Painlevé II τ-function as a Fredholm determinant

Harini Desiraju
joint work with Marco Bertola, Tamara Grava

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Painlevé equations

- Painlevé equations are certain second order non-linear equations of the form
$$u'' = R(u', u, t) \tag{1}$$
where R is a rational function, and cannot be solved in terms of known special functions.
- The Painlevé transcendents can be described as Nonlinear special functions.

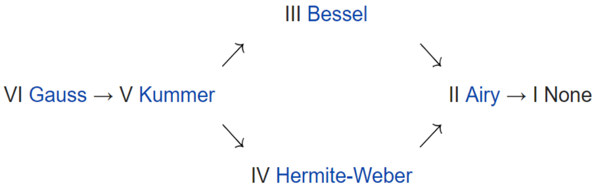


Figure: Painlevé transcendents: Wikipedia

Malgrange form and τ-function

Malgrange form: For a Riemann Hilbert problem on a contour Σ, depending on a parameter t,

$$\phi_+(z,t) = \phi_-(z,t)M(z,t); \quad \phi(\infty) = \mathbb{1} \tag{2}$$

Malgrange one-form is defined as

$$\omega_{\mathcal{M}} = \int_{\Sigma} \frac{dz}{2\pi i} \mathrm{Tr}\left[\phi^{-1}\phi' \dot{M} M^{-1}\right] \tag{3}$$

where $^{\cdot} \equiv \frac{\partial}{\partial t}$, $' \equiv \frac{\partial}{\partial z}$

τ-function:

$$\omega_{\mathcal{M}}(t) = d\log\tau(t) \tag{4}$$

For a Riemann Hilbert problem corresponding to an isomondromic problem, the τ-function is related to the solution of isomonodromic equation

$$u^2(t) \approx \frac{\partial^2}{\partial t^2}\log\tau[t]. \tag{5}$$

Zeros of the τ-function are the points where the Riemann Hilbert problem is not solvable.

A brief history

- Its, Izergin, Korepin, Slavnov'90 : Correlation function of Bose gas solves certain differential equation and the corresponding τ-function is Fredholm determinant of an integrable Kernel.
- Tracy, Widom '93 : Fredholm determinants of integrable kernels solve certain PDEs.
- Cafasso '08: The SSW τ-function Fredholm determinant of a particular combination of Toeplitz operators called the Widom constant.
- Cafasso, Lisovyy, Gavrylenko '17: The isomonodromic τ-function of certain Painlevé equations (VI, V, III) assume the form of Widom constant.

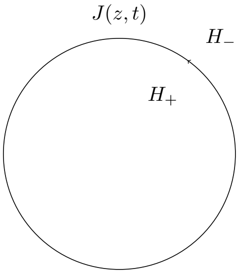
Widom constant

Consider a Riemann Hilbert problem defined on a unit circle.

$$\phi_+(z,t) = \phi_-(z,t)M(z,t); \quad \phi(\infty) = \mathbb{1} \tag{6}$$

$M(z,t)$ can be factorized in two different ways

$$M(z,t) = \phi_-^{-1}\phi_+ = \psi_+^{-1}\psi_- \tag{7}$$



- $L^2(S^1) = H_+ \oplus H_-$
- Define projection (Cauchy) operators $\Pi_\pm : L^2(S^1) \to H_\pm$
- Toeplitz operator is defined as $T_M = \Pi_+ M$

- Widom constant is defined as

$$\tau_W[M] = \det_{H_+}[T_M \circ T_{M^{-1}}] \tag{8}$$

- The zeros of T_J correspond to unsolvability of the RHP and the zeros of $T_{J^{-1}}$ correspond to the unsolvability of the dual RHP.
- Logarithmic derivatives of $\tau_W(t)$, Malgrange $\tau(t)$ coincide upto explicit terms

$$\partial_t \log \tau_W[t] = \partial_t \log \tau[t] + \text{elementary terms} \tag{9}$$

Can a generic τ-function of Painlevé II equation be expressed as a Fredholm determinant?

What is known?

$$\text{Painlevé II:} \quad u_{xx} = 2u^3 + xu \tag{10}$$

- Ablowitz-Segur family of solutions:

$$u(x) \approx \kappa Ai(x); \quad x \to +\infty, \quad \kappa \in \mathbb{C} \tag{11}$$

- Tracy, Widom '99: For the Ablowitz-Segur solutions,

$$u^2(x) = -\frac{\partial^2}{\partial x^2} \log \underbrace{\det\left[1 - \kappa^2 K_{Ai}|_{[x,\infty)}\right]}_{\tau(x)} \tag{12}$$

Relation to the Widom constant

⤙ The Ablowitz-Segur τ-function can be expressed as a Fredholm determinant of a combination of appropriate Toeplitz operators. Further, we can obtain a minor expansion of the Airy kernel. [arXiv:1906.11517v3]

RHP

$\Psi(\lambda)$ is piecewise holomorphic 2×2 matrix valued function such that

- $\Psi(\lambda)$ is holomorphic for $\lambda \in \mathbb{C} \; \cup \{\gamma_k\}$
- Boundary conditions on each Stokes' ray are

$$\Psi_+(\lambda) = \Psi_-(\lambda) S_k, \;\; \lambda \in \gamma_k \tag{13}$$

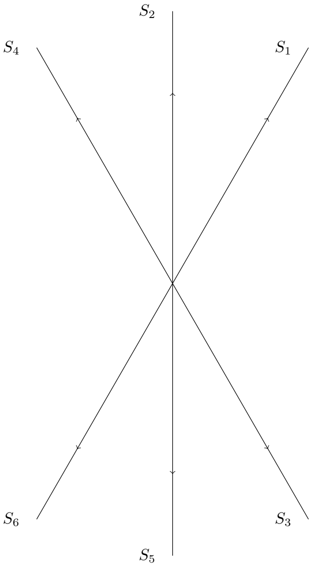
  Stokes' data satisfies the constraint
  $s_{k+3} = -s_k \;\; , \;\; s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$
- Asymptotic behaviour is specified by

$$\Psi(\lambda) e^{\theta(\lambda,x)\sigma_3} \to I \;\; ; \;\; \theta(\lambda, x) = i \left(\frac{4}{3}\lambda^3 + x\lambda \right), \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

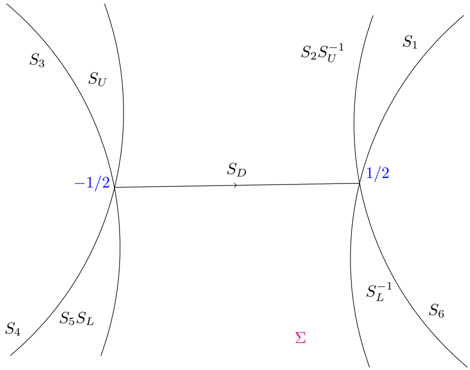
  Changing the coordinates $\lambda = (-x)^{1/2} z, \quad t = (-x)^{3/2}$ and defining the parameter $\nu = \frac{1}{2\pi} \log(1 - s_1 s_3)$

Disclaimer: All functions depend on z, t unless mentioned otherwise and all the solutions of RHPs are normalised at ∞.

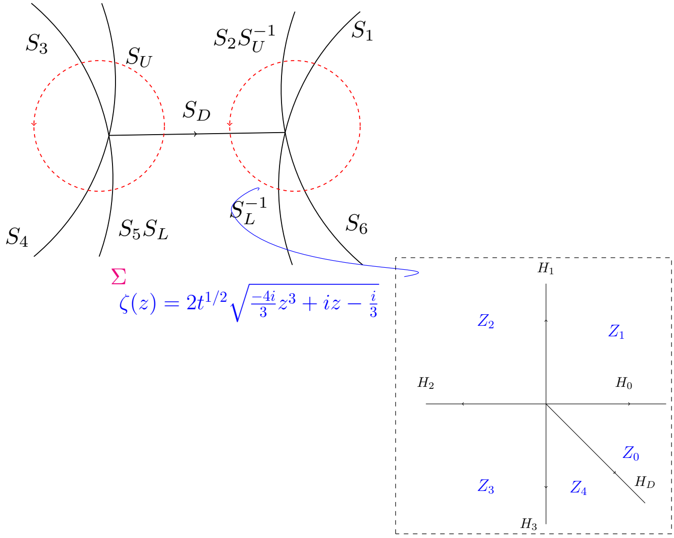
Painlevé II RHP contour



Painlevé II RHP contour

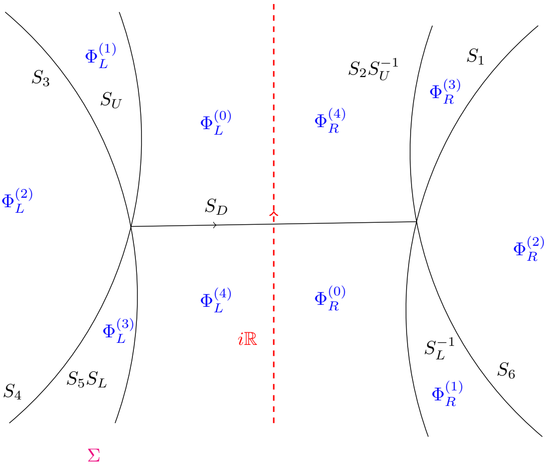


Painlevé II RHP contour



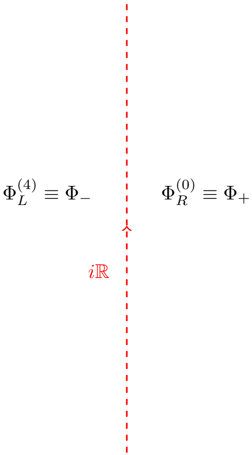
RHP of Parabolic Cylinder function

Painlevé II RHP contour



RHP of Painlevé II

Painlevé II RHP contour



Its-Izergin-Korepin-Slavnov (IIKS) kernel

Theorem

Given a RHP of the form

$$\Phi_+ = \Phi_- J \tag{15}$$

where the jump assumes the form $J = 1 - 2\pi i f(z) g^T(z)$ *with* $f^T(z)g(z) = 0$*; a Kernel*

$$K(z,w) = \frac{f^T(z)g(w)}{z-w} \tag{16}$$

can be constructed such that the RHP is solvable iff $(1-K)$ *is invertible.*

- The τ-function is then defined as

$$\tau[J] = \det[1-K] \tag{17}$$

The jump on $i\mathbb{R}$ is known

$$J = \Phi_-^{-1}\Phi_+ \tag{18}$$

Now the task reduces to constructing the appropriate integrable kernel $K(z,w)$.

Algorithm to construct Integrable kernels

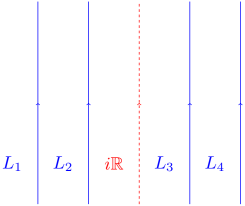
- Any $SL(2,\mathbb{C})$ can be decomposed into 'elementary' matrices. This is called the LULU decomposition.

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1+c-a}{a} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{b-1}{a} \\ 0 & 1 \end{bmatrix} \quad (19)$$

- This decomposition can be thought of defining a new RHP on a set of parallel contours

$$\Theta_+ = \Theta_- J_i; \quad \text{on} \quad L_i \quad (20)$$

with $J(z,t) = \prod_i J_i$



- $\chi_i(z)$ defines the characterstic function on L_i

Ref: Bertola, Marco. "The Malgrange form and Fredholm determinants." arXiv preprint arXiv:1703.00046 (2017).

- Using χ_i, the jump function can be brought to the desired form

$$J(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = 1 - 2\pi i f(z) g^T(z) \tag{21}$$

with

$$f(z) = \begin{bmatrix} \chi_2(z) + \frac{(b(z)-1)}{a(z)}\chi_4(z) \\ \frac{(1+c(z)-a(z))}{a(z)}\chi_1(z) + (a(z)-1)\chi_3(z) \end{bmatrix};$$

$$g(z) = \frac{1}{2\pi i}\begin{bmatrix} \chi_1(z) + \chi_3(z) \\ \chi_2(z) + \chi_4(z) \end{bmatrix}$$

- The kernel has the form

$$K(z,w) = \frac{f^T(z)g(w)}{z-w} \tag{22}$$

- The τ-function on the set of parallel lines:

$$\tau[J_i] = \det[1-K] \tag{23}$$

Painlevé II τ-function

$$\partial_t \log \tau_{PII} = \partial_t \log \det[1-K] - \int_{i\mathbb{R}} \frac{dz}{2\pi i} \left[(\partial_z a)c - a(\partial_z c)\right] \partial_t b$$
$$+ \int_{i\mathbb{R}} \frac{dz}{2\pi i} \operatorname{Tr}\left[\Phi_+' \Phi_+^{-1} \left(\dot{\Phi}_+ \Phi_+^{-1} - \dot{\Phi}_- \Phi_-^{-1}\right)\right] + \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t}\right] \tag{24}$$

where a, b, c, Φ_+, Φ_- are all functions of z, t and are given in terms of Parabolic Cylinder functions.

What's next?

- Understanding the pole distribution of Painlevé II transcendent.
- Expanding the Fredholm determinant in terms of minors and understanding it as some Partition function.
- Connection problem of Painlevé II.