Edge Universality for non-Hermitian Random Matrices

joint work with László Erdős and Dominik Schröder

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- In this talk I report on universality at the edge of non-Hermitian ensembles.

Model Assumptions

Let X be an $N \times N$ non-Hermitian matrix with entries i.i.d. real or complex centred random variables with variance N^{-1}, i.e.

$$x_{ab} \stackrel{d}{=} N^{-1/2}\chi, \qquad \forall a, b,$$

with $\mathbb{E}\chi = 0$, $\mathbb{E}|\chi|^2 = 1$, and $\mathbb{E}\Re\chi\Im\chi = 0$.

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- χ has bounded moments.
- Mild regularity assumption on the density of χ.

The Circular Law

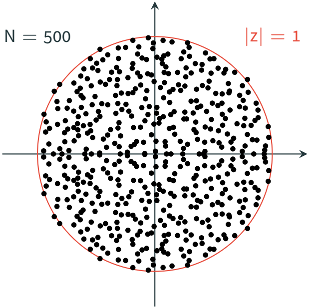

Figure 1: Real entries

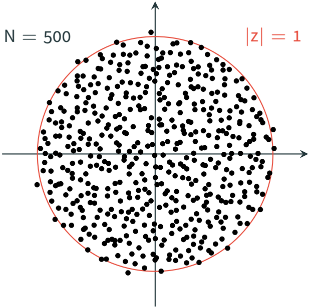

Figure 2: Complex entries with $\mathbf{E}\, x_{ab}^2 = 0$

- Circular law: Convergence of the density of states to the uniform distribution on the unit disk.
- Eigenvalues spacing $\sim N^{-1/2}$.
- Accumulation of $\sim \sqrt{N}$ eigenvalues on the real axis for real matrices

Circular Edge Universality

Theorem (C.-Erdős-Schröder, '19) For any $z_1, \ldots, z_k$ with $|1 - |z_j|^2| \lesssim N^{-1/2}$, and for any test function $F \colon \mathbb{C}^k \to \mathbb{C}$, we have

$$\int_{\mathbb{C}^k} F(\mathbf{w}) \left[p_k^{(N)} \left(\mathbf{z} + \frac{\mathbf{w}}{\sqrt{N}} \right) - p_{\mathbf{z}}^{(\infty, \mathrm{Gin}(\mathbb{F}))}(\mathbf{w}) \right] d\mathbf{w} = \mathcal{O}\left(N^{-c(k)} \right),$$

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- $p_{\mathbf{z}}^{(\infty, \mathrm{Gin}(\mathbb{F}))}$, the correlation function of the Gaussian case (Ginibre ensemble), is explicitly known [**Ginibre, '65; Metha, '67**] for the complex case and [**Edelman, '93; Akemann-Konziepier, '05; Forrester-Nagao, '07; Borodin-Sinclair, '08**] for the real case.

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- Previous universality result at the edge under the condition that the first four moments of the entries of X agree with the Gaussian moments [**Tao-Vu, ’12**].

Sketch of the proof

- Fix $|1-|z_0|^2| \lesssim N^{-1/2}$. We introduce the Ornstein-Uhlenbeck flow
$$dX_t = -\frac{1}{2}X_t\,dt + \frac{dB_t}{\sqrt{N}}, \quad X_0 = X,\ X_\infty \sim \mathrm{Gin}(N), \quad H_t^z := \begin{pmatrix} 0 & X_t - z \\ (X_t - z)^* & 0 \end{pmatrix},$$
and, denoting $G_t^z(i\eta) := (H_t^z - i\eta)^{-1}$, split Girko's hermitization formula as
$$\sum_{i=1}^{N} f(\sqrt{N}(\sigma_i - z_0)) = -\frac{1}{4\pi}\int_{\mathbb{C}} \Delta f(\sqrt{N}(z-z_0)) \left(\int_0^{\eta_0} + \int_{\eta_0}^{+\infty}\right) \Im \mathrm{Tr} G_t^z(i\eta)\, d\eta\, dz.$$

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- For $\eta_0 \geq N^{-3/4-\delta}$ we can control the flow of $\int_{\eta_0}^{\infty}$ for a long time using optimal cusp local law for the linearized matrix H_t^z **[Alt-Erdős-Krüger, '19]**:
$$|\langle \mathbf{x}, (G^z(i\eta) - M^z(i\eta))\mathbf{y}\rangle| \lesssim \frac{1}{N^{1/2}\eta^{1/3}} + \frac{1}{N\eta},$$
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- For $\eta_0 \leq N^{-1-\delta}$ we can use the classical smoothing result **[Sankar-Spielman-Teng, '05]** to exclude small eigenvalues λ_i^z of H^z:
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- **Remark:** If four moments matching is assumed [**Tao-Vu, '12**], the regime $\eta \in [N^{-1-\delta}, N^{-3/4-\delta}]$ can be covered by local law. $\Longrightarrow$ No gap.

Sketch of the proof

Without four moment assumption, to cover the regime $\eta \in [N^{-1-\delta}, N^{-3/4-\delta}]$, we need a completely new proof:

Theorem (C.-Erdős-Schröder, '19) Let $|1-|z|^2| \lesssim N^{-1/2}$. For the tail of the smallest eigenvalue of H^z, we have

$$\mathbb{P}\left(\min_i |\lambda_i^z| \leq N^{-3/4-\delta}\right) \leq N^{-c\delta},$$

where λ_i^z are the eigenvalues of H^z.

Estimate of the lower tail of $\min_i |\lambda_i^z|$.

- We use a supersymmetric representation

$$\mathrm{Tr}(H-w)^{-1} = i\int_{\mathbb{C}^N} \partial_\chi \langle \chi, \chi\rangle\, e^{-i\mathrm{Tr}(H-w)\Phi\Phi^*}\, ds, \quad \Phi := (s,\chi), \quad \Phi\Phi^* = ss^* + \chi\chi^*$$

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- Performing the expectation,

$$\mathbb{E}e^{-i\mathrm{Tr}(X-z)(X^*-\bar{z})\Phi\Phi^*} = \mathrm{sdet}\left(1 + i\frac{\Phi^*\Phi}{N}\right)^{-N} \exp\left(-N|z|^2 \, \mathrm{sTr}\left(1 + \frac{i}{N}\Phi^*\Phi\right)^{-1} \frac{i}{N}\Phi^*\Phi\right),$$

$$\mathrm{sdet}\begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = \frac{x^2}{xy - \tau\sigma}, \quad \mathrm{sTr}\begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = x - y.$$

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$$\int f(\Phi^*\Phi) = \frac{1}{2\pi i}\int_0^\infty dx\oint dy\,\partial_\tau\partial_\sigma\,\mathrm{sdet}^N(Q)f(Q), \quad Q := \begin{pmatrix} x & \sigma \\ \tau & y\end{pmatrix},$$

  to reduce the number of integrations to 2:

$$\mathbb{E}\mathrm{Tr}((X-z)(X-z)^*-w)^{-1} = \frac{N^2}{2\pi i}\int_0^\infty dx\oint dy\, e^{-Nf(x)+Nf(y)}y\cdot G(x,y)$$

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- The real case is more complicated, but also gives explicit formula as a 3-dim. integral

Thank you very much for your attention!