Eigenvalue Densities of Classical Matrix Ensembles

Anas A. Rahman
with P. J. Forrester

The School of Mathematics and Statistics,
The University of Melbourne

August, 2019

Today: Consider matrices X with random entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ such that $X = X^{\dagger}$ (real eigenvalues).

Random Matrix Ensembles

Today: Consider matrices X with random entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ such that $X = X^\dagger$ (real eigenvalues).

- The *Dyson index* $\beta = \#$(real components)$\in \{1, 2, 4\}$.

Random Matrix Ensembles

Today: Consider matrices X with random entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ such that $X = X^\dagger$ (real eigenvalues).

- The *Dyson index* $\beta = \#(\text{real components}) \in \{1, 2, 4\}$.
- Diagonalised by orthogonal, unitary, symplectic matrices, respectively.

Random Matrix Ensembles

Today: Consider matrices X with random entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ such that $X = X^\dagger$ (real eigenvalues).

- The *Dyson index* $\beta = \#$(real components)$\in \{1, 2, 4\}$.
- Diagonalised by orthogonal, unitary, symplectic matrices, respectively.
- The distribution of X is invariant under unitary conjugation $X \mapsto U^\dagger X U$.

Random Matrix Ensembles

Today: Consider matrices X with random entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ such that $X = X^\dagger$ (real eigenvalues).

- The *Dyson index* $\beta = \#(\text{real components}) \in \{1, 2, 4\}$.
- Diagonalised by orthogonal, unitary, symplectic matrices, respectively.
- The distribution of X is invariant under unitary conjugation $X \mapsto U^\dagger X U$.

Famous example: The Gaussian β Ensemble is defined by the PDF

$$P(X) = \frac{1}{C} \exp\left[-\tfrac{\beta}{2}\mathrm{Tr}(X^2)\right].$$

GβE Eigenvalue PDF

Diagonalising $X = U\Lambda U^\dagger$,

$$P(X) = \frac{1}{C}\prod_{i=1}^{N}\exp\left[-\frac{\beta}{2}\lambda_i^2\right]\ |\Delta(\Lambda)|^\beta\ f(U),$$

GβE Eigenvalue PDF

Diagonalising $X = U\Lambda U^{\dagger}$,

$$P(X) = \frac{1}{C}\prod_{i=1}^{N} \exp\left[-\tfrac{\beta}{2}\lambda_i^2\right] \; |\Delta(\Lambda)|^{\beta} \, f(U), \quad \Delta(\Lambda) = \prod_{1\le j<k\le N} (\lambda_j - \lambda_k)$$

(Vandermonde)

GβE Eigenvalue PDF

Diagonalising $X = U\Lambda U^\dagger$,

$$P(X) = \frac{1}{C}\prod_{i=1}^{N}\exp\left[-\tfrac{\beta}{2}\lambda_i^2\right] \; |\Delta(\Lambda)|^\beta \, f(U), \quad \Delta(\Lambda) = \prod_{1\le j<k\le N}(\lambda_j - \lambda_k)$$

(Vandermonde)

Thus, the *eigenvalue PDF* is

$$p(\lambda_1, \ldots, \lambda_N) = \frac{1}{C'}\prod_{i=1}^{N}\exp\left[-\tfrac{\beta}{2}\lambda_i^2\right] \; |\Delta(\Lambda)|^\beta$$

Some Ensemble Types

The GβE falls into a few categories:

Some Ensemble Types

The GβE falls into a few categories:

- **Invariant Ensembles:** $P(X) = P(UXU^\dagger)$ for $U \in O(N), U(N), Sp(N)$. This invariance of measure allows eigenvalue-eigenvector decoupling:

$$P(X) = g(\Lambda)|\Delta(\Lambda)|^\beta f(U), \quad X = U\Lambda U^\dagger.$$

The GβE falls into a few categories:

- **Invariant Ensembles:** $P(X) = P(UXU^{\dagger})$ for $U \in O(N), U(N), Sp(N)$. This invariance of measure allows eigenvalue-eigenvector decoupling:

$$P(X) = g(\Lambda)|\Delta(\Lambda)|^{\beta} f(U), \quad X = U\Lambda U^{\dagger}.$$

- **Orthogonal-Polynomial Ensembles:** $g(\Lambda) = \prod_{i=1}^{N} w(\lambda_i)$.

Some Ensemble Types

The GβE falls into a few categories:

- **Invariant Ensembles:** $P(X) = P(UXU^\dagger)$ for $U \in O(N), U(N), Sp(N)$. This invariance of measure allows eigenvalue-eigenvector decoupling:

$$P(X) = g(\Lambda)|\Delta(\Lambda)|^\beta f(U), \quad X = U\Lambda U^\dagger.$$

- **Orthogonal-Polynomial Ensembles:** $g(\Lambda) = \prod_{i=1}^N w(\lambda_i)$.
- **Classical Ensembles:** w is such that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(w(x)) = \frac{w'(x)}{w(x)} = -\frac{g(x)}{f(x)}$$

with $\deg(g) \leq 1$ and $\deg(f) \leq 2$.

Up to change of variables, there are exactly four ensembles satisfying this last criterion:

$$w(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \chi_{x>0}, & \text{Laguerre} \\ x^a (1-x)^b \chi_{0<x<1}, & \text{Jacobi} \\ 1/\left((1+\mathrm{i}x)^a (1-\mathrm{i}x)^{\bar{a}}\right), & \text{gCauchy} \end{cases},$$

where $\chi_A = 1$ if A is true, and 0 otherwise.

Up to change of variables, there are exactly four ensembles satisfying this last criterion:

$$w(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \chi_{x>0}, & \text{Laguerre} \\ x^a (1-x)^b \chi_{0<x<1}, & \text{Jacobi} \\ 1/\left((1+\mathrm{i}x)^a (1-\mathrm{i}x)^{\bar{a}}\right), & \text{gCauchy} \end{cases},$$

where $\chi_A = 1$ if A is true, and 0 otherwise.

Focus on Gaussian, Laguerre and Jacobi.

Definition (Eigenvalue Density)

For a matrix ensemble Z with N eigenvalues residing in I, the eigenvalue density is

$$\rho^{(Z)}(\lambda) := N \int_{I^{N-1}} p^{(Z)}(\lambda, \lambda_2, \ldots, \lambda_N) \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_N$$

Eigenvalue Densities

Definition (Eigenvalue Density)

For a matrix ensemble Z with N eigenvalues residing in I, the eigenvalue density is

$$\rho^{(Z)}(\lambda) := N \int_{I^{N-1}} p^{(Z)}(\lambda, \lambda_2, \ldots, \lambda_N) \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_N$$

Definition (Global Scaling)

$$\tilde{\rho}^{(G)}(\lambda) := \frac{1}{\sqrt{N}} \rho^{(G)}(\sqrt{N}\lambda)$$
$$\tilde{\rho}^{(L)}(\lambda) := \rho^{(L)}(N\lambda)$$
$$\tilde{\rho}^{(J)}(\lambda) := \frac{1}{N} \rho^{(J)}(\lambda)$$

Eigenvalue Densities

Definition (Eigenvalue Density)

For a matrix ensemble Z with N eigenvalues residing in I, the eigenvalue density is

$$\rho^{(Z)}(\lambda) := N \int_{I^{N-1}} p^{(Z)}(\lambda, \lambda_2, \ldots, \lambda_N) \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_N$$

Definition (Global Scaling)

$$\tilde{\rho}^{(G)}(\lambda) := \tfrac{1}{\sqrt{N}} \rho^{(G)}(\sqrt{N}\lambda)$$
$$\tilde{\rho}^{(L)}(\lambda) := \rho^{(L)}(N\lambda)$$
$$\tilde{\rho}^{(J)}(\lambda) := \tfrac{1}{N} \rho^{(J)}(\lambda)$$

$\lim_{N\to\infty} \tilde{\rho}^{(Z)}$ has compact domain and integrates to 1.

Eigenvalue Densities

As $N \to \infty$, these densities tend to the Wigner semicircle law, the Marchenko-Pastur law, and a form first obtained by Wachter, respectively.

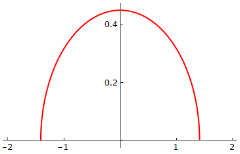


Figure: GUE limit

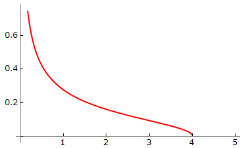


Figure: LUE limit

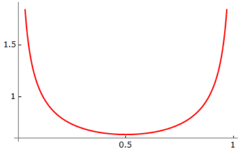


Figure: JUE limit

Eigenvalue Densities

We present the densities for finite N:

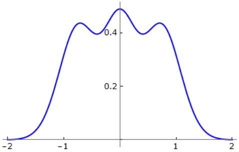


Figure: GUE ($N = 3$)

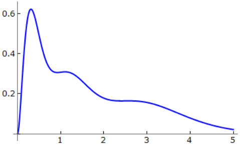


Figure: LUE ($N = 3$)

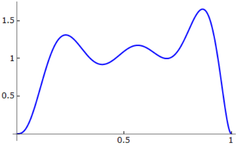
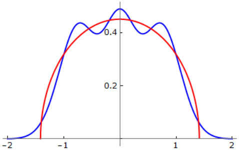
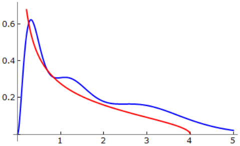
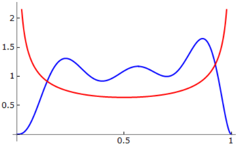


Figure: JUE ($N = 3$)







Eigenvalue Densities

We present the densities for finite N:

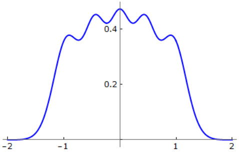


Figure: GUE ($N = 5$)

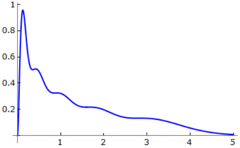


Figure: LUE ($N = 5$)

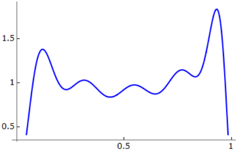
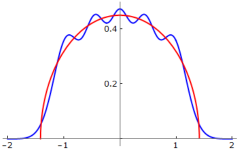
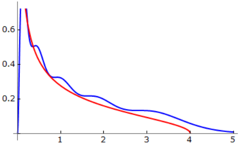
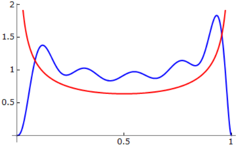


Figure: JUE ($N = 5$)







Eigenvalue Densities

We present the densities for finite N:

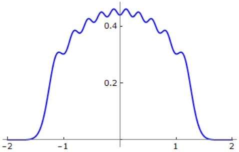


Figure: GUE ($N = 10$)

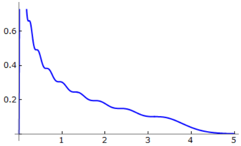


Figure: LUE ($N = 10$)

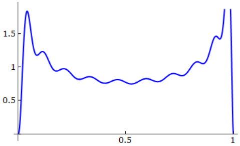
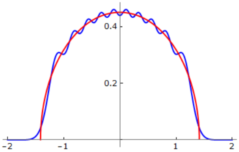
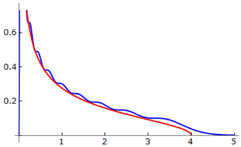
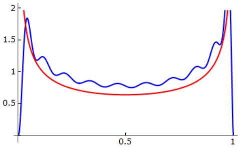


Figure: JUE ($N = 10$)







The Gaussian-Laguerre-Jacobi Trio

It is natural to study these three classical ensembles together:

$$\begin{array}{ccc} & \text{Jacobi} & \\ \swarrow & & \searrow \\ \text{Gaussian} & & \text{Laguerre} \end{array}$$

It is natural to study these three classical ensembles together:

$$\begin{array}{ccc} & \text{Jacobi} & \\ \swarrow & & \searrow \\ \text{Gaussian} & & \text{Laguerre} \end{array}$$

- Letting $x = y/b$,

$$b^a w^{(J)}(x) = b^a x^a (1-x)^b = y^a (1 - y/b)^b \xrightarrow{b\to\infty} y^a e^{-y} = w^{(L)}(y)$$

The Gaussian-Laguerre-Jacobi Trio

It is natural to study these three classical ensembles together:

$$\begin{array}{ccc} & \text{Jacobi} & \\ \swarrow & & \searrow \\ \text{Gaussian} & & \text{Laguerre} \end{array}$$

- Letting $x = y/b$,

$$b^a w^{(J)}(x) = b^a x^a (1-x)^b = y^a (1 - y/b)^b \overset{b\to\infty}{\longrightarrow} y^a e^{-y} = w^{(L)}(y)$$

- Letting $x = \frac{1}{2}(1 + y/\sqrt{L})$ and $a = b = L$, we see

$$4^{2L} w^{(J)}(x) = (1 - y^2/L)^L \overset{L\to\infty}{\longrightarrow} e^{-y^2} = w^{(G)}(y).$$

Differential Equations

Goal: Analogues of Götze & Tikhomirov's ('05) result for the LUE eigenvalue density:

$$0 = x^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3}\rho^{(L)}(x) + 4x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho^{(L)}(x) + \left[(a+2N)x - a^2\right]\rho^{(L)}(x) - \left[x^2 - 2(a+2N)x + a^2 - 2\right] x \frac{\mathrm{d}}{\mathrm{d}x}\rho^{(L)}(x)$$

Goal: Analogues of Götze & Tikhomirov's ('05) result for the LUE eigenvalue density:

$$0 = x^3 \tfrac{\mathrm{d}^3}{\mathrm{d}x^3}\rho^{(L)}(x) + 4x^2 \tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\rho^{(L)}(x) + \left[(a+2N)x - a^2\right]\rho^{(L)}(x) \\ - \left[x^2 - 2(a+2N)x + a^2 - 2\right] x \tfrac{\mathrm{d}}{\mathrm{d}x}\rho^{(L)}(x)$$

From Forrester's ('93) work on $1/r^2$ quantum many body systems, we have DEs for $\rho^{(J)}(x)$ with $\beta \in \mathbb{N}$:

$$0 = x^{N+1}(1-x)^{N+1}\frac{\mathrm{d}^{N+1}}{\mathrm{d}x^{N+1}}\rho^{(J)}(x) + \cdots$$

Goal: Analogues of Götze & Tikhomirov's ('05) result for the LUE eigenvalue density:

$$0 = x^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3} \rho^{(L)}(x) + 4x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \rho^{(L)}(x) + \left[(a+2N)x - a^2\right] \rho^{(L)}(x) - \left[x^2 - 2(a+2N)x + a^2 - 2\right] x \frac{\mathrm{d}}{\mathrm{d}x} \rho^{(L)}(x)$$

From Forrester's ('93) work on $1/r^2$ quantum many body systems, we have DEs for $\rho^{(J)}(x)$ with $\beta \in \mathbb{N}$:

$$0 = x^{N+1}(1-x)^{N+1} \frac{\mathrm{d}^{N+1}}{\mathrm{d}x^{N+1}} \rho^{(J)}(x) + \cdots$$

DEs are of order $N+1$.

Dualities

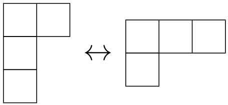
- A $\beta \leftrightarrow N$ duality gives DEs of order $\beta + 1$, for $\beta \in 2\mathbb{N}$, (need $|\Delta|^{\beta} = (\Delta)^{\beta}$),

$$\left\langle \prod_{i=1}^{N} (\lambda_i - \lambda)^{n} \right\rangle_{\beta,N}^{(a,b)} \propto \left\langle \prod_{i=1}^{n} (1 - \lambda_i \lambda)^{N} \right\rangle_{4/\beta,n}^{(a',b')} .$$

Dualities

- A $\beta \leftrightarrow N$ duality gives DEs of order $\beta + 1$, for $\beta \in 2\mathbb{N}$, (need $|\Delta|^\beta = (\Delta)^\beta$),

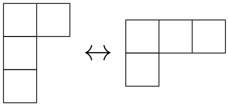
$$\left\langle \prod_{i=1}^{N} (\lambda_i - \lambda)^{n} \right\rangle_{\beta,N}^{(a,b)} \propto \left\langle \prod_{i=1}^{n} (1 - \lambda_i \lambda)^{N} \right\rangle_{4/\beta,n}^{(a',b')}.$$



Dualities

- A $\beta \leftrightarrow N$ duality gives DEs of order $\beta + 1$, for $\beta \in 2\mathbb{N}$, (need $|\Delta|^{\beta} = (\Delta)^{\beta}$),

$$\left\langle \prod_{i=1}^{N} (\lambda_i - \lambda)^{n} \right\rangle_{\beta, N}^{(a,b)} \propto \left\langle \prod_{i=1}^{n} (1 - \lambda_i \lambda)^{N} \right\rangle_{4/\beta, n}^{(a',b')}.$$



- A $\beta \leftrightarrow 4/\beta$ duality for $\beta \in \mathbb{R}_+ \implies \mathbb{H} \leftrightarrow \mathbb{R}$,

$$m_k^{(J)}(\beta, N, a, b) := \left\langle \sum_{i=1}^{N} \lambda_i^k \right\rangle_{\beta, N}^{(a,b)}$$

$$= (-2/\beta)^k \, m_k^{(J)}(4/\beta, -\beta N/2, -2a/\beta, -2b/\beta)$$

For example, for the LOE and LSE,

$$\begin{aligned}\mathcal{D}_{\beta,N}^{(L)} = &\boxed{4x^5\frac{\mathrm{d}^5}{\mathrm{d}x^5} + 40x^4\frac{\mathrm{d}^4}{\mathrm{d}x^4}} - \left[5\left(\tfrac{x}{\kappa-1}\right)^2 - 10\left(a_\beta + 4N_\beta\right)\tfrac{x}{\kappa-1} + 5\tilde{a} - 88\right]x^3\frac{\mathrm{d}^3}{\mathrm{d}x^3} \\ &- \left[16\left(\tfrac{x}{\kappa-1}\right)^2 - 38\left(a_\beta + 4N_\beta\right)\tfrac{x}{\kappa-1} + 22\tilde{a} - 16\right]x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ &+ \tfrac{1}{(\kappa-1)^2}\left[\left(\tfrac{x}{\kappa-1}\right)^2 - 4\left(a_\beta + 4N_\beta\right)\left(\tfrac{x}{\kappa-1}\right) + 2\left(2\left(a_\beta + 4N_\beta\right)^2 + \tilde{a} - 2\right)\right]x^3\frac{\mathrm{d}}{\mathrm{d}x} \\ &- \left[4(\tilde{a}-3)\left(a_\beta + 4N_\beta\right)\tfrac{x}{\kappa-1} - \tilde{a}^2 + 14\tilde{a} + 16\right]x\frac{\mathrm{d}}{\mathrm{d}x} - \left(a_\beta + 4N_\beta\right)\left(\tfrac{x}{\kappa-1}\right)^3 \\ &+ \left(2\left(a_\beta + 4N_\beta\right)^2 + \tilde{a}\right)\left(\tfrac{x}{\kappa-1}\right)^2 - (3\tilde{a}+4)\left(a_\beta + 4N_\beta\right)\tfrac{x}{\kappa-1} + \tilde{a}^2\end{aligned}$$

Then, for $\beta = 1$ *and* 4*,*

$$\mathcal{D}_{\beta,N}^{(L)}\rho_{(1),\beta,N}^{(L)}(x) = 0$$

What Has Now Been Done

There are order $\max\{\beta, 4/\beta\} + 1$ order DEs for the eigenvalue densities of the following matrix ensembles:

β / Ensemble	2/3	1	2	4	6
Gaussian	new	W-F	H-T	W-F	new
Laguerre		new	G-T	new	
Jacobi		new	new	new	

We can now generalise certain formalisms applied to the GUE and LUE in the past.

Applications

We can now generalise certain formalisms applied to the GUE and LUE in the past.

- Systematic way to derive generalisations of the Harer-Zagier moment expansions

Applications

We can now generalise certain formalisms applied to the GUE and LUE in the past.

- Systematic way to derive generalisations of the Harer-Zagier moment expansions
- Systematic way to generate $1/N$ expansions for the resolvents $W_1(x) = \sum_{k=0}^{\infty} m_k/x^{k+1}$

Applications

We can now generalise certain formalisms applied to the GUE and LUE in the past.

- Systematic way to derive generalisations of the Harer-Zagier moment expansions
- Systematic way to generate $1/N$ expansions for the resolvents $W_1(x) = \sum_{k=0}^{\infty} m_k / x^{k+1}$
- Provides DE characterisation of the densities at the hard and soft edges

Applications

We can now generalise certain formalisms applied to the GUE and LUE in the past.

- Systematic way to derive generalisations of the Harer-Zagier moment expansions
- Systematic way to generate $1/N$ expansions for the resolvents $W_1(x) = \sum_{k=0}^{\infty} m_k / x^{k+1}$
- Provides DE characterisation of the densities at the hard and soft edges
- Götze and Tikhomirov used DEs to study the rate of convergence to the Wigner semicircle law

Thanks for your time.

References contained in:
http://bit.ly/ZiFLinearDEs

To appear: A.A. Rahman and P.J. Forrester, "*Linear Differential Equations for the Resolvents of the Classical Matrix Models*"