On the Correlation Functions of the Characteristic Polynomials of Non-Hermitian Random Matrices with Independent Entries

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Random matrices

Examples

- The adjacency matrices of random Erdős–Rényi graphs

$$M_{jk} = \begin{cases} 1 - \delta_{jk} & \text{with probability } \frac{p_n}{n}; \\ 0 & \text{with probability } 1 - \frac{p_n}{n}. \end{cases}$$

- Wigner ensemble

$$M^T = M \text{ or } M^* = M$$

$$M_{jk} = n^{-1/2} w_{jk},$$

$$\{w_{jk}\} - \text{i.i.d.}, \quad \mathbf{E}\{w_{jk}\} = 0, \quad \mathbf{E}\{|w_{jk}|^2\} = 1$$

- Non-Hermitian random matrices with independent entries (Ginibre ensemble if entries are Gaussian)

$$M_{jk} = n^{-1/2} w_{jk},$$

$$\{w_{jk}\} - \text{i.i.d.}, \quad \mathbf{E}\{w_{jk}\} = \mathbf{E}\{w_{jk}^2\} = 0, \quad \mathbf{E}\{|w_{jk}|^2\} = 1$$

Problems of the random matrix theory

1. Global regime

$$N_n(\Delta) = \#\{\lambda_j^{(n)} \in \Delta,\, j = 1, \ldots, n\}/n$$

2. Local regime — distribution of eigenvalues in a small neighbourhood of a fixed point (for non-Hermitian matrices — of order $\frac{1}{\sqrt{n}}$)

A logarithmic potential of a measure in the plane

$$P_\mu(z) = \int \log|z - \zeta| \; d\mu(\zeta).$$

$$P_{N_n}(z) = \sum \log\left|z - \lambda_j^{(n)}\right| = \frac{1}{2}\log\det(M_n - z)(M_n - z)^*$$

For the 2-point correlation function it is enough to consider

$$\mathbf{E}\left\{\prod_{j=1}^{2} \frac{\det\left((M_n - z_j)(M_n - z_j)^* + \delta_j\right)}{\det\left((M_n - z_j)(M_n - z_j)^* + \varepsilon_j\right)}\right\}$$

The correlation functions of the characteristic polynomials

Let M_n be some ensemble of non-Hermitian random matrices. Consider the m^{th} correlation function of characteristic polynomials

$$F_m(Z) = \mathbf{E}\left\{ \prod_{j=1}^{m} \det\left(M_n - z_j\right)\left(M_n - z_j\right)^* \right\}, \quad Z = \operatorname{diag}\{z_j\}_{j=1}^{m}$$

$$z_j = z_0 + \frac{\zeta_j}{\sqrt{n}}, \qquad |z_0| < 1$$

$$F_m(Z) \xrightarrow[n\to\infty]{} ?$$

Some results

- Brezin and Hikami (2000, 2001)
- Strahov and Fyodorov (2002, 2003); Fyodorov and Khoruzhenko (2006)
- T. Shcherbina (2011, 2013, 2014, 2015, 2016, 2017)
- Recher, Kieburg, Guhr, and Zirnbauer (2012)
- Akemann and Vernizzi (2003); Webb and Wong (2019)
- and other

The main result

Theorem (A.:19 (published in JSP))

Let the common distribution of the entries of M_n have $2m$ finite moments. Then the m^{th} correlation function of the characteristic polynomials satisfies the asymptotic relation

$$\lim_{n\to\infty} n^{-\frac{m^2-m}{2}} \frac{F_m(Z)}{F_1(z_1)\cdots F_1(z_m)}$$

$$= \exp\left\{\frac{m^2-m}{2}\left(1-|z_0|^2\right)^2\kappa_4\right\}\frac{\det(K(\zeta_j,\zeta_k))_{j,k=1}^m}{|\Delta(\zeta_1,\ldots,\zeta_m)|^2},$$

where $\kappa_4 = \mathbf{E}\{|x_{11}|^4\} - 2$, $\Delta(\zeta_1,\ldots,\zeta_m)$ is a Vandermonde determinant and

$$K(z,w) = e^{-|z|^2/2-|w|^2/2+z\bar{w}}.$$

Moments of the characteristic plynomials

Theorem (Webb and Wong:19)

Let $\Re\gamma > -2$ Then

$$\mathbf{E}\left\{|\det(M_n - z_0)|^\gamma\right\} = n^{\frac{\gamma^2}{8}} e^{\frac{\gamma}{2}n(|z_0|^2-1)} \frac{(2\pi)^{\frac{\gamma}{4}}}{G(1+\frac{\gamma}{2})}(1+o(1)),$$

where G is the Barnes G-function. $(G(1)=1,\ G(z+1)=\Gamma(z)G(z))$.

Theorem (A.:19 (published in JSP))

Let the common distribution of the entries of M_n have $2m$ finite moments. Then

$$\mathbf{E}\left\{|\det(M_n - z_0)|^{2m}\right\}$$
$$= e^{\frac{m^2-m}{2}(1-|z_0|^2)^2\kappa_4} n^{\frac{m^2}{2}} e^{mn(|z_0|^2-1)} \frac{(2\pi)^{m/2}}{\prod_{j=1}^{m-1} j!}(1+o(1)).$$

Grassmann variables

Let $\{\psi_j, \overline{\psi}_j\}_{j=1}^n$ be a set of anti-commuting variables, i.e.

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0.$$

Then for an arbitrary $n \times n$ matrix A

$$\int \exp\left\{ -\sum_{j,k=1}^{n} \overline{\psi}_j A_{jk} \psi_k \right\} \prod_{j=1}^{n} d\overline{\psi}_j d\psi_j = \det A.$$