A scale of boundary conditions for the random normal matrix model

Joint work with Yacin Ameur and Nam-Gyu Kang

Seong-Mi Seo (KIAS)

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Randomness in Physics and Mathematics

Random normal matrix ensembles

2D Coulomb gas model

Consider n point charges on $\mathbb{C}$ influenced by an external potential Q.

- The energy of a configuration $(\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$ is given by

$$H_n(\zeta_1, \cdots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j).$$

- The Boltzmann-Gibbs distribution at inverse temperature $\beta > 0$:

$$\frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{2} H_n(\zeta_1, \cdots, \zeta_n)}.$$

- $\beta = 2$: eigenvalues of random normal matrix model

$$d\mathbf{P}_n(\zeta_1, \cdots, \zeta_n) = \frac{1}{Z_n} \prod_{l \leq k} |\zeta_l - \zeta_k|^2 e^{-n \sum_{j=1}^{n} Q(\zeta_j)} \prod_{j=1}^{n} dA(\zeta_j).$$

[Chau-Zaboronsky '98, Wiegmann-Zabrodin '00, Elbau-Felder '05].

Global properties

Q: a smooth potential satisfying $\liminf_{\zeta\to\infty} Q(\zeta)/\log|\zeta| > 2$.

- Convergence to the equilibrium: $n \to \infty$

$$\frac{1}{n}\sum_{j=1}^{n}\delta_{\zeta_j} \to \sigma.$$

- $\sigma = \sigma_Q$ is the Frostman's equilibrium measure.
- σ has compact support $S = S_Q$ (droplet).
- $d\sigma(\zeta) = \mathbf{1}_S \Delta Q(\zeta)\, dA(\zeta)$ where $\Delta = \partial\bar{\partial}$ and $dA(x+iy) = dx\,dy/\pi$.

Determinantal point process ($\beta = 2$)

- k**-point correlation function** (for $1 \leq k \leq n$) of the point process:

$$\mathbf{R}_{n,k}(\zeta_1, \cdots, \zeta_k) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2k}} \mathbf{P}_n \left[\mathcal{N}(D(\zeta_j, \epsilon)) \geq 1 \text{ for all } 0 \leq j \leq k \right],$$

where $\mathcal{N}(D)$ is the number of eigenvalues in $D \subset \mathbb{C}$.

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- $\mathbf{R}_{n,k}(\zeta_1, \cdots, \zeta_k) = \det \left(\mathbf{K}_n(\zeta_i, \zeta_j) \right)_{i,j=1}^{k},$

  where $\mathbf{K}_n : \mathbb{C}^2 \to \mathbb{C}$ is called a **correlation kernel**.

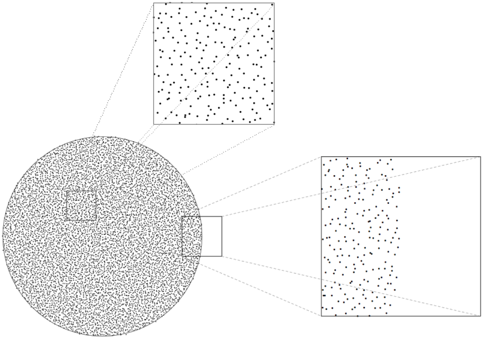
- Correlation kernel $\mathbf{K}_n$ is expressed by

$$\mathbf{K}_n(\zeta, \eta) = \sum_{j=0}^{n-1} p_{n,j}(\zeta) \overline{p_{n,j}(\eta)} e^{-nQ(\zeta)/2 - nQ(\eta)/2}.$$

  where $p_{n,j}$ is a j-th orthonormal polynomial with respect to $e^{-nQ} dA$.

Local properties

Define a rescaled eigenvalue system $\{z_j\}_1^n$ at a point in the droplet S.



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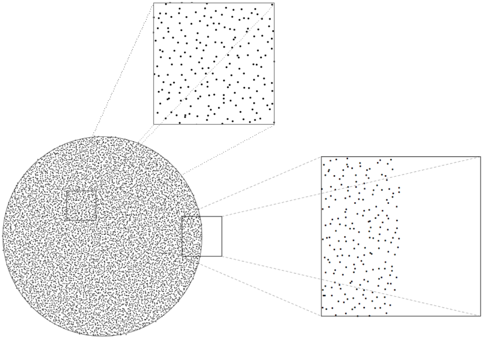

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$\star$ Bulk scaling ($p \in \operatorname{Int} S$)

$$z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p).$$

$\star$ Edge scaling ($p \in \partial S$)

$$z_j = \sqrt{n\Delta Q(p)}\, e^{-i\theta}(\zeta_j - p),$$

$e^{i\theta}$: outer normal to ∂S at p.

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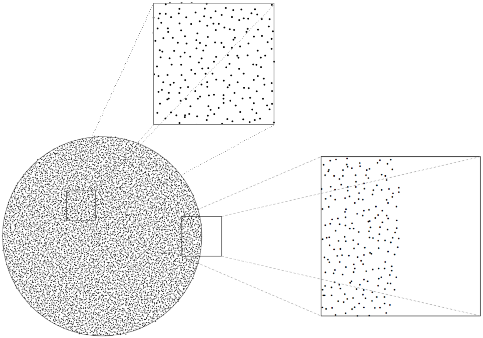

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Universality phenomenon: in the large n limit the microscopic behavior of eigenvalues does not depend on the specific potential.

Universality results

- Bulk scaling limit ($p \in \operatorname{Int} S$) [Ameur-Hedenmalm-Makarov, 2008]
  Rescaled system $z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p)$ converges to the determinantal point process with correlation kernel

$$G(z,w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2} \quad \text{(Ginibre kernel).}$$

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$$G(z,w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2} \quad \text{(Ginibre kernel).}$$

- Edge scaling limit ($p \in \partial S$) [Hedenmalm-Wennman, 2017]
  Rescaled system $z_j = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta_j - p)$ converges to the determinantal point process with correlation kernel

$$G(z,w) \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{z+\bar{w}}{\sqrt{2}}\right).$$

- Ginibre case: Forrester-Honner (1999).
- Translation invariant limits are characterized: Ameur-Kang-Makarov (2014).

Edge scaling limits

$R = \lim R_{n,1}$: the limiting 1-point function of the rescaled system $\{z_j\}$.

$$R(z) = G(z,z) \cdot \frac{1}{2}\operatorname{erfc}(2\operatorname{Re} z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(\xi - 2\operatorname{Re} z)^2/2} d\xi.$$

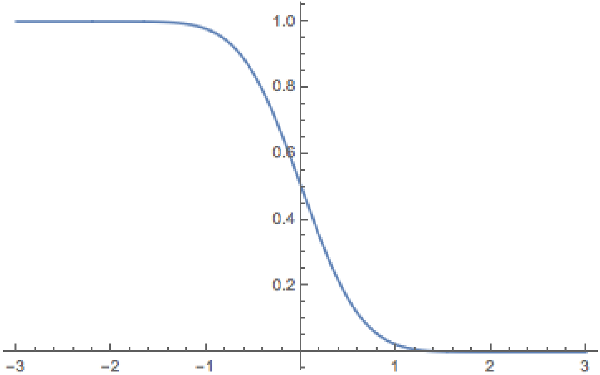


Figure. The graph of R restricted to $\mathbb{R}$.

Free boundary/ Hard edge

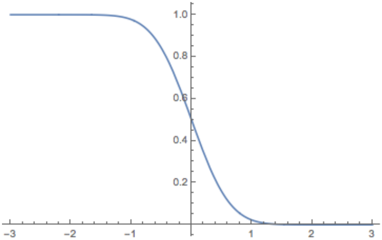
Random normal matrix models with two boundary conditions.

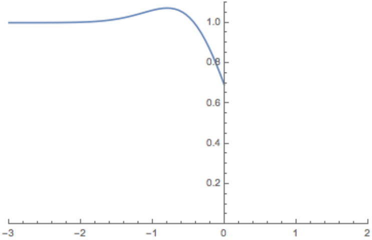
⋆ Free boundary RNM ensemble

- external potential Q
- Free boundary plasma function
  $\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(\xi - z)^2/2} d\xi$
- limiting correlation kernel
  $G(z,w)\, \varphi(z+\bar{w})$

⋆ Hard edge RNM ensemble

- $Q^{\mathrm{H}} = Q + \infty \cdot \mathbf{1}_{\mathbb{C}\setminus S}$
- Hard edge plasma function
  $H(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-(\xi - z)^2/2}}{\Phi(\xi)} \, d\xi$
- limiting correlation kernel
  $G(z,w) H(z+\bar{w}) 1_{\mathbb{L}}(z) 1_{\mathbb{L}}(w)$





Obstacle functions

- $\check{Q}$ is the solution of the obstacle problem: the maximal subharmonic function $\leq Q$ which grows like $\log|\zeta|^2 + O(1)$ when $|\zeta| \to \infty$.
- Logarithmic potential of the equilibrium measure σ

$$\check{Q}(\zeta) = -2\int_{\mathbb{C}} \log\frac{1}{|\zeta-\eta|}\,d\sigma(\eta) + \gamma,$$

  where γ is a constant.
- Properties:
  - $\check{Q} \leq Q$ on $\mathbb{C}$.
  - $S \subset S^* = \{Q = \check{Q}\}$ and $S^* \setminus S$ is a null set for $|\Delta Q| dA$.
  - $\check{Q}$ is harmonic on $\mathbb{C} \setminus S$ and $\check{Q}(\zeta) = 2\log|\zeta| + O(1)$ when $\zeta \to \infty$.

A scale of boundary conditions

Construct the random normal matrix model which interpolates between the free boundary case and the hard edge case.

- Assumptions:
  - Q is real analytic and strictly subharmonic in S^*.
  - The boundary ∂S^* is a smooth, simple, and closed curve.
- External potential $Q_t := \check{Q} + t(Q - \check{Q})$ for $t > 0$.

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- External potential $Q_t := \check{Q} + t(Q - \check{Q})$ for $t > 0$.
- $t = 1$: $Q_t = Q$ (free boundary)
- $t = \infty$: $Q_t = Q^{\mathrm{H}}$ (hard edge)
- $t = 0$: $Q_t = \check{Q}$

Exterior estimates

- $\{\zeta_j\}_1^n$: RNM ensembles associated with Q_t.
- $\mathbf{K}_n^t$: correlation kernel of $\{\zeta_j\}_1^n$ and $\mathbf{R}_{n,1}^t(\zeta) = \mathbf{K}_n^t(\zeta,\zeta)$.
- Exterior estimate:

$$\mathbf{R}_{n,1}^t(\zeta) \le C\, n\, e^{-n(Q_t(\zeta) - \check{Q}(\zeta))}, \quad \zeta \in \mathbb{C}.$$

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- $p \in \partial S$ and $h \ge 0$
  $(Q_t-\check{Q})(p+e^{i\theta}h) = t(Q-\check{Q})(p+e^{i\theta}h) = 2t\Delta Q(p)\, h^2+O(|h|^3), \quad h \to 0.$
- Rescaled one point function at p

$$R_{n,1}^t(z) = \frac{1}{n\Delta Q(p)}\mathbf{R}_{n,1}^t(\zeta) \le Ce^{-2t(\operatorname{Re} z)^2}, \quad z \in \mathcal{K},$$

where $z = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta - p)$ and $\mathcal{K}$ is compact in $\{z : \operatorname{Re} z > 0\}$.

Edge plasma functions

- Edge plasma function S_t:

$$S_t(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-(\xi - z)^2/2}}{\Phi_t(\xi)}\, d\xi$$

where

$$\Phi_t(\xi) = \varphi(\xi) + \frac{e^{\frac{1-t}{2t}\xi^2}}{\sqrt{t}} \left(1 - \varphi\left(\frac{\xi}{\sqrt{t}}\right)\right), \quad \varphi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\xi}^{\infty} e^{-u^2/2}\, du.$$

- Correlation kernel

$$K^t(z, w) = G(z, w)\, S_t(z + \bar{w})\, e^{(1-t)(\mathrm{Re}\, z)^2 \mathbf{1}_{\mathrm{Re}\, z > 0}} e^{(1-t)(\mathrm{Re}\, w)^2 \mathbf{1}_{\mathrm{Re}\, w > 0}}.$$

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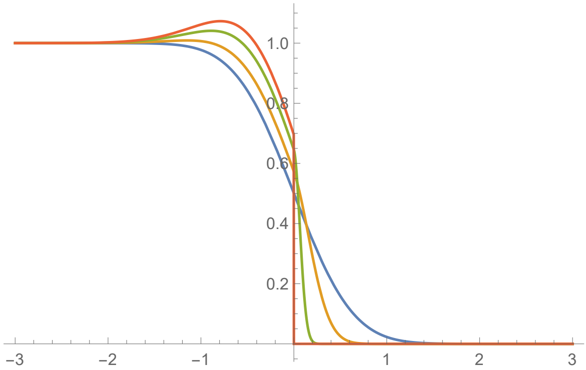
- $t = 1$: $\Phi_1 = 1$ and $S_1 = \varphi$ (free boundary plasma function)
  $K(z,w) = G(z,w)\, \varphi(z+\bar{w})$.

- $t \to \infty$: $\Phi_\infty = \varphi$ and $S_\infty = H$ (hard edge plasma function)
  $K^{\mathrm{H}}(z,w) = G(z,w) H(z+\bar{w}) \mathbf{1}_{\{\mathrm{Re}\, z<0\}} \mathbf{1}_{\{\mathrm{Re}\, w<0\}}$.

One point density of the rescaled system

R^t: the limiting one point function of the rescaled eigenvalue system $\{z_j\}$.

Graphs of $R^t(x) = K^t(x,x) = S_t(2x)\, e^{2(1-t)x^2 \mathbf{1}_{\{x>0\}}}$ for $x \in \mathbb{R}$. $(t \geq 1)$



- $t = 1$ (blue)
- $t = 5$ (orange)
- $t = 50$ (green)
- $t = \infty$ (red)

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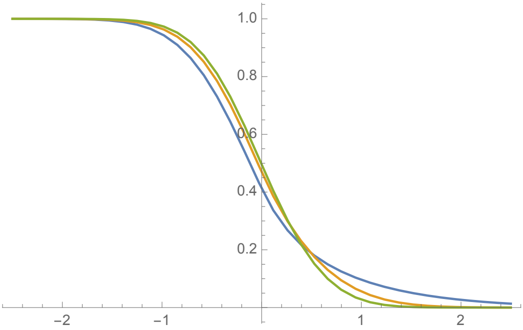
Graphs of $R^t(x) = K^t(x,x) = S_t(2x)\, e^{2(1-t)x^2 \mathbf{1}_{\{x>0\}}}$ for $x \in \mathbb{R}$. $(t < 1)$



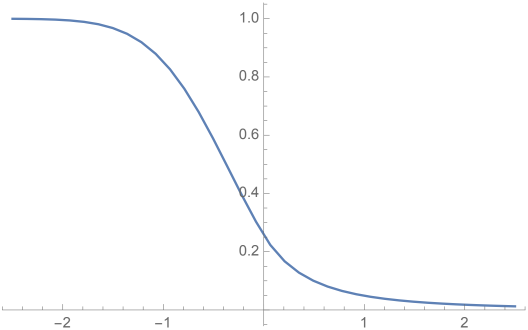
- $t = 0.1$ (blue)
- $t = 0.5$ (orange)
- $t = 0.9$ (green)

Exterior estimates: $R^t(x) \le Ce^{-2tx^2}, \quad x > 0.$

One point density of the rescaled system

Ultra-soft edge ($t \to 0$):

$$R^0(x) = \lim_{t\to 0} R^t(x) = e^{2x^2 \mathbf{1}_{x>0}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{(\xi-2x)^2/2}}{\varphi(\xi) - e^{-\xi^2/2}/(\sqrt{2\pi}\xi)} d\xi.$$



$$R^0(x) = \frac{1}{4x^2} + O(x^{-3}), \;\; x \to \infty.$$

Universality for radially symmetric potentials

Theorem [Ameur–Kang–S., 2019]

Suppose that Q is radially symmetric. For $0 < t < \infty$, the rescaled process $\{z_j\}_1^n$ converges to the determinantal point process with correlation kernel

$$K^t(z,w) = G(z,w) S_t(z+\bar{w}) e^{(1-t)((\operatorname{Re} z)^2 \mathbf{1}_{\{\operatorname{Re} z>0\}} + (\operatorname{Re} w)^2 \mathbf{1}_{\{\operatorname{Re} w>0\}})}$$

with locally uniform convergence of correlation functions.

- In the free boundary case ($t = 1$), the above theorem is proved for general potentials. [Hedenmalm–Wennman, 2017]
- Our approach: the recent method of Hedenmalm and Wennman to approximate orthogonal polynomials near the boundary of the droplet.

Approximate quasi-polynomials

Fix j with $n - \sqrt{n}\log n \leq j \leq n-1$ and write $\tau = j/n$.

- $\check{Q}_\tau$: the obstacle function pertaining to Q s.t. $\check{Q}_\tau \sim \tau \log|z|^2$ at ∞.
- $S_\tau = \{z \in \mathbb{C} : Q(z) = \check{Q}_\tau(z)\}$.
- Let ϕ_τ: $S_\tau^c \to \mathbb{D}^c$ be the conformal map s.t. $\phi_\tau(\infty) = \infty$, $\phi_\tau'(\infty) > 0$.

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- $\mathcal{Q}_\tau$: the bounded holomorphic function on S_τ^c s.t. $\operatorname{Re}\mathcal{Q}_\tau = Q$ on ∂S_τ.
- $\mathcal{H}_\tau$: the bounded holomorphic function on S_τ^c s.t.

  $\operatorname{Re}\mathcal{H}_\tau = \frac{1}{2}\log \Delta Q - \log \Phi_\tau$ on ∂S_τ where $\Phi_\tau(\zeta) = \Phi_t\left(\frac{j-n}{\sqrt{n}} \cdot \frac{\phi_\tau'(\zeta)}{\sqrt{\Delta Q(\zeta)}}\right)$

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The approximate quasi-polynomial of degree j is defined in a neighborhood of S_τ^c by

$$F_{n,j} = \left(\frac{n}{2\pi}\right)^{\frac{1}{4}} \sqrt{\phi_\tau'}\, \phi_\tau^j\, e^{n\mathcal{Q}_\tau/2}\, e^{\mathcal{H}_\tau/2}.$$

Error function approximation

The one point function of the system $\{\zeta_j\}$ can be approximated by

$$\mathbf{R}_n^t(\zeta) = \sum_{j=0}^{n-1} |p_{n,j}(\zeta)|^2 e^{-nQ_t(\zeta)} = \sum_{j=n-\sqrt{n}\log n}^{n-1} |F_{j,n}(\zeta)|^2 e^{-nQ_t(\zeta)} (1 + O(n^{-\frac{1}{2}+\delta}))$$

for all ζ with $\operatorname{dist}(\zeta, \partial S) = O(n^{-1/2})$.

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A gaussian ridge around ∂S_τ:

$$|F_{j,n}(\zeta)|^2 e^{-nQ_t(\zeta)} \sim \frac{\sqrt{n\Delta Q(\zeta)}}{\Phi_\tau(\zeta)} e^{-n(Q_t - \check{Q}_\tau)(\zeta)}.$$

Error function approximation

Rescaled one point function: for $z = \sqrt{n\Delta Q(p)}\, e^{-i\theta}(\zeta - p)$

$$R_n^t(z) = \frac{\mathbf{R}_n^t(\zeta)}{n\Delta Q(p)} \sim \frac{|\phi'(p)|}{\sqrt{2\pi n\Delta Q(p)}} \sum_{k=1}^{\sqrt{n}\log n} \frac{e^{-\frac{1}{2}(\xi_k - 2\,\mathrm{Re}\, z)^2}}{\Phi_t(\xi_k)} e^{2(1-t)(\mathrm{Re}\, z)^2 \mathbf{1}_{\{\mathrm{Re}\, z>0\}}}$$

where $\xi_k = -\frac{k}{\sqrt{n}} \cdot \frac{|\phi'(p)|}{\sqrt{\Delta Q(p)}}$.

By the Riemann sum approximation,

$$\begin{aligned}\lim_{n\to\infty} R_n^t(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-\frac{1}{2}(\xi - 2\,\mathrm{Re}\, z)^2}}{\Phi_t(\xi)}\, d\xi \cdot e^{2(1-t)(\mathrm{Re}\, z)^2 \mathbf{1}_{\{\mathrm{Re}\, z>0\}}} \\ &= S_t(2\,\mathrm{Re}\, z)\; e^{2(1-t)(\mathrm{Re}\, z)^2 \mathbf{1}_{\{\mathrm{Re}\, z>0\}}}.\end{aligned}$$

Thank you for your attention!