Local correlations of two-dimensional Coulomb gases on an ellipse

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Outline

On Random Matrices

- Random Matrix Theory, RMT
- Boltzman factor form
- Determinantal point process, DPP

Our result

- The model, asociated planar OP and spectral density

Outline

Given a matrix M whose entries are taken randomly from a known distribution, what would the distribution for its eigenvalues be?

Remark To define a matrix Ensemble one has to specify two things

- The space S on which the matrices vary
- A probability density function over S

Examples

Gaussian Unitary Ensemble, GUE

- $S = \{H \in Mat_{\mathbb{C}}(n) : H = H^\dagger\}$
  with independent, identicaly distributed (i.i.d) Gaussian entries
- $P(H)[dH] = \exp[-n\, tr(H^2)][dH]$
  where the flat measure (volume element)
  $[dH] = \prod_i dx_{ii} \prod_{i<j} dx_{ij} dy_{ij}$

Ex.GUE

what is the jpdf of its eigenvalues?

Since every complex $n \times n$ Hermitian matrix H is unitarily diagonalizable

$$H = UDU^{\dagger}$$

$H \to (U \in U(n), D)$ is not one to one, namely

$$UDU^{\dagger} = VDV^{\dagger} \ if \ U^{\dagger}V = diag(\lambda_1, \cdots, \lambda_n), \ \lambda_i \in U(1)$$

we therefore have to restrict the unitary matrices to the coset space $U(n)/U(1)^n$

$$e^{-n \ tr(H^2)}[dH] = C_n \Delta^2(x) e^{-n \sum x_i} d^n x \ d\mu(U)$$

then the desired joint p.d.f of all eigenvalues is given by

$$dP(x_1, \ldots, x_n) = \frac{1}{Z_n} \Delta^2(x) \prod_{i=1}^{n} e^{-n x_i^2} dx_i$$

Ex. Complex Ginibre Ensemble

what is the jpdf of its eigenvalues?

- $S = Mat_{\mathbb{C}}(n)$
  with i.i.d Gaussian $N_{\mathbb{C}}(0, 1/n)$ entries
- $P(J)[dJ] = \exp[-n\, tr(JJ^{\dagger})][dJ]$

A Schur decomposition, $J = U(D + T)U^{\dagger}$, can be use to obatain the jpdf of its eigenvalues

$$dP(z_1, \ldots, z_n) = \frac{1}{Z_n} |\Delta_n(z)|^2 \prod_{i=1}^{n} e^{-n|z|^2} dA(z_i)$$

with $dA(z)$ the planar Lebesgue measure

Ex. Elliptic Ginibre Ensemble

what is the jpdf of its eigenvalues?

- $S = \{J = aJ_1 + ibJ_2 : J_1, J_2 \text{ are indep. } GUE\}$
- $P(J)[dJ] = \exp[-n\, tr(A\, JJ^\dagger - \frac{B}{2}(J^2 + J^{\dagger 2}))]\,[dJ]$
  where $A = \frac{a^2+b^2}{2a^2b^2}$, $B = \frac{a^2-b^2}{2a^2b^2}$, $a > b > 0$

Schur decomposition implies

$$dP(z_1, \ldots, z_n) = \frac{1}{Z_n} |\Delta_n(z)|^2 \prod_{i=1}^{n} e^{-n\,h(z_i)} dA(z_i)$$

where $h(z) := A|z|^2 - B\, Re(z^2)$

- The Ginibre ensemble is recovered choosing $a = b$
- The GUE is obtained by formally taking the limit $b \to 0$

Outline

Boltzman factor form

Each eigenvalue jpdf can be written in its Boltzman factor form

$$e^{-\frac{\beta}{2}\sum_i V(z_i)}|\Delta_n(z)|^{\beta} = e^{-\frac{\beta}{2}\mathcal{E}_V(z)}$$

where

$$\mathcal{E}_V : \mathbb{C}^n \to \bar{\mathbb{R}},\ \mathcal{E}_V(z) = \sum_i V(z_i) + \sum_{i \neq j} log \frac{1}{|z_j - z_i|}$$

- $log \frac{1}{|z_j - z_i|}$ Electrostatic repulsion, $V : \mathbb{C} \to \bar{\mathbb{R}}$ external fiel

Assuming this charger gas be in a thermodynamical equilibrium at temperature T, so that the probability density of the positions of the n charges is given by

$$P(z_1, \ldots, z_n) = C\, e^{-\frac{1}{kT}\mathcal{E}_V(z)},\ k \text{ Boltzman constant}$$

is the Coulomb gas model corresponding to the matrix ensemble

Outline

Partition function & orthogonal polynomials

$$\Delta_n(z) = \prod_{i<j}(z_j - z_i) = \det_{1\le i,j\le n}\left[z_i^{j-1}\right] = \det_{1\le i,j\le n}[p_{j-1}(z_i)]$$

$$\begin{aligned} Z_n &= \int |\Delta_n(z)|^2 \prod_{i=1}^{n} w(z_i)\,dA(z_i) \\ &= n! \det_{1\le i,j\le n}\left[\int p_{i-1}(z)\overline{p_{j-1}(z)}w(z)\,dA(z)\right] \\ &= n! \prod_{j=0}^{n-1} h_j \end{aligned}$$

k-point correlation function

By using $det\, A = det\, A^t$ and Dyson's theorem

$$\begin{aligned}\rho(z_1, z_2, \ldots, z_k) &:= \frac{n!}{(n-k)!}\int \prod_{i=k+1}^{n} dA(z_i) P(z_1, z_2, \cdots, z_n) \\ &= \det_{1\le i,j\le k}\left[K_n(z_i, \bar{z}_j)\right]\end{aligned}$$

can be written in a determinantal form, in terms of the kernel K_n of the orthogonal polynomials w.r.t $w\, dA$

$$K_n(z_i, \bar{z}_j) = \sqrt{w(z_i)w(z_j)} \sum_{l=0}^{n-1} \frac{1}{h_l} p_l(z_i)\overline{p_l(z_j)}$$

- $\rho(z) = w(z) \sum_{l=0}^{n-1} \frac{1}{h_l} |p_l(z)|^2$, spectral density

Outline

Eigenvalue density on an ellipse

Let $\beta = 2$ and

$$dP(z_1, \ldots, z_n) = |\Delta_n(z)|^2 \prod_{i=1}^{n} w(z_i) dA(z_i)$$

- We impose the particles to be confined to an ellipse, which is given in the following parametrisation

$$E = \{z \in \mathbb{C} : h(z) = (\mathrm{Re}\, z)^2/a^2 + (\mathrm{Im}\, z)^2/b^2 < 1\}$$

whre $h(z) := A|z|^2 - B\, Re(z^2) = (\mathrm{Re}\, z)^2/a^2 + (\mathrm{Im}\, z)^2/b^2$

- We define a one-particle weight function

$$w(z) = (1 - h(z))^\alpha$$

which is real and non-negative, $w(z) = w(\bar{z}) \geq 0 \ (\forall z \in E)$

Orthogonal polynomials on an ellipse

For any non-negative integer n let us define the polynomials

$$p_n^{(\alpha)}(z) \quad := \quad \frac{1}{\sqrt{h_n}} C_n^{(1+\alpha)}\left(\frac{z}{c}\right) , \qquad (1)$$

where $C_n^{(1+\alpha)}(x)$ are the standard Gegenbauer polynomials on the real line having real coefficients. The constant $c = \sqrt{a^2 - b^2} > 0$ is then the right focus of the ellipse E.

$$h_n \quad := \quad \frac{1+\alpha}{1+\alpha+n} C_n^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right) > 0 ,$$

Theorem (Akemann,P., Nagao, Vernizzi, arXiv:1905.02397)

The set of polynomials $\{p_n^{(\alpha)}\}_{n\in\mathbb{N}}$ *defined in* (1) *forms a orthonormal basis for the weighted Bergman space* A^2_α.

where

$$A^2_\alpha = \{f \in H(E) : \int_E |f(z)|^2 (1-h(z))^\alpha dA(z) < \infty\}$$

Local correlations at weak non-Hermiticity

Let $a^2 = (1+\tau)/2\tau$ and $b^2 = (1-\tau)/2\tau$ and thus $a^2 - b^2 = 1$, with foci located at $z = \pm 1$.

$$\frac{1}{\tau} = 1 + \frac{s^2}{2N^2}, \quad 0 < s < \infty, \; z_j = x_j + iy_j = \frac{\hat{z}_j}{N}, \; j = 1, 2,$$

$N \to \infty$, and the weak non-Hermiticity parameter s is kept fixed.

$$\begin{aligned} K_{\mathrm{Bulk}}(\hat{z}_1, \hat{z}_2) &= \lim_{N\to\infty} \frac{1}{N^2} K_N\left(\frac{\hat{z}_1}{N}, \frac{\hat{z}_2}{N}\right) \\ &= \frac{2}{s\pi^{\frac{3}{2}}\Gamma(a+1)} \left(1 - \frac{4\hat{y}_1^2}{s^2}\right)^{\frac{a}{2}} \left(1 - \frac{4\hat{y}_2^2}{s^2}\right)^{\frac{a}{2}} \\ &\quad \times \int_0^1 dc \frac{(cs/2)^{a+\frac{1}{2}}}{I_{a+\frac{1}{2}}(cs)} \cos(c(\hat{z}_1 - \bar{\hat{z}}_2)). \end{aligned}$$

- $s \to 0$ sine-kernel
- $s \to \infty$ limiting kernel at strong non-Hermiticity
- By taking $\alpha \to \infty$ in the K_{strong} we obtain the Ginibre kernel

Thank you for your attention!