The Replica Symmetric Phase of Random Constraint Satisfaction Problems

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Randomness in Physics and Mathematics
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Joint work with Amin Coja–Oghlan and Tobias Kapetanopoulos

Overview

- Random Constraint Satisfaction Problems
- Solution space geometry
- Results

Boolean formulae

$$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$$

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Variable assigment: Configuration $\underline{x} \in \{0,1\}^V$

Understand properties of the set $SOL \subset \{0,1\}^V$ of assignments that satisfy all constraints.

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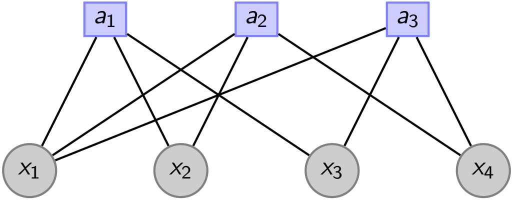

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Important quantities:

- The number of solutions: *partition function* $Z(G)$
- The uniform measure on solutions: *Boltzmann distribution* μ_G:
  If $Z(G) > 0$, set

$$\mu_G(\sigma) = \frac{\mathbf{1}\{\sigma \text{ is a solution}\}}{Z(G)}$$

- Inspect $Z(\mathbb{G})$ and $\mu_{\mathbb{G}}$ for increasing *density* parameters $d > 0$.

Evolution of the solution space of rCSPs

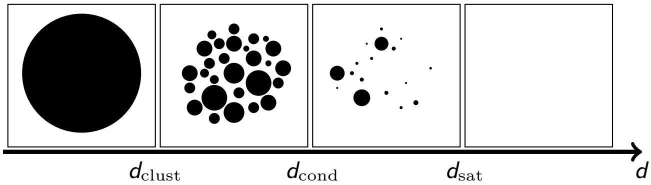

Figure: An adaption of a conjecture by Krząkała, Montanari, Ricci-Tersenghi, Semerjian and Zdeborová '07.

Theorem (Coja-Oghlan, Kapetanopoulos, Müller '19+)

Let $d > 0$. With γ a $\mathrm{Po}(d)$-random variable, $\rho_1^{(\pi)}, \rho_2^{(\pi)}, \ldots$ chosen from $\pi \in P(\Omega)$ and $\psi_1, \psi_2, \ldots \in \Psi$ chosen from P, all mutually independent, let

$$\mathcal{B}(d, P, \pi) = \mathrm{E}\left[q^{-1}\xi^{-\gamma}\Lambda\left(\sum_{\sigma \in \Omega}\prod_{i=1}^{\gamma}\sum_{\tau \in \Omega^k} \mathbf{1}\{\tau_k = \sigma\}\psi_i(\tau)\prod_{j=1}^{k-1}\rho_{ki+j}^{(\pi)}(\tau_j)\right) - \frac{d(k-1)}{k\xi}\Lambda\left(\sum_{\tau \in \Omega^k}\psi_1(\tau)\prod_{j=1}^{k}\rho_j^{(\pi)}(\tau_j)\right)\right],$$

$$d_{\mathrm{cond}} = \inf\left\{d > 0 : \sup_{\pi \in \mathcal{P}_*^2(\Omega)} \mathcal{B}(d, P, \pi) > \ln q + \frac{d}{k}\ln\xi\right\}.$$

Then $1/(k-1) \leq d_{cond}(q) < \infty$ and for all $d < d_{cond}(q)$ we have

$$\sqrt[n]{Z(\mathbb{G})} \quad \xrightarrow{n\to\infty} \quad q\xi^{d/k} \quad \textit{in probability.}$$

By constrast, for any $d > d_{cond}(q)$ there exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \mathrm{P}\left[\sqrt[n]{Z(\mathbb{G})} > q\xi^{d/k} - \varepsilon\right]^{\frac{1}{n}} < 1 - \varepsilon.$$

Limiting Distribution

Let $(K_\ell)_{\ell \geq 1}$ be Poisson variables with means $\mathrm{E}[K_\ell] = \frac{1}{2\ell}(d(k-1))^\ell$ and let $(\psi_{\ell,i,j})_{\ell,i,j \geq 1}$ be a sequence of samples from P, all mutually independent.

Theorem (Coja-Oghlan, Kapetanopoulos, Müller '19+)

Suppose that $0 < d < d_{\mathrm{cond}}$ and $\mathbb{G}$ is a random factor graph model that satisfies certain assumptions. Let

$$\mathcal{K} = \exp\left(\frac{d(k-1)(1-\mathrm{tr}(\Phi))}{2} + \mathbf{1}\{k=2\}\frac{d^2(1-\mathrm{tr}(\Phi^2))}{4}\right)$$

$$\times \prod_{\ell=2+\mathbf{1}\{k=2\}}^{\infty} \exp\left(\frac{(d(k-1))^\ell}{2\ell}\left(1-\mathrm{tr}(\Phi^\ell)\right)\right) \prod_{i=1}^{K_\ell} \mathrm{tr} \prod_{j=1}^{\ell} \Phi_{\psi_{\ell,i,j}}$$

Then $\mathcal{K} > 0$ almost surely and

$$\frac{Z(\mathbb{G})}{q^{n+\frac{1}{2}}\xi^{\boldsymbol{m}}} \prod_{\lambda \in \mathrm{Eig}(\Phi)\setminus\{1\}} \sqrt{1-d(k-1)\lambda} \quad \xrightarrow{n\to\infty} \quad \mathcal{K}$$

in distribution.

Examples covered

- random k–NAESAT
- "balanced" satisfiability
- random (hyper)graph coloring
- "parity–majority" (cryptography)