Direct and inverse scattering for the Sturm-Liouville operator with unbounded potentials and initial value problem for the Korteweg-de Vries equation

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Korteweg-de Vries equation

$$u_t(x,t) + u(x,t)u_x(x,t) + \frac{1}{12}u_{xxx}(x,t) = 0, \quad x \in \mathbb{R}, t \geq 0.$$

Lax pair representation

$$\varphi_{xx} + 2u\varphi = \lambda\varphi, \qquad \text{(Sturm-Liouville equation (SL))}$$

$$\varphi_t = \frac{u_x}{6}\varphi - \frac{\lambda + u}{3}\varphi_x.$$

For the vector $\Phi = \begin{pmatrix} \varphi \\ \varphi_x \end{pmatrix}$, the Lax pair takes the form

$$\Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda - 2u(x,t) & 0 \end{pmatrix} \Phi, \quad \Phi_t = \begin{pmatrix} \frac{u_x}{6} & \frac{-\lambda - u}{3} \\ \frac{-\lambda^2}{3} + \frac{u\lambda}{3} + \frac{4u^2 + u_{xx}}{6} & \frac{-u_x}{6} \end{pmatrix} \Phi.$$

Properties of Sturm-Liouville equation for u : $\int_{\mathbb{R}} |u(x)|(1+|x|)\mathrm{d}x < \infty$

- spectrum is $(-\infty, 0] \cup \{\lambda_j\}_{j=1}^N$, discrete spectrum is finite and simple;
- there is a bijection to a set of *scattering data*, consisting of the *reflection coefficient* $r(\lambda)$, and pairs $(\lambda_j, \gamma_j)_{j=1}^N$;
- given an initial function $u_0(x)$ for the KdV equation, the solution can be obtained in the following way: find the set of *scattering data*, and then for every fixed x, solve the Marchenko integral equations (Fredholm) with respect to $y \geq x$,

$$K(x,y,t) + F(x+y,t) + \int_x^{+\infty} K(x,z,t)F(y+z,t)\mathrm{d}z = 0, \quad y \geq x,$$

where

$$F(s,t) = \frac{1}{2\pi}\int_{\mathbb{R}} r(k)\mathrm{e}^{-\mathrm{i}kx - \frac{2\mathrm{i}t}{3}k^3}\mathrm{d}k + \sum_j \gamma_j^2 \mathrm{e}^{x\sqrt{\lambda_j} - \frac{t}{3}\lambda_j^{3/2}}, \qquad k = \mathrm{i}\sqrt{\lambda},$$

Then $u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x,t)$ is the solution of the KdV with initial function $u_0(x)$.

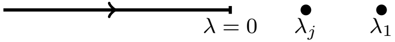
Alternatively, $u(x,t)$ can be obtained from the solution of the following RHP:

Riemann-Hilbert problem 1

Find 2×2 *matrix-valued* $\Phi(x,t;\lambda)$, *which*

- *analyticity: piece-wise meromorphic in* $\lambda\in\mathbb{C}\setminus(-\infty,0]$, *with simple poles at* λ_j, $j=1,\dots,N$;
- *jumps:* $\Phi_+ = \Phi_- \begin{pmatrix} -\mathrm{i}\,r(\lambda) & \dfrac{-\mathrm{i}}{\mathrm{i}\,r(\lambda)} \\ -\mathrm{i}(1-|r(\lambda)|^2) & \end{pmatrix}$, $\quad \lambda\in(-\infty,0)$;
- *poles at* λ_j : $Res_{\lambda_j}\Phi(x,t;\lambda) = \Phi(x,t;\lambda)\begin{pmatrix}0 & 0\\ \gamma_j & 0\end{pmatrix}$;
- *asymptotics as* $\lambda\to\infty$*:*
  $\Phi(x,t;\lambda) = \frac{1}{\sqrt{2}}\lambda^{-\sigma_3/4}(\sigma_3+\sigma_1)\left(I+\frac{\mathrm{diag}}{\sqrt{\lambda}}+\mathcal{O}(\lambda^{-1})\right)\mathrm{e}^{(x\lambda^{1/2}-\frac{t}{3}\lambda^{3/2})\sigma_3}$,

where $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $s^{\sigma_3}=\begin{pmatrix}s&0\\0&s^{-1}\end{pmatrix}$, $\quad \mathrm{e}^{s\sigma_3}=\begin{pmatrix}\mathrm{e}^{s}&0\\0&\mathrm{e}^{-s}\end{pmatrix}$.



Reconstruction of $u(x,t)$ from $\Phi(x,t;\lambda)$.

Then the asymptotics of $\Phi(x,t;\lambda)$ as $\lambda \to \infty$ has the structure

$$\Phi(x,t;\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -H_1 & 1 \end{pmatrix} \left(I + \frac{1}{\lambda} \begin{pmatrix} \frac{H_1^2+u}{2} & -H_1 \\ -\frac{H_0}{3} + \frac{H_1^3}{3} + H_1 u + \frac{u_x}{4} & \frac{H_1^2+u}{-2} \end{pmatrix} + \mathcal{O}(\lambda^{-2}) \right)$$
$$\cdot \lambda^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathrm{e}^{(x\lambda^{1/2} - \frac{t}{3}\lambda^{3/2})\sigma_3}.$$

where $(H_1(x,t))_x = u(x,t)$, and $u(x,t)$ is the solution of the KdV, which at the time $t=0$ has the *scattering data* consisting of the reflection coefficient $r(\lambda)$ and the set $\{\lambda_j, \gamma_j\}_{j=1}^N$.

Here

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad and \quad \mathrm{e}^{a\sigma_3} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Let now $u_0(x) \sim \sqrt[3]{-6x}$ as $x \to \pm\infty$.

Questions:

- What is the spectrum of $\mathrm{d}_x^2 + 2u_0(x)$?
- Is there a bijection to some *scattering data*, which lives in the complex plane of the spectral parameter λ?
- Can this *scattering data* be used to solve the *initial value problem* for the KdV equation?

If $u_0(x)$ is a smooth compactly supported perturbation of the function $U(x, t_0)$, which will be defined below, then

Answers:

- the spectrum is $(-\infty, +\infty)$, it is one-folded continuous spectrum, no discrete spectrum or embedded eigenvalues.
- there is a bijection to a class of reflection coefficients $r(\lambda)$, discontinuous across the real line;

Given the spectral functions we can define the following RHP:

Riemann-Hilbert problem 2

To find a 2×2 *matrix-valued function* $\widehat{\mathbb{F}}(x,t;\lambda)$, *which*

1. *is meromorphic in* $\lambda \in \mathbb{C} \setminus \Sigma_{\mathbb{F}}$, *with finite number of poles at* λ_j, $\overline{\lambda_j}$, $j = i, \dots, J$. *Here* $\Sigma_{\mathbb{F}} = \mathbb{R} \cup \gamma_3 \cup \gamma_{-3}$; $\gamma_3 = (\mathrm{e}^{6\pi\mathrm{i}/7}, 0)$, $\gamma_{-3} = (\mathrm{e}^{-6\pi\mathrm{i}/7}, 0)$
2. *has the jumps as in the picture;*
3. *has the following pole conditions at the points* λ_j, $\overline{\lambda_j}$, $j = 1, \dots, J$:

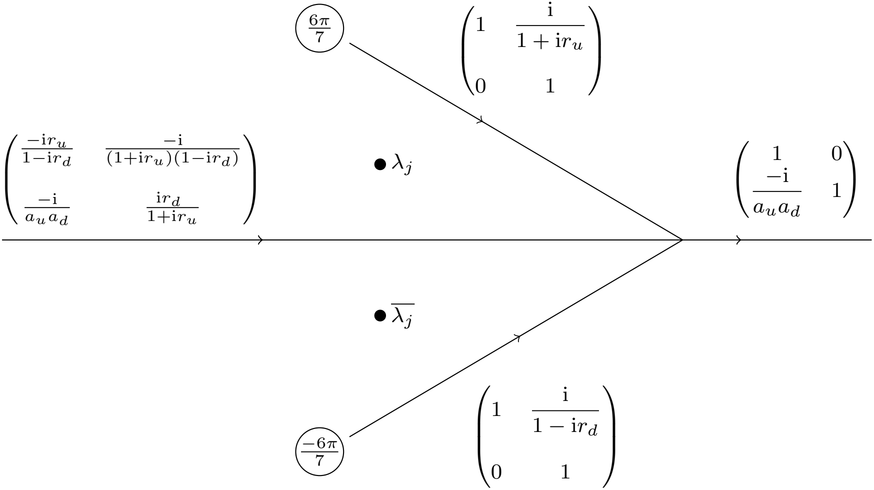
$$\mathbb{F}_{[2]}(\lambda) + \frac{\mathrm{i}}{1 + \mathrm{i}r_u(\lambda)}\mathbb{F}_{[1]}(\lambda) = \mathcal{O}(1) \quad \textit{and} \quad \mathbb{F}_{[1]}(\lambda) = \mathcal{O}(1) \quad \textit{for} \quad \lambda \to \lambda_j,$$

$$\mathbb{F}_{[2]}(\lambda) - \frac{\mathrm{i}}{1 - \mathrm{i}r_d(\lambda)}\mathbb{F}_{[1]}(\lambda) = \mathcal{O}(1) \quad \textit{and} \quad \mathbb{F}_{[1]}(\lambda) = \mathcal{O}(1) \quad \textit{for} \quad \lambda \to \overline{\lambda_j},$$

4. *has the following asymptotics as* $\lambda \to \infty$, *which are uniform w.r.t.* $\arg\lambda \in [-\pi, \pi]$:

$$\mathbb{F}(x,t;\lambda) = \frac{1}{\sqrt{2}}\lambda^{-\sigma_3/4}(\sigma_3 + \sigma_1)\left(I + \frac{\mathrm{diag}}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1})\right)\mathrm{e}^{\theta(x,t;\lambda)\sigma_3},$$

where $\theta = \theta(x,t;\lambda) = x\lambda^{1/2} - \frac{t}{3}\lambda^{3/2} + \frac{1}{105}\lambda^{7/2}$.


$\frac{6\pi}{7}$
$\begin{pmatrix} 1 & \frac{\mathrm{i}}{1+\mathrm{i}r_u} \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \frac{-\mathrm{i}r_u}{1-\mathrm{i}r_d} & \frac{-\mathrm{i}}{(1+\mathrm{i}r_u)(1-\mathrm{i}r_d)} \\ \frac{-\mathrm{i}}{a_u a_d} & \frac{\mathrm{i}r_d}{1+\mathrm{i}r_u} \end{pmatrix}$
λ_j
$\begin{pmatrix} 1 & 0 \\ \frac{-\mathrm{i}}{a_u a_d} & 1 \end{pmatrix}$
$\overline{\lambda_j}$
$\begin{pmatrix} 1 & \frac{\mathrm{i}}{1-\mathrm{i}r_d} \\ 0 & 1 \end{pmatrix}$
$\frac{-6\pi}{7}$

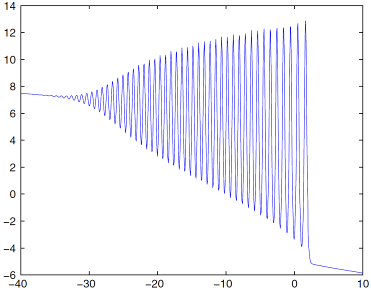
Reconstruction of $u(x,t)$ from $\Phi(x,t;\lambda)$.

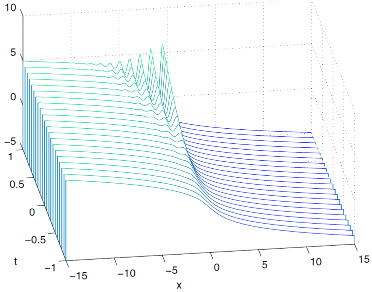
Then the asymptotics of $\mathbb{F}(x,t;\lambda)$ as $\lambda \to \infty$ has the structure

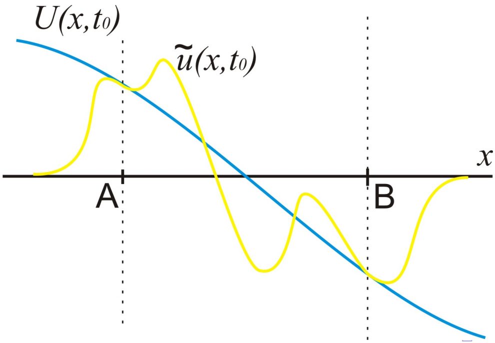
$$\mathbb{F}(x,t;\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -H_1 & 1 \end{pmatrix} \left(I + \frac{1}{\lambda} \begin{pmatrix} \frac{H_1^2+u}{2} & -H_1 \\ -\frac{H_0}{3} + \frac{H_1^3}{3} + H_1 u + \frac{u_x}{4} & \frac{H_1^2+u}{-2} \end{pmatrix} + \mathcal{O}(\lambda^{-2}) \right)$$
$$\cdot \lambda^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathrm{e}^{(x\lambda^{1/2} - \frac{t}{3}\lambda^{3/2} + \frac{1}{105}\lambda^{7/2})\sigma_3}.$$

where $(H_1(x,t))_x = u(x,t)$, and $u(x,t)$ is the solution of the KdV, which at the time $t=0$ has the *reflection coefficient* $r(\lambda)$.

$U(x, t = 4)$, from T. Grava, A. Kapaev, C. Klein, '15







Motivation 1.

$u_t + au_x = 0$, the solution is given by $u(x,t) = u_0(x - at)$.
For $u_t + uu_x = 0$, $\quad u$ - heuristically, is a speed of a point. Hence the profile will become more steep, and will break at a point x^*, t^*.
The solution is given in a parametric form, by the method of characteristics,

$$u(x,t) = u_0(\xi),$$
$$x = tu_0(\xi) + \xi.$$

For $u_t + uu_x + \varepsilon^2 u_{xxx} = 0$, look for $\varepsilon \to 0$. Near the point x^*, t^*, the solution is described by a function $U(x,t)$, which

- solves KdV
- solves P_I^2
- behaves as $\sqrt[3]{-6x}$ as $x \to \pm\infty$.

Motivation 1

$u_t + uu_x + \varepsilon^2 u_{xxx} = 0.$

$$u(x,t,\varepsilon) = u_c + \left(\frac{2\varepsilon^2}{k^2}\right)^{1/7} U\left(\frac{x - x_c - 6u_c(t-t_c)}{(8k\varepsilon^6)^{1/7}}, \frac{6(t-t_c)}{(4k^3\varepsilon^4)^{1/7}}\right) + \mathcal{O}(\varepsilon^{4/7}),$$

in the double scaling limit where $\varepsilon \to 0$ and at the same time

$$\lim \frac{x - x_c - 6u_c(t-t_c)}{(8k\varepsilon^6)^{1/7}} = X, \lim \frac{6(t-t_c)}{(4k^3\varepsilon^4)^{1/7}} = T,$$

with $X, T \in \mathbb{R}$. The constant k is given by $k = -f_-'''(u_c)$, where f_- is the inverse function of the decreasing part of the initial function $u_0(x)$, which is assumed to be real analytic and with a single negative hump.
B.Dubrovin '06, T.Claeys, T.Grava ' 08

Here x_c, t_c, u_c are points of gradient catastrophe of the Hopf equation $u_t + u_x u = 0.$

Motivation 2

Question: In which cases of initial function the KdV is solvable?
Answer: If initial function $u_0(x)$ is a perturbation of an exact solution of KdV.
Examples:

- (rapidly) vanishing initial function
- $u_0 \to c,\ x \to= \pm\infty$
- $u_0 \to c_-,\ x \to -\infty$ and $u_0 \to c_+,\ x \to +\infty$ (step-like constant)
- $u_0(x) \to q_p(x, 0),\ x \to \pm\infty$, where $q_p(x, t)$ is a periodic (finite-gap, quasi-periodic) solution.
- $u_0 \to q_{p,\pm}(x, 0),\ x \to \pm\infty$ (step-like periodic) (I. Egorova, G. Teschl)

Known examples of exact solutions of KdV:

- $u_\pm(x,t) \equiv 0$. The corresponding continuous spectrum is two folded $(-\infty, 0]$.

- $u_\pm(x,t) \equiv c_\pm$, where $c_\pm$ are constants. The continuous spectrum is partially one or two folded.

- $u_\pm(x,t)$ are the so-called *finite gap* (quasi periodic) solutions of KdV, who bear their name after the form of the spectrum (B. Dubrovin, S. Novikov, P. Lax, A. Its, V. Matveev, V. Marchenko, B. Levitan, H. Knörer, E. Trubowitz). The solutions of the Lax pair are the Baker-Akhiezer functions, which are meromorphic functions on the corresponding Riemann surface. The typical spectrum has the following shape:

- $u_\pm(x,t) = U(x,t)$, where the $U(x,t) \sim \sqrt[3]{-x/6}$ as $x \to \pm\infty$ is some particular function, defined through a Riemann-Hilbert problem. The corresponding spectrum is one folded real line $\mathbb{R}$.

Details of analysis for decaying $u_0(x)$.

- Find solutions of Sturm-Liouville equation for non-perturbed function, i.e. for 0:

$$e_r^{(0)}(x,\lambda) = \mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{-x\sqrt{\lambda}}, \quad e_l^{(0)}(x,\lambda) = \mathrm{e}^{-ikx} = \mathrm{e}^{x\sqrt{\lambda}}, \qquad \sqrt{\lambda} = -\mathrm{i}k.$$

- Find relation between non-perturbed Jost solutions,
$e_l^{(0)}(x,\lambda - \mathrm{i}0) = e_r^{(0)}(x,\lambda + \mathrm{i}0), \quad \lambda \in (-\infty, 0).$
- Find solutions of Sturm-Liouville equation for perturbed function, i.e. for $u_0(x)$.
Integral equation,

$$f_r(x,\lambda) = \mathrm{e}^{-x\sqrt{\lambda}} + \int_x^{\infty} \frac{\sin(\mathrm{i}(x-y)\sqrt{\lambda})}{x-y} f_r(y,\lambda)\mathrm{d}y.$$

Integral representation,

$$f_r(x,\lambda) = \mathrm{e}^{-x\sqrt{\lambda}} + \int_x^{+\infty} K(x,y)\mathrm{e}^{-y\sqrt{\lambda}}\mathrm{d}y,$$

where $K(x,y)$ is an integrable function (in y.)

Details of analysis for decaying $u_0(x)$: Riemann-Hilbert problem

Observe that the function

$$\Phi(x;\lambda) = \frac{1}{\sqrt{2}\sqrt[4]{\lambda}} \begin{pmatrix} \frac{1}{a(\lambda)} f_l(x,\lambda) & f_r(x,\lambda) \\ \frac{1}{a(\lambda)} \partial_x f_l(x,\lambda) & \partial_x f_r(x,\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

has the same jumps, pole and asymptotics, as in RHP. That is, it solves the RHP for $t = 0$.

- jumps: $\Phi_+ = \Phi_- \begin{pmatrix} -\mathrm{i}\, r(\lambda) & \frac{-\mathrm{i}}{\mathrm{i}\, r(\lambda)} \\ -\mathrm{i}(1 - |r(\lambda)|^2) & \end{pmatrix}, \quad \lambda \in (-\infty, 0);$
- poles at λ_j : $Res_{\lambda_j} \Phi(x,t;\lambda) = \Phi(x,t;\lambda) \begin{pmatrix} 0 & 0 \\ \gamma_j & 0 \end{pmatrix};$
- asymptotics as $\lambda \to \infty$:
  $\Phi(x,t;\lambda) = \frac{1}{\sqrt{2}} \lambda^{-\sigma_3/4} (\sigma_3 + \sigma_1) \left(I + \frac{\mathrm{diag}}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1}) \right) \mathrm{e}^{x\lambda^{1/2}\sigma_3},$

Details of analysis in the growing $\sqrt[3]{-6x}$ case.

- What is $U(x,t)$?
- What are the solutions of non-perturbed SL equation, which correspond to the function $U(x,t)$?
- What is the spectrum?
- For a perturbation $u_0(x)$ of $U(x,t_0)$, what are the corresponding solutions of SL equation?
- Spectral functions?
- How to reconstruct $u_0(x)$ from spectral functions?

To define $U(x,t)$ and the corresponding solutions of SL equation, take $r \equiv 0$ in the RHP 2.

Riemann-Hilbert problem 3

To find a 2×2 *matrix-valued function* $\mathbb{E}(x,t;\lambda)$, *which*

1. *is piece-wise analytic in* $\lambda \in \mathbb{C}\setminus\Sigma_{\mathbb{E}}$. *Here* $\Sigma_{\mathbb{E}} = \mathbb{R}\cup\gamma_3\cup\gamma_{-3}$; $\gamma_3 = (\mathrm{e}^{6\pi\mathrm{i}/7},0)$, $\gamma_{-3} = (\mathrm{e}^{-6\pi\mathrm{i}/7},0)$
2. *has the jumps as in the picture;*
3. *has the following asymptotics as* $\lambda\to\infty$, *which are uniform w.r.t.* $\arg\lambda\in[-\pi,\pi]$:

$$\mathbb{E}(x,t;\lambda) = \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left(I+\frac{diag}{\sqrt{\lambda}}+\overline{o}(\lambda^{-1})\right)\mathrm{e}^{\theta\sigma_3},$$

where $\theta = x\sqrt{\lambda}-\frac{t}{3}\lambda^{3/2}+\frac{1}{105}\lambda^{7/2}$.

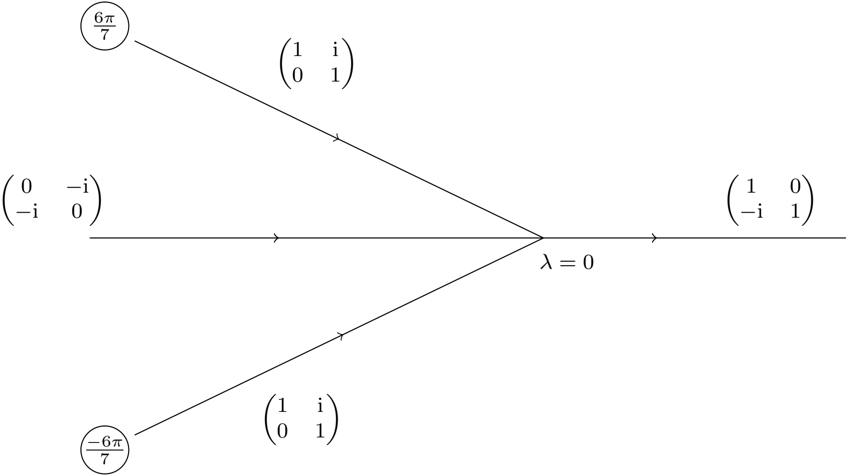


Figure: Contour of the RHP for the function $\mathbb{E}(x, t; \lambda)$.

Observe that since the jump matrix does not depend on x, t, λ, the derivatives of $\mathbb{E}$ w.r.t. x, t, λ satisfy the same jumps, and hence

$$\mathbb{E}_x \mathbb{E}^{-1} =: \mathfrak{U}, \qquad \mathbb{E}_t \mathbb{E}^{-1} =: \mathfrak{V}, \qquad \mathbb{E}_\lambda \mathbb{E}^{-1} =: \mathfrak{W}$$

are entire functions of variable λ. From expansion as $\lambda \to \infty$ (Claeys Vanlessen'08)

$$\mathbb{E}(x,t;\lambda) = \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \left(I + \sum_{j=1}^{J-1} \begin{pmatrix} a_j(x,t) & b_j(x,t) \\ c_j(x,t) & d_j(x,t) \end{pmatrix} \lambda^{-j} + \mathcal{O}(\lambda^{-J}) \right) \cdot$$
$$\cdot \frac{\lambda^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathrm{e}^{(x\lambda^{1/2} - \frac{t}{3}\lambda^{3/2} + \frac{1}{105}\lambda^{7/2})\sigma_3},$$

we find that

$$\mathbb{E}_x \mathbb{E}^{-1} = \begin{pmatrix} 0 & 1 \\ \lambda - 2U(x,t) & 0 \end{pmatrix}, \quad \mathbb{E}_t \mathbb{E}^{-1} = \begin{pmatrix} \frac{U_x}{6} & \frac{-\lambda - U}{3} \\ \frac{-\lambda^2}{3} + \frac{U\lambda}{3} + \frac{4U^2 + U_{xx}}{6} & \frac{-U_x}{6} \end{pmatrix}$$

$\mathbb{E}_\lambda \mathbb{E}^{-1} =$
a polynomial of 3rd degree in λ, coefficients expressed in terms of U, U_x, U_{xx}.

and some of the coefficients a_j, b_j, c_j, d_j are related to the function $U = U(x,t)$ in the following way:

$$U(x,t) = 2a_1 - b_1^2 = -b_{1x}, \quad U_x := 2(3a_1b_1 - b_1^3 - b_2 + c_1), \tag{1}$$

$$x + \frac{U^3}{15} - \frac{U_x^2}{120} + \frac{UU_{xx}}{60} = \frac{-2a_1^2 + a_2 + 4a_1b_1^2 - b_1^4 - 2b_1b_2 + b_1c_1 - d_2 + 3b_{1t}}{3}, \tag{2}$$

$$b_1 = \frac{1}{480}\left(240tU^2 - 20U^4 - 20U(24x + U_x^2) + U_{xx}^2 - 2U_xU_{xxx}\right). \tag{3}$$

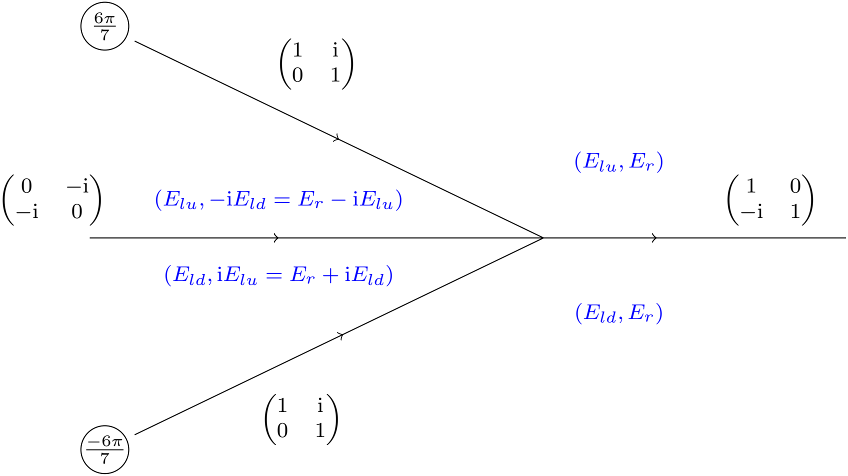
The consistency condition of the system

$$\begin{cases} \mathbb{E}_x = \mathfrak{U}\mathbb{E}, \\ \mathbb{E}_t = \mathfrak{V}\mathbb{E}, \end{cases}$$

i.e.

$$\mathfrak{U}_t - \mathfrak{V}_x + [\mathfrak{U}, \mathfrak{V}] = 0,$$

gives that $U(x,t)$ satisfies the KdV equation.



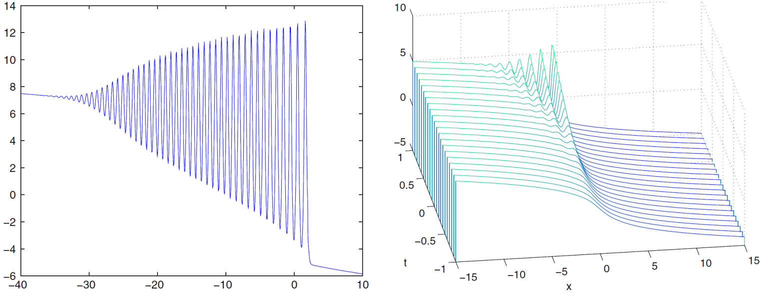
We define solutions E_l, E_r of SL equation corresponding to $U(x,t)$ from the columns of $\mathbb{E}$.

Theorem. Claeys, Vanlessen '08

- $U(x,t)$ is real-valued and pole-free for $x, t \in \mathbb{R}$,
- for fixed $t \in \mathbb{R}$, the function $U(x,t)$ has the following asymptotic behavior:

$$U(x,t) = \sqrt[3]{-6x} + \frac{2t}{\sqrt[3]{-6x}} + \frac{8t^3}{3\cdot(6x)^{5/3}} + \mathcal{O}(|x|^{-2}), \; x \to \pm\infty.$$

$U(x, t = 4)$, from T. Grava, A. Kapaev, C. Klein, '15



Solutions of the SL equation

If

$$\int_{-\infty}^{+\infty} \frac{|u_{t_0}(x) - U(x,t_0)| \, \mathrm{d}x}{1 + \sqrt[6]{|x|}} < \infty.$$

then there exist solutions $F_l(x;\lambda), F_r(x;\lambda)$ of SL equation, determined by their large x asymptotics:

$$F_l(x,t_0;\lambda) = E_l(x,t_0;\lambda)(1 + o\,(1), \quad x \to -\infty,$$
$$F_r(x,t_0;\lambda) = E_r(x,t_0;\lambda)(1 + o\,(1), \quad x \to -\infty,$$

Properties of F_l, F_r

1. Analyticity: $F_l(x,t_0;\lambda)$ is analytic in $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and continuous up to the boundary. Denote

$$F_l \equiv \begin{cases} F_{lu}, \Im\lambda > 0, \\ F_{ld}, \Im\lambda < 0. \end{cases}$$

   $F_r(x,t_0;\lambda)$ is an entire function.

2. Symmetry:

$$\overline{F_l(x,t_0;\overline{\lambda})} = F_l(x,t_0;\lambda), \qquad \overline{F_r(x,t_0;\overline{\lambda})} = F_r(x,t_0;\lambda).$$

3. Determinant:
$$\det(F_{lu}, F_{ld}) = W\{f_{lu}, f_{ld}\} = -\mathrm{i}.$$

4. Additional smoothness: if $u_{t_0}(x) \in C^n(\mathbb{R})$, then $F_l(x;\lambda), F_r(x;\lambda) \in C^{n+2}(\mathbb{R})$.

Large λ behavior of F_l, F_r.

If $\exists A < B$ such that

$$u_{t_0}(x) = U(x, t_0), \qquad \text{for } x < A \text{ and for } x > B,$$

then, for any fixed $x \in \mathbb{R}$, uniformly w.r.t. $\arg\lambda \in [-\pi, 0] \cup [0, \pi]$,

$$F_l(x, t_0; \lambda) = E_l(x, t_0; \lambda)\left(1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})\right), \quad \lambda \to \infty,$$

and for any fixed $\varepsilon > 0$ uniformly w.r.t. $\arg\lambda \in [-\pi + \varepsilon, \pi - \varepsilon]$,

$$F_r(x, t_0; \lambda) = E_r(x, t_0; \lambda)\left(1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})\right), \quad \lambda \to \infty,$$

Remark about non-compactly supported perturbations

It is not trivial to obtain large λ asymptotics of F_l, F_r in the case of non-compactly supported perturbations, because of the presence of the term $\frac{1}{\sqrt{\lambda - \lambda_0(y)}}$ in the corresponding integral equations, in which both λ and $\lambda_0(y)$ might be large, but their difference might be small.
Here $\lambda_0(y)$ is a real solution of $\lambda_0^3 - 24t\lambda_0 + 48y = 0$.

Spectral function $a(\lambda)$.

On the real line $\lambda \in \mathbb{R}$, we have three solutions F_{lu}, F_{ld}, F_r of the SL equation. The relation between them is:

$$F_r(x,t_0;\lambda) = \mathrm{i}a_d(\lambda)F_{lu}(x,t_0;\lambda) - \mathrm{i}a_u(\lambda)F_{ld}(x,t_0;\lambda),$$

where the function $a(\lambda)$ is analytic in analytic in $\lambda \in \mathbb{C} \setminus \mathbb{R}$, continuous up to the boundary, and is expressed in terms of F_l, F_r as follows:

$$a(\lambda) := \det(F_r, F_l), \quad a_u(\lambda) := \det(F_r, F_{lu}),\ a_d(\lambda) := \det(F_r, F_{ld}),$$

$$a(\lambda) \equiv \begin{cases} a_u(\lambda), \Im\lambda > 0, \\ a_d(\lambda), \Im\lambda < 0. \end{cases}$$

Question: Is there only one spectral function?

Spectral function $b(\lambda)$

Let the functions $h_1(x;\lambda)$, $h_2(x;\lambda)$ be solutions of SL, corresponding to $u_0(x)$ for $A < x < B$, i.e.

$$h_{xx} + 2u_0(x)h = \lambda h, \quad A < x < B,$$

and let their Wronskian

$$W(\lambda) = \{h_1, h_2\} \equiv h_1 h_{2x} - h_2 h_{1x}$$

not be identically 0.
Then the functions f_l, f_r have the form

$$f_l(x, t_0, \lambda) = \begin{cases} e_l(x, t_0, \lambda), & x < A, \\ \frac{1}{W(\lambda)} \left[\{e_l, h_2\}_A \, h_1(x;\lambda) - \{e_l, h_1\}_A \, h_2(x;\lambda) \right], & A < x < B, \\ b(\lambda) e_r(x, t_0, \lambda) + a(\lambda) e_l(x, t_0, \lambda), & x > B, \end{cases}$$

and

$$f_r(x, t_0, \lambda) = \begin{cases} \mathrm{i}(a_d - a_u) e_l(x, t_0, \lambda) + a(\lambda; t_0) e_r(x, t_0, \lambda), & x < A, \\ \frac{1}{W(\lambda)} \left[\{e_r, h_2\}_B \, h_1(x, t_0, \lambda) - \{e_r, h_1\}_B \, h_2(x, t_0, \lambda) \right], & A < x < B, \\ e_r(x, t_0, \lambda), & x > B. \end{cases}$$

The function $b(\lambda)$ is determined through the large $x \to +\infty$ behaviour of f_l,

$$b(\lambda) \equiv \begin{cases} b_u(\lambda), \Im\lambda > 0, \\ b_d(\lambda), \Im\lambda < 0, \end{cases}$$

Define $r(\lambda) = \frac{b(\lambda)}{a(\lambda)}$.

Properties of $a(\lambda), b(\lambda), r(\lambda)$.

1. Functions $a(\lambda), b(\lambda)$ are analytic in $\mathbb{C}\setminus\mathbb{R}$, and continuous up to the boundary.
2. The restrictions $a_u(\lambda),\ b_u(\lambda)$ of $a(\lambda), b(\lambda)$ to the upper half-plane $\Im\lambda > 0$, can be extended analytically to $\mathbb{C}$;
   the restrictions $a_d(\lambda),\ b_d(\lambda)$ of $a(\lambda),\ b(\lambda)$ to the lower half-plane $\Im\lambda < 0$ are related to a_u, b_u as $\overline{a_u(\overline{\lambda})} = a_d(\lambda),\ \overline{b_u(\overline{\lambda})} = b_d(\lambda)$.
3. Symmetries:
   $$a(\lambda) = \overline{a(\overline{\lambda})}, \qquad b(\lambda) = \overline{b(\overline{\lambda})}.$$
4. Large $\lambda\to\infty$ behavior: uniformly w.r.t. $\arg\lambda\in[-\pi,0]\cup[0,\pi]$,
   $$a(\lambda) = 1 + \frac{1}{\sqrt{\lambda}}\int_A^B (U(x,t_0) - u_0(x))\mathrm{d}x + \overline{o}(\frac{1}{\sqrt{\lambda}}).$$
5. Function $a(\lambda)$ does not vanish nowhere, i.e. $a_u(\lambda)\neq 0$ for $\Im\lambda\geq 0$.
   Moreover, $a_u(\lambda)\neq 0$ for $\lambda\in\mathbb{C}$.

Properties of $a(\lambda), b(\lambda), r(\lambda)$ (continuation)

6. Large $\lambda \to \infty$ behaviour of $r(\lambda)$.
   If $u_0 \in BV_{loc}$, then for $\lambda \to \infty$, uniformly w.r.t. $\arg\lambda \in [-\pi, \pi]$,

$$r_u(\lambda) = \mathcal{O}(\frac{1}{\lambda}) \cdot \mathrm{e}^{2(B\lambda^{1/2} - \frac{t_0}{3}\lambda^{3/2} + \frac{1}{105}\lambda^{7/2})\sigma_3}$$

   Furthermore, if $u_0 \in BV_{loc}^{(N)}$, then

$$r_u(\lambda) = \mathcal{O}(\lambda^{-\frac{N}{2}-1}) \cdot \mathrm{e}^{2(B\lambda^{1/2} - \frac{t_0}{3}\lambda^{3/2} + \frac{1}{105}\lambda^{7/2})\sigma_3}.$$

   Furthermore, if $u_0 \in BV_{loc}^{(\infty)}$, then for any N,

$$r_u(\lambda) = \mathcal{O}(\lambda^{-\frac{N}{2}-1}) \cdot \mathrm{e}^{2(B\lambda^{1/2} - \frac{t_0}{3}\lambda^{3/2} + \frac{1}{105}\lambda^{7/2})\sigma_3}.$$

7. As $s \to \pm\infty$, $s \in \mathbb{R}$,

$$\Im r_u(s) = \mathcal{O}(|s|^{-1}).$$

8. For $s \in \mathbb{R}$,

$$\Im r_u(s) < \frac{1}{2} \quad \text{and} \quad \Im r_d(s) > -\frac{1}{2}.$$

Properties of $a(\lambda), b(\lambda), r(\lambda)$ (continuation 2)

9. $r_u(\lambda) - \overline{r_u(\overline{\lambda})} \neq \mathrm{i}$ for all $\lambda \in \mathbb{C}$;
10. As $\lambda \to \infty$, uniformly w.r.t. $\arg\lambda \in [-\pi, \pi]$,

$$a_u(\lambda; t_0) - \overline{a_u(\overline{\lambda}; t_0)} = \mathcal{O}(\frac{1}{\lambda}) \cdot \mathrm{e}^{-2\theta(A, t_0; \lambda)}$$

11. $r_u(\lambda) - r_d(\lambda) = \mathrm{i}\left(1 - \dfrac{1}{a_u(\lambda) a_d(\lambda)}\right).$
12. Functions a_u, a_d can be expressed in terms of r_u in the following way:

$$a_u(\lambda) = \exp\left\{\frac{-1}{2\pi\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\ln(1 - 2\Im r_u(s))\mathrm{d}s}{s - \lambda}\right\}, \quad \text{for } \Im\lambda > 0,$$

$$= \frac{1}{1 + \mathrm{i}\left(r_u(\lambda) - \overline{r_u(\overline{\lambda})}\right)} \cdot \exp\left\{\frac{-1}{2\pi\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\ln(1 - 2\Im r_u(s))\mathrm{d}s}{s - \lambda}\right\}, \quad \text{for } \Im\lambda < 0.$$

Remark

It follows from the jump relation of $r(\lambda)$ that the function

$$\mathcal{E}(\lambda) = \frac{b}{a} - \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1 - \frac{1}{a_u(s)a_d(s)}}{s-\lambda} ds}_{\mathcal{O}(\frac{1}{\sqrt{\lambda}})}$$

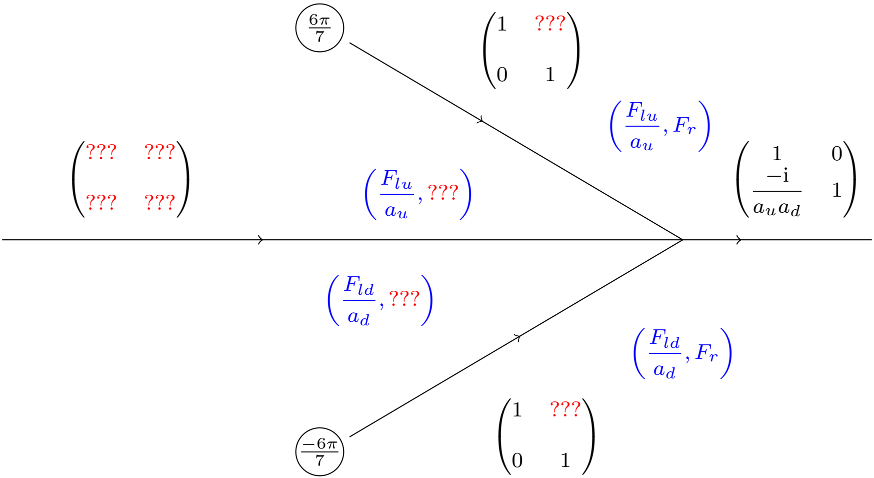
is an entire function in the whole complex plane. It satisfies the symmetry condition $\overline{\mathcal{E}\left(\overline{\lambda}\right)} = \mathcal{E}(\lambda)$, and, if $u_{t_0}(x) = c$ for $A < x < B$, then it has the uniform w.r.t. $\arg\lambda$ asymptotics as $\lambda \to \infty$

$$\mathcal{E}(\lambda) = \mathrm{e}^{2\theta(B,t_0;\lambda)}\mathcal{O}(\lambda^{-1}) + \mathcal{O}(\lambda^{-1/2}).$$

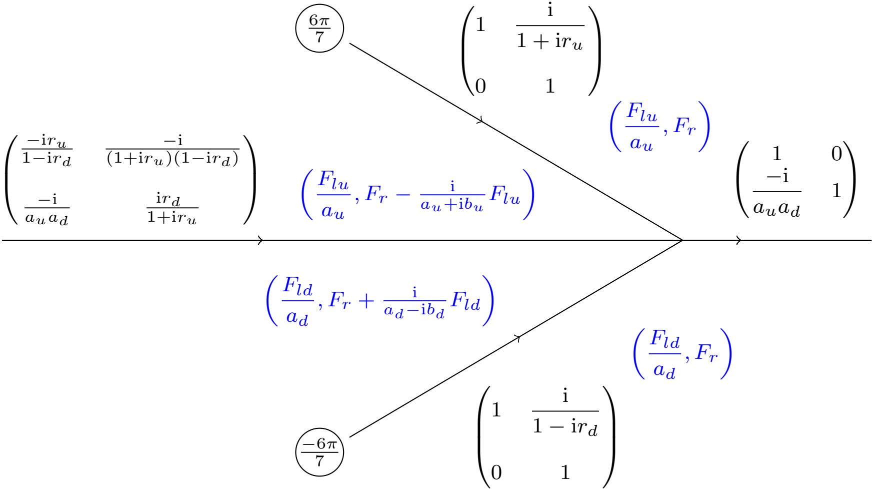
The existence of an entire function with such uniform asymptotics at infinity is quite remarkable.

Reconstruction of $u_0(x)$ from $r_u(\lambda)$.

Idea: to construct a piece-wise meromorphic function out of F_l, F_r, which would solve a RHP, and the RHP would depend only on $a(\lambda), b(\lambda), r(\lambda)$.



Fill in the gaps as follows. Observe that the obtained jumps behave well for large λ.



Poles at λs such that $r_u(\lambda) = \mathrm{i}$, $\arg\lambda \in (\frac{6\pi}{7}, \pi)$.

RHP (appropriate for all real t and x.)

To find a 2×2 matrix-valued function $\mathbb{F}(x,t;\lambda)$, which

1. is analytic in $\lambda \in \mathbb{C} \setminus \Sigma$,
2. has the jump $\mathbb{F}_+ = \mathbb{F}_- J_{\mathbb{F}}$ across Σ as in the Figure.
3. has the following pole conditions at the roots of $r_u = \mathrm{i}$, $r_d = -\mathrm{i}$:
   for $\lambda^* \in II$, $\Im\lambda^* > 0$ such that $r_u(\lambda^*) = \mathrm{i}$,

$$\mathbb{F}_{[2]}(\lambda) + \frac{\mathrm{i}}{1 + \mathrm{i} r_u(\lambda)} \mathbb{F}_{[1]} = \mathcal{O}(1) \quad \text{and} \quad \mathbb{F}_{[1]} = \mathcal{O}(1) \quad \text{for} \quad \lambda \to \lambda^*,$$

$$\mathbb{F}_{[2]}(\lambda) - \frac{\mathrm{i}}{1 - \mathrm{i} r_d(\lambda)} \mathbb{F}_{[1]} = \mathcal{O}(1) \quad \text{and} \quad \mathbb{F}_{[1]} = \mathcal{O}(1) \quad \text{for} \quad \lambda \to \overline{\lambda^*},$$

4. has the following asymptotics as $\lambda \to \infty$, which is uniform w.r.t. $\arg\lambda \in [-\pi, 0] \cup [0, \pi]$:

$$\mathbb{F}(x,t;\lambda) = \frac{1}{\sqrt{2}} \lambda^{-\sigma_3/4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + \frac{diag}{\sqrt{\lambda}} + \overline{o}\left(\frac{1}{\sqrt{\lambda}}\right) \right) \mathrm{e}^{\theta(x,t;\lambda)\sigma_3},$$

Remark

Let us notice that the set of data of the RHP is the function $r_u(\lambda)$, which is an entire function. Thus, it contains also all the information about the points λ_j, $\Im\lambda_j > -0$, where $r_u(\lambda_j) = \mathrm{i}$.

Solvability of the RHP and smoothness of its solution w.r.t. x, t

1. Reformulate the RHP as a Singular Integral Equation (SIE). $M_+ = M_- J$,
$M = I + \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{M_+(s)(I - J^{-1}(s))\mathrm{d}s}{s - \lambda}$, $\quad M_+ = I + \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{M_+(s)(I - J^{-1}(s))\mathrm{d}s}{(s - \lambda)_+}$.
2. Prove that the singular integral operator is Fredholm: (to find a pseudoinverse).
3. Prove that the index of the operator is 0: (continuity).
4. Prove that the homogeneous RHP has only zero solution.
Relies on:

An entire function $g(\lambda)$, which has the large $\lambda \to \infty$ asymptotics $\mathcal{O}(\lambda^{-3/4})\mathrm{e}^{2\theta(x,t;\lambda)}$ uniformly w.r.t. $\arg \lambda \in [-\pi, \pi]$, equals 0 identically.

5. Observe that the SIE admits formal differentiation w.r.t. x, t, and the corresponding SIE are solvable.

Summary: Scheme of integration of the initial value problem:

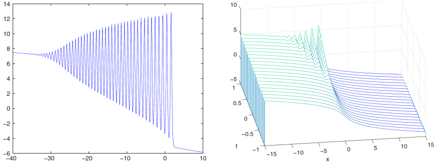
Usual scheme for integrating the ivp for KdV consists of two steps:

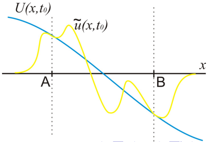
- **Forward scattering transform:** Given $u_0(x)$, construct the solutions of the Lax pair at the time $t = 0$, and construct the associated spectral functions, and then
- **Inverse scattering transform:** Given the spectral functions, plug in the evolution in time t and reconstruct the solution $u(x,t)$ of the ivp.

This is well-known in the case of an initial function which is
- a perturbation of zero (Gardner Green Kruskal Miura),
- periodic initial function (V. Marchenko, B. Levitan),
- step-like perturbations of finite-gap functions (E. Khruslov, I. Egorova, G. Teschl).

Our goal here is to develop such a theory for $u_0(x)$, which is a perturbation of $U(x, t_0)$.

$U(x, t = 4)$, from T. Grava, A. Kapaev, C. Klein, '15





Summary: Main features of analysis

- the Jost solutions of the Lax operator are not similar:
  left solution $f_l(x;\lambda) \in \mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ is discontinuous across $\lambda \in \mathbb{R}$,
  right solution $f_r(x;\lambda) \in \mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ is an entire function;
- as a consequence, there is only one (scattering) relation between $f_\pm$;
- the spectrum is the real line (one-folded, no discrete spectrum).
- only one spectral function, $a(\lambda)$ is determined through that (scattering) relation,

$$f_r(x;\lambda) = \mathrm{i}a(\lambda - \mathrm{i}0) f_l(x;\lambda + \mathrm{i}0) - \mathrm{i}a(\lambda + \mathrm{i}0) f_r(x;\lambda - \mathrm{i}0);$$

- another spectral function, $b(\lambda)$, is determined through asymptotics as $x \to +\infty$ of $f_l(x;\lambda)$;
- both $a(\lambda), b(\lambda)$ are $\in \mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ and discontinuous across $\lambda \in \mathbb{R}$;
- to reconstruct solution $u(x,t)$ of KdV from $a(\lambda)$, $b(\lambda)$, one needs to define a piece-wise meromorphic matrix-valued function in the complex plane, using as entries linear combinations of f_l, f_r;
- we use compactness of perturbation in order to construct the above matrix;
- poles of the conjugation problem solved by the above matrix are caused not by zeros of $a(\lambda)$ ($a(\lambda) \neq 0$ everywhere), but by zeros of $a(\lambda) + ib(\lambda)$ in the upper half-plane.
- instead of $|R|^2 + |T|^2 = 1$, or $|r|^2 \equiv \frac{|b|^2}{|a|^2} = 1 - \frac{1}{|a|^2}$,we have
  $-\mathrm{i}\left(r(\lambda + \mathrm{i}0) - r(\lambda - \mathrm{i}0)\right) = 1 - \frac{1}{|a(\lambda)|^2}, \lambda \in \mathbb{R}.$

Analog of the inverse Fourier transform formula

For a smooth compactly supported function $\varphi(x)$,

$$\varphi(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e_r(x; t_0, \lambda) \int\limits_{\mathbb{R}} \varphi(y) e_r(y; t_0, \lambda) \mathrm{d}y \mathrm{d}\lambda.$$

Remark: in other cases of initial functions (perturbations of zero, constant, step-constant, periodic, step-periodic) it is also used to derive integral representation for the Jost solutions.

It is a natural generalization of usual Fourier transform, and of the transform involving the Airy function.

For Airy functions: no term $\lambda^{7/2}$.

$$\delta(x-y) = \int_{\mathbb{R}} Ai(x+\lambda) Ai(y+\lambda) \mathrm{d}\lambda.$$