KU LEUVEN

The local universality of Muttalib-Borodin ensembles

L.D. Molag

KU Leuven

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- The Muttalib-Borodin ensemble with parameter $\theta > 0$ and weight function w is the j. p. d. f. on $[0, \infty)$

$$\frac{1}{Z_n} \prod_{j<k} (x_k - x_j)(x_k^\theta - x_j^\theta) \prod_{j=1}^{n} w(x_j).$$

- We will consider an n-dependent weight function

$$w(x) = x^\alpha e^{-nV(x)}$$

with $\alpha > -1$ and an external field V that has enough increase at infinity.

- It was introduced by Muttalib in 1994 to better model disordered conductors in the metallic regime.

- It is an example of a determinantal p. p., i.e., we have a correlation kernel $K_{V,n}^{\alpha,\theta}(x,y)$ such that

$$\frac{1}{Z_n}\prod_{j<k}(x_k-x_j)(x_k^\theta-x_j^\theta)\prod_{j=1}^{n}w(x_j)=\det\left(K_{V,n}^{\alpha,\theta}(x_i,x_j)\right)_{i,j=1}^{n}$$

In fact, it is a biorthogonal ensemble and from this one derives

$$K_{V,n}^{\alpha,\theta}(x,y)=w(y)\sum_{j=0}^{n-1}p_j(x)q_j(y^\theta)$$

where p_j, q_j are polynomials of degree j satisfying

$$\int_0^\infty p_j(x)q_k(x^\theta)w(x)dx=\delta_{j,k},\qquad j,k=0,1,\dots,n-1.$$

- Borodin in 1999 computed the hard edge scaling limit for the Laguerre case, namely if $V(x) = x$, then

$$\lim_{n\to\infty} \frac{1}{n^{1+1/\theta}} K_{V,n}^{\alpha,\theta}\left(\frac{x}{n^{1+1/\theta}}, \frac{y}{n^{1+1/\theta}}\right) = \mathbb{K}^{(\alpha,\theta)}(x,y)$$

with limiting kernel

$$\mathbb{K}^{(\alpha,\theta)}(x,y) = \theta y^\alpha \int_0^1 J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(ux) J_{\alpha+1,\theta}\left((uy)^\theta\right) u^\alpha du$$

where

$$J_{a,b}(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!\Gamma(a+bj)}.$$

- He discovered a new (universality) kernel (like sine, Bessel, etc.).

Recently there is renewed interest for the Muttalib-Borodin ensemble.

- There are actual RM models where the density of eigenvalues is described by the MB ensemble.
  Cheliotis (2014) *Triangular random matrices and biorthogonal ensembles*
  Forrester, Wang (2017) *Muttalib-Borodin ensembles in random matrix theory - realisations and correlation functions*
- The scaling limit that Borodin found (related to the *Meijer G-kernel*) turns op in several places.
  Bertola, Gekhtman and Szmigielski (2009) *Cauchy-Laguerre two-matrix model and the Meijer-G random point field*
  Akemann, Ipsen, Kieburg (2013) *Products of Rectangular Random Matrices: Singular Values and Progressive Scattering*
  Kuijlaars, Stivigny (2014) *Singular values of products of random matrices and polynomial ensembles*

- Our goal was to generalize Borodin's result for the MB ensemble to general external fields V.
- I prove that, for general external fields and $\theta = 1/r$, with r a positive integer, we get

$$\lim_{n\to\infty} \frac{1}{n^{1+1/\theta}} K_{V,n}^{\alpha,\theta}\left(\frac{x}{n^{1+1/\theta}}, \frac{y}{n^{1+1/\theta}}\right) = \mathbb{K}^{(\alpha,\theta)}(x,y)$$

- General means that V is analytic around $[0,\infty)$, there is also some generic condition on V, having to do with the corresponding equilibrium measure.
- The $\theta = \frac{1}{2}$ case is published together with Arno Kuijlaars (2018).

- To prove the result for $\theta = 1/r$ we identify the MB ensemble with a multiple orthogonal polynomial ensemble. p_n is the unique monic polynomial of degree n that satisfies

$$\int_0^\infty p_n(x) x^k w_i(x) dx = 0, \qquad i = 1, \ldots, r, \quad k = 0, \ldots, \lfloor \tfrac{n-i}{r} \rfloor,$$

where $w_i(x) = x^{(i-1)\theta} w(x)$ for $i = 1, \ldots, r$.

- Such MOP ensembles can be related to $(r+1) \times (r+1)$ RH problems.

RH-Y1 $Y : \mathbb{C} \setminus [0,\infty) \to \mathbb{C}^{(r+1)\times(r+1)}$ is analytic.

RH-Y2 Y has boundary values for $x \in (0,\infty)$, that satisfy

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) & \dots & w_r(x) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

RH-Y3 As $|z| \to \infty$

$$Y(z) = \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right)\right) \operatorname{diag}\left(z^n, z^{-\frac{n}{r}}, \dots, z^{-\frac{n}{r}}\right).$$

- We use the Deift-Zhou method of steepest descent

$$Y \to X \to T \to S \to R.$$

- A vector equilibrium problem was needed (Kuijlaars 2016).
- For the global parametrix an $r+1$ sheeted Riemann-Surface was needed corresponding to the algebraic equation $(1-\xi)\xi^r z = c_V$.
- The local parametrix problem was new and needed to be solved using special functions, namely

$$G^{r+1,0}_{0,r+1}\left(\begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{r}, \ldots, -\alpha - \frac{r-1}{r} \end{matrix} \middle| z \right)$$

- Matching in higher dimensional RHPs.

- We want to find the same result for general $\theta > 0$.
- To describe the physical model that Muttalib treated one actually gets a better approximation when one substitutes

$$\frac{1}{Z_n}\prod_{i<j}(x_i - x_j)(x_i^\theta - x_j^\theta)\prod_j w(x_j)$$
$$\to \frac{1}{\tilde{Z}_n}\prod_{i<j}(x_i - x_j)(\log x_i - \log x_j)\prod_j w(x_j).$$

This essentially corresponds to the limit $\theta \to 0$. Since we know the asymptotic expressions for $\theta = 1/r$ we might be able to shine a light on this situation by letting $r = r_n \to \infty$ in a suitale way.