Universality for random permutations

Randomness in Physics and Mathematics - ZIF Bielefeld University

Slim Kammoun

Supervisors: Mylène Maïda and Adrien Hardy

Laboratoire Paul Painlevé, Université de Lille

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Plan

1. Longest increasing subsequence and Ulam–Hammersley problem.
2. The first arrows of random Young tableaux (edge)
3. The Vershik-Kerov-Logan-Shepp shape

Longest increasing subsequence

- $\mathfrak{S}_n$: symmetric group, (the group of permutations of $\{1, \ldots, n\}$).
- $(\sigma(i_1), \ldots, \sigma(i_k))$ increasing subsequence of σ of length k if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$.
- $\ell(\sigma)$: the length of the longest increasing subsequence of σ.
- For example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & \color{red}{1} & \color{red}{2} & \color{red}{4} & 7 & \color{red}{5} & \color{red}{6} \end{pmatrix}.$$

$\ell(\sigma) = 5.$

Conjecture (Ulam (1961))

If $\sigma_n \sim U_{\mathfrak{S}_n}$, then

$$\lim_{n \to \infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} = c.$$

Longest increasing subsequence

Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} = 2$$

and

$$\frac{\ell(\sigma_n)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 2.$$

Theorem (Baik, Deift, and Johansson (1999))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s \right) = F_2(s).$$

F_2: CDF of the GUE Tracy-Widom distribution.

	GUE + Wigner (with a good control on moments)	Uniform permutation
First particle	T.W	T.W
Edge	Soft edge (Airy)	Soft edge (Airy)
Global convergence	Semi circular	VKLS
Fluctuations	Gaussian	Gaussian
Bulk	Sine process	Discrete sine process

Longest increasing subsequence

Theorem (K (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* n, σ_n *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

Then for all $s \in \mathbb{R}$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s). \tag{TW}$$

Ewens' case

Definition (Ewens distribution)

Let $\theta \geq 0$. If $\sigma_n \sim Ew(\theta)$ then

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{k=1}^{n-1}(\theta + k)}.$$

- $\theta = 1$: uniform distribution.
- $\theta = 0$: uniform distribution on permutations with a unique cycle.
- $\mathbb{E}(\#(\sigma_n)) = 1 + \sum_{k=1}^{n-1} \frac{\theta}{\theta+k} \sim \theta \log(n)$.

Ewens' case

Corollary

Assume that $\sigma_n \sim Ew(\theta_n)$*. If*

$$\lim_{n\to\infty} \frac{\theta_n \log(n)}{n^{\frac{1}{6}}} = 0. \tag{H'2}$$

Then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s). \tag{TW}$$

Other applications: Ewens-Pitman, virtual permutations (Kingman), etc.

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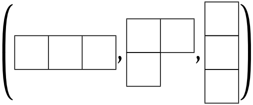
Young diagram

Definition (Young diagram)

$\lambda = (\lambda_i)_{i \geq 1} \in \mathbb{N}^{\mathbb{N}^*}$ is a Young diagram of size n if

- $\forall i \geq 1,\ \lambda_{i+1} \leq \lambda_i$,
- $\sum_{i=1}^{\infty} \lambda_i = n$.

Example: Young diagrams of size 3 are
$\mathbb{Y}_3 = (3, \underline{0}), (2, 1, \underline{0}), (1, 1, 1, \underline{0})$
or .

Young tableau

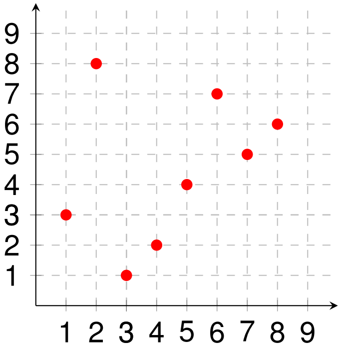
Definition (Young tableau)

A Young tableau of shape λ is a filling of the boxes of λ using the entries $\{1, 2, \ldots, n\}$ and the entries in each row and each column are increasing.

- Example: Young tableaux of shape $\ydiagram{3,1}$ are
  $\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}, \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}.$
- $dim(\lambda) = \#$ Young tableaux of shape λ.
- Example: $dim\left(\ydiagram{3,1}\right) = 3.$
- $dim(\lambda) =$ dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by λ.
- $\sum_{\lambda \in \mathbb{Y}_n} dim(\lambda)^2 = \#(\mathfrak{S}_n) = n!.$

Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



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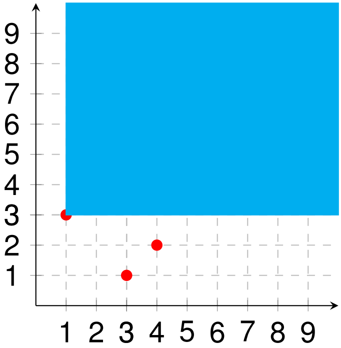

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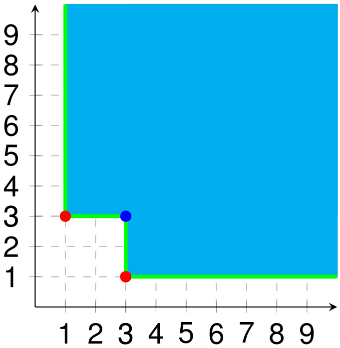

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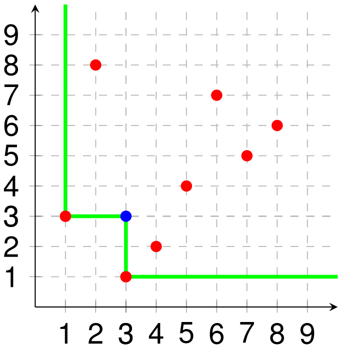

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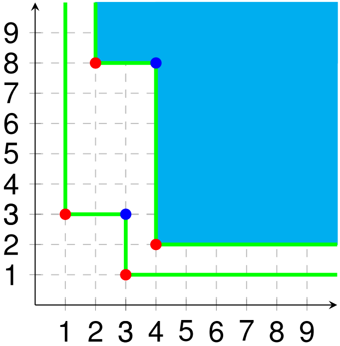

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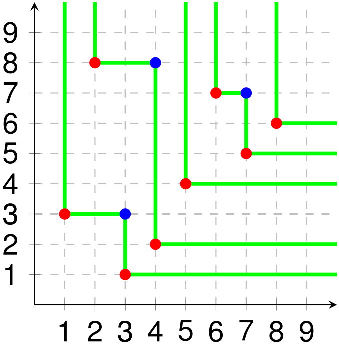
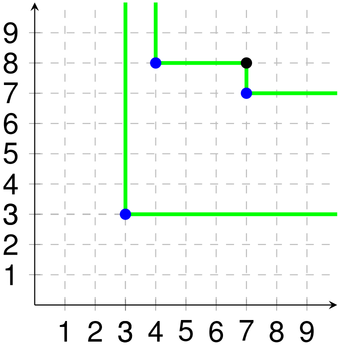


1	2	4	5	6	,	1	2	5	6	8

Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



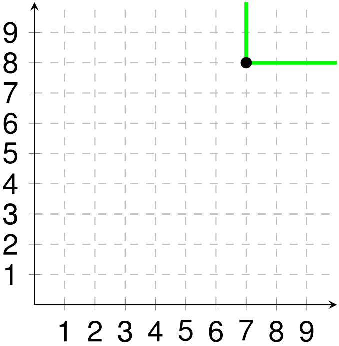
1	2	4	5	6
3	7			

,

1	2	5	6	8
3	4			

Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



1	2	4	5	6
3	7			
8				

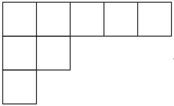
,

1	2	5	6	8
3	4			
7				

Robinson-Schensted correspondence

- One-to-one correspondence between permutations and pairs of standard Young tableaux of the same shape.
- We denote by $\lambda(\sigma) := (\lambda_i(\sigma))_{i\geq 1}$ the shape of the image of σ by this correspondence. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix} \quad \text{then} \quad \lambda(\sigma) =$$



.

- $\ell(\sigma) = \lambda_1(\sigma)$.

Edge: Plancherel case

Theorem (Borodin, Okounkov, and Olshanski (2000))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then* $\forall k \geq 1, \forall s_1, s_2, \ldots, s_k \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k, \ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = \mathbb{P}(\forall i \leq k, \ \xi_i \leq s_i).$$

$\{\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k \geq \ldots\}$: *Airy ensemble.*

	GUE + Wigner (with a good control on moments)	Uniform permutation
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Edge	Soft edge (Airy)	Soft edge (Airy)
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Fluctuations	Gaussian	Gaussian
Bulk	Sine process	Discrete sine process

Edge: generalization

Theorem (K (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ satisfies:

- *For all positive integer n, σ_n is invariant under conjugation i.e. $\forall \sigma, \rho \in \mathfrak{S}_n$,*

$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$

- *The number of cycles is such that: For all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

Then for all $s \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i).$$

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Russian notations

- Rotate the diagram by $\frac{3\pi}{4}$.
- Complete the high function by $x \to |x|$.
- We denote by L_λ the resulting function.

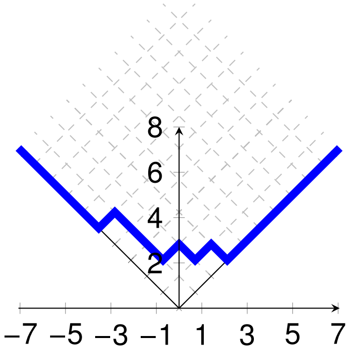

Figure: $L_{(5,2,1,\underline{0})}$

Vershik-Kerov-Logan-Shepp shape

Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))

If $\sigma_n \sim U_{\mathfrak{S}_n}$, *then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| < \varepsilon\right) = 1,$$

where

$$\Omega(s) := \begin{cases} \frac{2}{\pi}\left(s \arcsin(s) + \sqrt{1-s^2}\right) & \textit{if } |s| < 1 \\ |s| & \textit{if } |s| \geq 1 \end{cases}.$$

Vershik-Kerov-Logan-Shepp shape

Ω is strongly related to the semi-circular law.
We denote by

$$\omega(s) := \frac{\Omega(2s) - |2s|}{2}.$$

We have

$$\exp\left(\int_{\mathbb{R}} \frac{d\omega(u)}{u - \frac{1}{x}}\right) = \int_{\mathbb{R}} \frac{d\mu_{sc}(u)}{1 - ux}$$

with

$$d\mu_{sc}(u) := \frac{\sqrt{4 - u^2}}{2\pi} du.$$

Vershik-Kerov-Logan-Shepp shape

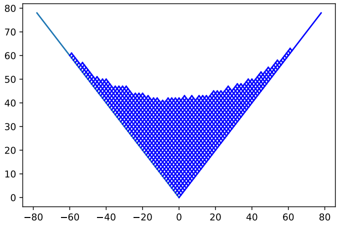

Figure: Typical Young diagram under the Plancherel distribution

	GUE + Wigner (with a good control on moments)	Uniform permutation
First particle	T.W	T.W
Edge	Soft edge (Airy)	Soft edge (Airy)
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Limit shape

Theorem (K (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* n, σ_n *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n} > \varepsilon\right) = 0. \tag{H3}$$

Then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left|\frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s)\right| < \varepsilon\right) = 1.$$

Bulk

Theorem (Borodin, Okounkov, and Olshanski (2000))

For any $|\alpha| < 2$, under Plancherel measure,

$$\{\lambda_i - i - \alpha\sqrt{n}\}_{i\geq 1} \rightarrow Sin_\alpha.$$

Sin_α D.P.P with kernel

$$K_\alpha(x,y) = \begin{cases} \frac{sin(arccos(\frac{\alpha}{2})(x-y))}{\pi(x-y)} & if\, x \neq y \\ \frac{arcos(\frac{\alpha}{2})}{\pi} & if\, x = y \end{cases}$$

	GUE + Wigner (with a good control on moments)	Uniform permutation
First particle	T.W	T.W
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Conclusion

	GUE + Wigner (with a good control on moments)	Plancherel	Random permutations invariant under conjugation (with a good control on cycles' number)
First particle	T.W	T.W	T.W
Edge	Soft edge (Airy)	Soft edge (Airy)	Soft edge (Airy)
Global convergence	Semi circular	VKLS	VKLS
Fluctuations	Gaussian	Gaussian	??
Bulk	Sine process	Discrete sine process	??

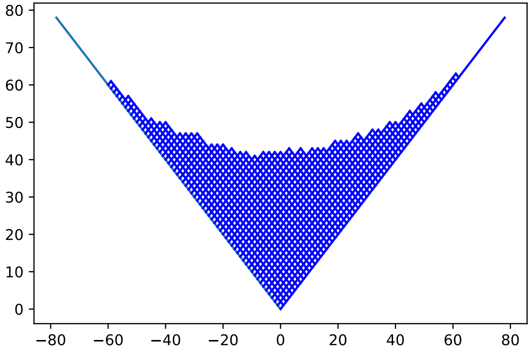
- Independance ~ invarience under conjugation ?
- Moments ~ cycles' structure ?

Conjectures

- We need only $\frac{\#(\sigma_n)}{n^{\frac{1}{2}}} \xrightarrow{\mathbb{P}} 0$ to obtain Tracy-Widom fluctuations.
- For any sequence of random permutations invariant under conjugation, for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{2\sqrt{n}} L_{\lambda(\sigma_n)}\left(2s\sqrt{n - fix(\sigma_n)}\right) - \sqrt{1 - \frac{fix(\sigma_n)}{n}}\Omega(s) \right| < \varepsilon \right) = 1.$$

- Under a good control on cycles we have discrete sine process (Bulk).



Thank you for your attention

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