when $n \to \infty$.

Moreover, this convergence gives an identity in law which provides an explicit expression of the distribution of $GMC^\gamma(f)$ when $\gamma \leq 1$ (subcritical and critical chaos) and $f$ is a trigonometric polynomial. This expression does not need to pass to the limit as in the original definition of the GMC.
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4. Our main result
Definition of the orthogonal polynomials on the unit circle (OPUC)

- If $\mu$ is a probability measure on the unit circle, the Gram-Schmidt procedure applied on $L^2(\mu)$ to the sequence $(z^k)_{k \geq 0}$ gives a sequence $(\Phi_k)_{0 \leq k < m}$ of monic orthogonal polynomials, $m$ being the (finite or infinite) cardinality of the support of $\mu$. If $m < \infty$, the procedure stops after $\Phi_{m-1}$ since all $L^2(\mu)$ is spanned: we then define

$$\Phi_m(z) := \prod_{\lambda \in \text{Supp}(\mu)} (z - \lambda),$$

which vanishes in $L^2(\mu)$. 

For convenience, we also introduce the polynomials $\Phi^*_k(z) := z^k \Phi^*_k(1/\bar{z})$, which are obtained from the $\Phi_k$ by conjugating the coefficients and reversing their order.
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which are obtained from the $\Phi_k$ by conjugating the coefficients and reversing their order.
Szegö recursion

There exists a sequence \((\alpha_j)_{0 \leq j < m}\) of complex numbers, \(|\alpha_j| = 1\) if \(j = m - 1 < \infty\), \(|\alpha_j| < 1\) otherwise, called Verblunsky coefficients, such that the polynomials above satisfy the so-called Szegö recursion: for \(j < m\),

\[
\Phi_{j+1}(z) = z\Phi_j(z) - \overline{\alpha_j}\Phi^*_j(z),
\]

\[
\Phi^*_{j+1}(z) = -\alpha_j z\Phi_j(z) + \Phi^*_j(z).
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  \[ \Phi_{j+1}^*(z) = -\alpha_jz\Phi_j(z) + \Phi_j^*(z). \]

- Moreover, Killip and Nenciu have found an explicit probability distribution for the Verblunsky coefficients, for which one can recover the characteristic polynomial of the Circular Beta Ensemble.
OPUC and $C_{\beta E}$

- Killip and Nenciu have discovered an explicit distribution for Verblunsky coefficients so that $X_n$, the characteristic polynomial of $C_{\beta E_n}$, is a $\Phi_n^*$!
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**Theorem**

- Let $(\alpha_j)_{j \geq 0}$ be independent random variables, rotationally invariant and such that
  \[ P(|\alpha_j|^2 \in dx) = (\beta/2)(j + 1)(1 - x)^{(\beta/2)(j+1) - 1} \]
  and let $\eta$ be independent and uniform on the unit circle.

- Let $(\Phi_j, \Phi_j^*)_{j \geq 0}$ be a sequence of OPUC obtained from the coefficients $(\alpha_j)_{j \geq 0}$ and the Szegö recursion.

Then we have the equality in law *between random polynomials*:

\[ X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z). \]
Killip and Nenciu have discovered an explicit distribution for Verblunsky coefficients so that $X_n$, the characteristic polynomial of $C_{\beta}E_n$, is a $\Phi_n^*$!

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$$

**Remark**

This result provides a way to define $C_{\beta}E_n$ for all values of $n$ on a single probability space.
An infinite dimensional version of the $C\beta E$

If a measure defines Verblunsky coefficients, the converse is also true:

**Theorem (Verblunsky 1930)**

The Verblunsky coefficients define a bijective correspondence between the probability measures on $\mathbb{U}$ and the sequences $(\alpha_j)_{0 \leq j < m}$ of complex numbers, such that one of the following holds:

- The sequence is infinite (i.e. $m = \infty$) and all the coefficients $\alpha_j$ are in the open unit disc.
- The sequence is finite (i.e. $m < \infty$) and all the coefficients $\alpha_j$ are in the open unit disc, except the last one $\alpha_{m-1}$ which is on the unit circle.
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For $n \geq 1$, the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \eta)$ corresponds to a measure $\mu_n^\beta$ whose support follows the $C\beta E$ of order $n$, if the distribution of $(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \eta)$ is the same as in the result by Killip and Nenciu.
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- For $n \geq 1$, the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \eta)$ corresponds to a measure $\mu_n^\beta$ whose support follows the $C_\beta E$ of order $n$, if the distribution of $(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \eta)$ is the same as in the result by Killip and Nenciu.
- The infinite sequence $(\alpha_j)_{j \geq 0}$ corresponds to a random probability measure $\mu^\beta$ with infinite support, which can be viewed as an infinite-dimensional version of the $C_\beta E$. 
A puzzling question

By continuity properties of the Verblunsky map, $\mu_n^\beta$ (supported by $C\beta E_n$) converges to $\mu^\beta$. 

Since $\Phi^* n$ is closely related to the characteristic polynomial of the $C\beta E$, this result reminds the convergence result by Nikula, Saksman and Webb.
A puzzling question

- By continuity properties of the Verblunsky map, $\mu_n^\beta$ (supported by $C_\beta E_n$) converges to $\mu^\beta$.
- Moreover, a general result by Bernstein and Szeg"{o} shows that the probability measure $\tilde{\mu}_n^\beta$ associated to the infinite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, 0, 0, 0, \ldots)$ has a density proportional to $|\Phi_n^*|^{-2}$ with respect to the Lebesgue measure.

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Again using continuity properties, we deduce that almost surely,

$$\frac{\int_0^{2\pi} |\Phi_n^*(e^{i\theta})|^{-2} d\theta}{\int_0^{2\pi} |\Phi_n^*(e^{i\theta})|^{-2} d\theta} \to^\mu \mu^\beta.$$
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- By continuity properties of the Verblunsky map, \( \mu_n^\beta \) (supported by \( C \beta E_n \)) converges to \( \mu^\beta \).
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- Again using continuity properties, we deduce that almost surely,

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\int_0^{2\pi} \frac{|\Phi^*_n(e^{i\theta})|^{-2} d\theta}{\int_0^{2\pi} |\Phi^*_n(e^{i\theta})|^{-2} d\theta} \xrightarrow{n \to \infty} \mu^\beta.
\]

- Since \( \Phi^*_n \) is closely related to the characteristic polynomial of the \( C \beta E \), this result reminds the convergence result by Nikula, Saksman and Webb.
- Question: what is the distribution of \( \mu^\beta \)?
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4 Our main result
The main result of our article presented here is the following:

**Theorem (Chhaibi-N. 2019)**

For $\beta \geq 2$, let $\gamma = \sqrt{\frac{2}{\beta}} \leq 1$ (subcritical or critical regime) and let $\mu^\beta$ be the random probability measure whose Verblunsky coefficients $(\alpha_j)_{j \geq 0}$ have the same law as in the result by Killip and Nenciu. Then, the random measure $C_0 \mu^\beta$ has the same law as $\text{GMC}^\gamma$, where

$$C_0 = \left(2 \mathbb{1}_{\beta=2} + 1 - \frac{2}{\beta}\right) \left(1 - |\alpha_0|^2\right)^{-1} \prod_{j=0}^{\infty} (1 - |\alpha_j|^2)^{-1} \left(1 - \frac{2}{\beta(j+1)}\right)$$

In particular, $\mu^\beta$ has the same law as $\text{GMC}^\gamma$, normalized into a probability measure. Moreover, we have for $\beta > 2$ and $\gamma < 1$ (subcritical regime), the almost sure convergence:

$$\frac{1}{\mathbb{E}} \left| \Phi_n^*(e^{i\theta}) \right|^{-2} d\theta \underset{n \to \infty}{\longrightarrow} C_0 \mu^\beta \overset{\text{law}}{\equiv} \text{GMC}^\gamma.$$
Consequences

- $(C_{\beta}E_n; \beta \geq 2, n \in \mathbb{N}^*)$ can all be coupled upon constructing $(GMC^\gamma; 0 < \gamma \leq 1)$. In order to do that, one considers the support of the measure whose $n - 1$ first Verblunsky coefficients are the same as those of $GMC^\gamma$, the $n$-th and last Verblunsky coefficient being an independent, uniform variable $\eta$ on the unit circle. With a suitable precise definition, $C_{\beta}E_n$ corresponds to the $n$-quadrature points of $GMC^\gamma$. 

For $\gamma \geq 1$, the total mass of $GMC^\gamma$ has the same law as $C_0$, which gives an alternative proof of a conjecture by Fyodorov and Bouchaud, proven by Remy in 2017. Indeed, $GMC^\gamma(U) = K_{\beta}(1 - |\alpha_0|^2) - \prod_{j=1}^{\infty} (1 - |\alpha_j|^2)^{-1} (1 - 2\beta(j + 1))L = K'_{\beta}e^{-2\beta}$. Where $K_{\beta}$ and $K'_{\beta}$ depend only on $\beta$, and $e$ is a standard exponential variable.
Consequences

- \((C\beta E_n ; \beta \geq 2, n \in \mathbb{N}^*)\) can all be coupled upon constructing\((GMC^\gamma ; 0 < \gamma \leq 1)\). In order to do that, one considers the support of the measure whose \(n - 1\) first Verblunsky coefficients are the same as those of\(GMC^\gamma\), the \(n\)-th and last Verblunsky coefficient being an independent, uniform variable \(\eta\) on the unit circle. With a suitable precise definition, \(C\beta E_n\) corresponds to the \(n\)-quadrature points of \(GMC^\gamma\).

- For \(\gamma \geq 1\), the total mass of \(GMC^\gamma\) has the same law as \(C_0\), which gives an alternative proof of a conjecture by Fyodorov and Bouchaud, proven by Remy in 2017. Indeed,

\[
GMC^\gamma(\mathbb{U}) = K_\beta (1 - |\alpha_0|^2)^{-1} \prod_{j=1}^{\infty} (1 - |\alpha_j|^2)^{-1} \left( 1 - \frac{2}{\beta(j + 1)} \right) \overset{\mathcal{L}}{=} K'_\beta \ e^{-\frac{2}{\beta}}.
\]

where \(K_\beta\) and \(K'_\beta\) depend only on \(\beta\), and \(e\) is a standard exponential variable.
One can also describe all moments

\[
c_k = \frac{1}{\text{GMC}_\gamma(U)} \int_0^{2\pi} e^{ik\theta} \text{GMC}_\gamma(d\theta).
\]

via universal expressions in terms of the Verblunsky coefficients. For example:

\[
\begin{align*}
    c_1 & = \alpha_0, \\
    c_2 & = \alpha_0^2 + \alpha_1(1 - |\alpha_0|^2), \\
    c_3 & = (\alpha_0 - \alpha_1 \overline{\alpha_0})[\alpha_0^2 + \alpha_1(1 - |\alpha_0|^2)] \\
    & \quad + \alpha_1 \alpha_0 + \alpha_2(1 - |\alpha_0|^2)(1 - |\alpha_1|^2).
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Up to a normalization constant depending only on $\beta$, the definition of the measure $C_0\mu^\beta$ makes sense for all values of $\beta > 0$, This gives an alternative construction of the supercritical GMC, which can be proven to give an atomic measure. It would be interesting to relate this construction to the articles by Barral-Jin-Rhodes-Vargas and Madaule-Rhodes-Vargas.
Open questions

Our result brings forth other questions:

- At critical $\beta = 2$, can we relate our result to the Fyodorov-Hiary-Keating conjecture on the maximal modulus of the characteristic polynomial $X_n$ on the unit circle?
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Joseph Najnudel (School of Mathematics, University of Bristol, UK)

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- The critical exponent for the GMC occurs when $\beta = 2$, which is also a special value in the random matrix theory side, since it corresponds to the CUE. Can we understand better why these special situations occur for the same value of $\beta$?
- More generally, some duality seems to occur between the exponents $\beta$ and $4/\beta$, both in the random matrix and the GMC points of view. Which precise results can we get in this direction, and how can we relate the two points of views for this question?
Acknowledgements

Thank you for your attention!