On branching random walks on periodic lattices

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A random walk

Suppose there is one particle at a point $v \in \mathbb{Z}^d$ at time $t = 0$. This particle will either move to a point $u \neq v$, $u \in \mathbb{Z}^d$ or will remain at v over a short period of time δt.

a probability of the transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of the transition $v \to v$

$$p(v, v, \delta t) = 1 + a(v, v)\delta t + o(\delta t).$$

The transition intensity 1

The value $a(v,u)$ is called the transition intensity between v and u.

$(i)\ a(v,u) \geqslant 0, \quad v \neq u;$

$(ii)\ a(v,v) < 0;$

$(iii)\ \sum_{u \in \mathbb{Z}^d} a(v,u) = 0;$

The transition intensity 2

Let $g_1, \ldots, g_d$ be a family of linearly independent (not necessarily orthogonal) vectors with integer coordinates. By a lattice we mean a set

$$\Gamma = \Big\{ g \in \mathbb{Z}^d : g = \sum_{j=1}^{d} n_j g_j,\ n_j \in \mathbb{Z},\ j = 1, \ldots, d \Big\}.$$

(iv) $a(v,u) = a(u,v) = a(v+g, u+g), \quad \forall g \in \Gamma;$

(v) $\sum_{u \in \mathbb{Z}^d} \|u\|^2 |a(v,u)| < \infty,\ v \in \mathbb{Z}^d;$

(vi) the graph $G = (\mathbb{Z}^d, \mathcal{E})$ with the vertex set $\mathbb{Z}^d$ and edge set

$$\mathcal{E} = \{(v,u) :\ a(v,u) > 0,\ v, u \in \mathbb{Z}^d\}$$

is connected.

A branching source

Suppose there is one particle at a point with branching source and it can't move anywhere from there. We assume that a particle can generate several descendants over a short period of time δt.

a probability of generating $k \neq 1$ descendants

$$p_k = b_k \delta t + o(\delta t).$$

a probability of generating $k = 1$ descendant

$$p_1 = 1 + b_1 \delta t + o(\delta t).$$

The branching intensity

The value b_k is called a branching intensity into k descendants.

(1) $b_k \geqslant 0, \quad k \neq 1;$

(2) $b_1 \leqslant 0;$

(3) $\sum\limits_{k=0}^{+\infty} b_k = 0;$

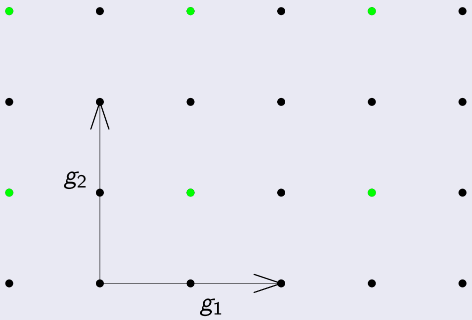
(4) $\beta = \sum\limits_{k=1}^{+\infty} k b_k < \infty;$

Locations of the branching sources

Let branching intensities depend on the point v and value $\beta(v)$ be periodic with respect to the lattice Γ.

(5) $\beta(v+g) = \beta(v), \quad g \in \Gamma.$

Example



$g_1 = (2, 0)$

$g_2 = (0, 2)$

We assume that each new particle evolves according to the same law independently of other particles.

Each particle located at a point $v \in \mathbb{Z}^d$ at time t can either move to a point $u \neq v$ or remain at the source and produce $k \neq 1$ descendants located at the point v (for $k = 0$ we assume that the number of descendants is 0; that is, the particle dies) or remain unchanged (that is, no changes occur) over a short period of time $[t; t + \delta t)$.

a probability of transition $v \to u$

$$p(v, u, \delta t) = a(v, u)\delta t + o(\delta t),$$

a probability of generating $k \neq 1$ offsprings in v

$$p_k(v, \delta t) = b_k(v)\delta t + o(\delta t).$$

a probability of remaining unchanged

$$p(v, \delta t) = 1 + a(v, v)\delta t + b_1(v)\delta t + o(\delta t).$$

Finite number of branching sources

Yarovaya E., et. all [1998-2007]

One source. Asymptotic behavior of all moments. Limit theorems.

Vatutin V. Topchii V. [2005]

Limit theorem for catalytic BRW on $\mathbb{Z}$ with one source of branching in super critical case.

Rytova A. Yarovaya E. [2018]

Extinction probability for sub and super critical cases.

Khristolyubov I. Yarovaya E. [2018-2019]

N sources. Asymptotic behavior of all moments in subcritical and supercritical cases. Limit theorems.

Mean value of particles

By $M(v,u,t)$ we denote the mean number of particles at a point u at time t, provided that at the initial time $t=0$ there was one particle at a point v. The function $M(v,u,t)$ satisfies the following Cauchy problem:

The Cauchy problem

$$\begin{cases} M_t'(v,u,t) = \mathcal{A}M(v,u,t), \\ M(v,u,0) = \delta_u(v). \end{cases}$$

The operator $\mathcal{A}$

$$\mathcal{A} = \mathcal{A}_0 + Q,$$

$$(\mathcal{A}_0 f)(v) = \sum_{w\in\mathbb{Z}^d} a(v,w)f(w),$$

$$(Qf)(v) = \beta(v)f(v).$$

Properties of $\mathcal{A}$

The operator $\mathcal{A} : \ell^2(\mathbf{Z}^d) \to \ell^2(\mathbf{Z}^d)$ satisfies the following properties:

- $\mathcal{A}$ is self-adjoint;
- $\mathcal{A}$ is bounded;
- $\mathcal{A}_0$ is non-positive;
- $\mathcal{A}$ is periodic with respect to the lattice Γ.

Connection with a discrete Laplacian

If condition (v) is replaced with stronger condition that for any $v \in \mathbb{R}^d$ there are only finite number of transition probabilities $a(v, u)$ are not zero, then operator $-\mathcal{A}_0$ is a discrete combinatorial Laplacian on the graph G defined in (vi). In this case operator $-\mathcal{A}$ is a discrete Schödinger with periodic potential.

A discrete Laplacian

Sunada T., Sy P. [1992]

Local spectral properties near the left edge of the spectrum.

Higuchi Y., Shirai T. [2001]

Local spectral properties near the left edge for the magnetic Shrödinger.

Higuchi Y., Shirai T. [2004]

Direct integral decomposition. Spectral properties.

Higuchi Y., Nomura Y. [2008]

Direct integral decomposition. Conditions for absolute continuity of the spectrum (absence of flat bands).

Korotyaev E., et. all [2010-2018]

Direct integral decomposition with explicit form of fiber operators. Estimates for spectrum of Laplacian with different types of perturbations.

Asymptotic behaviour of $M(v, u, t)$

Theorem

The function $M(v, u, t)$ has the following asymptotic behaviour as $t \to \infty$

$$M(v,u,t) = e^{\lambda_1(0)t} t^{-\frac{d}{2}} c_0(v,u)\Big(1 + O(t^{-1})\Big)$$

where

$$c_0(v,u) = \frac{|\mathcal{C}|}{(2\pi)^{\frac{d}{2}}} \frac{\psi_1(v',0)\psi_1(u',0)}{\sqrt{\left|\det\left\{\frac{\partial^2\lambda_1(\theta)}{\partial\theta^2}\Big|_{\theta=0}\right\}\right|}}$$

where $v = v' + \gamma_v$, $u = u' + \gamma_u$, $v', u' \in \Omega$, $\gamma_v, \gamma_u \in \Gamma$.

If $a(v, u)$ decays fast enough, then

$$M(v,u,t) \overset{as}{=} e^{\lambda_1(0)t} t^{-\frac{d}{2}} \sum_{k=0}^{\infty} c_k(v,u) t^{-k}.$$

Notations

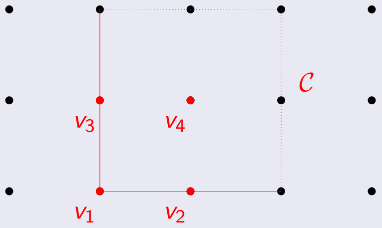
Fundamental cell

$$\mathcal{C} = \{x \in \mathbf{R}^d : x = \sum_{j=1}^{d} x_j g_j,\ 0 \leqslant x_j < 1, j = 1, \ldots, d\}.$$

Fundamental vertex set

$$\Omega = \mathcal{C} \cap \mathbb{Z}^d, \qquad \Omega = \{v_1, \ldots, v_p\}.$$

Example



$$g_1 = (2, 0)$$

$$g_2 = (0, 2)$$

An auxiliary matrix family

We introduce a new variable θ with range such that $2\pi\theta \in \mathcal{C}$.

$$A(\theta) = \begin{pmatrix} \widetilde{a}_{11}(\theta) + \beta_1 & \widetilde{a}_{12}(\theta) & \cdots & \widetilde{a}_{1p}(\theta) \\ \widetilde{a}_{21}(\theta) & \widetilde{a}_{22}(\theta) + \beta_2 & \cdots & \widetilde{a}_{2p}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{p1}(\theta) & \widetilde{a}_{p2}(\theta) & \cdots & \widetilde{a}_{pp}(\theta) + \beta_p \end{pmatrix},$$

where the functions $\widetilde{a}_{jk}(\theta)$ and constants β_j is defined by

$$\widetilde{a}_{jk}(\theta) = \sum_{g \in \Gamma} e^{-i\langle g, \theta\rangle} a(v_j + g, v_k), \qquad \beta_j = \beta(v_j).$$

Connection between $\mathcal{A}$ and $A(\theta)$

Let the eigenvalues of the matrix family $A(\theta)$ be ordered in non-increasing order for every parameter θ: $\lambda_1(\theta) \geqslant \ldots \geqslant \lambda_p(\theta)$.

$$\sigma(\mathcal{A}) = \bigcup_{j=1}^{p} \bigcup_{2\pi\theta \in \mathcal{C}} \lambda_j(\theta),$$

Properties of $\lambda_1(\theta)$

Theorem

For $\lambda_1(\theta)$ the following statements hold:

a) *the right edge of the spectrum of $\mathcal{A}$ coincides with $\lambda_1(0)$, i.e.*

$$\max \sigma(\mathcal{A}) = \lambda_1(0);$$

b) *the distance between the right edge of the spectrum of $\mathcal{A}$ and the right edge of the second spectral band is positive, i.e.*

$$\lambda_1(0) - \sup_{\theta \in \widetilde{\mathcal{C}}} \lambda_2(\theta) > 0;$$

c) *the determinant of the Hessian matrix of $\lambda_1(\theta)$ does not vanish at $\theta = 0$, i.e.*

$$\det \left\{ \frac{\partial^2 \lambda_1(\theta)}{\partial \theta^2} \Big|_{\theta=0} \right\} \neq 0;$$

d) *$\lambda_1(0)$ is not an eigenvalue of $\mathcal{A}$.*

Properties of $\lambda_1(0)$

Theorem

$\lambda_1(0)$ is positive if either $\sum_{j=1}^{p} \beta_j > 0$ or for some $m = 1, \dots, p$ $\beta_m \geqslant \|A_0(0)\|$.

Theorem

If we add the coupling constant μ then

- *for $\mu \to 0$*
$$\lambda_1(0) = \frac{\mu}{p} \sum_{j=1}^{p} \beta_j + O(\mu^2).$$
- *for $\mu \to \infty$*
$$\lambda_1(0) = \mu \max_{j=1,\dots,p} \beta_j + O(1).$$

The second moment (supercritical case)

Let $M_2(v,u,t)$ be the second moment for our BRW. Assume that additionally to (4) and (5) following conditions are satisfied:

$(4')$ $\beta_2(v) = \sum\limits_{k=2}^{+\infty} k(k-1)b_k(v) < \infty$;

$(5')$ $\beta_2(v+g) = \beta_2(v), \quad g \in \Gamma$.

The Cauchy problem for the second moment

$$\begin{cases} M_{2t}^{'}(v,u,t) & = (\mathcal{A}M_2)(v,u,t) + \beta_2(v)M^2(v,u,t), \\ M_2(v,u,0) & = \delta_u(v), \end{cases}$$

Asymptotic behaviour of $M_2(v, u, t)$

Theorem

Suppose that $\lambda_1(0) > 0$. Then the function $M_2(v, u, t)$ has the following asymptotic behavior as $t \to \infty$

$$M_2(v, u, t) = \frac{e^{2\lambda_1(0)t} t^{-d} d_0(v, u)}{\lambda_1(0)}(1 + O(t^{-1})),$$

where the $d_0(v, u)$ is defined by

$$d_0(v, u) = \frac{|\mathcal{C}|^2 \psi_1(v', 0)|\psi_1(u', 0)|^2}{2^{d/2}(2\pi)^d \left| \det \left\{ \frac{\partial^2 \lambda_1(\theta)}{\partial \theta^2} \Big|_{\theta=0} \right\} \right|} \sum_{w \in \Omega} \psi_1(w, 0)\beta_2(w)|\psi_1(w, 0)|^2$$

References

- Higuchi Y., Nomura Y. Spectral structure of the Laplacian on a covering graph //European Journal of Combinatorics. – 2009. – V. 30.2. – P. 570-585.
- Higuchi Y., Shirai T. Some spectral and geometric properties for infinite graphs //Contemporary Mathematics. – 2004. – V. 347. – P. 29-56.
- Khristolyubov I., Yarovaya E. A Limit Theorem for Supercritical Branching Random Walks with Branching Sources of Varying Intensity //arXiv preprint arXiv:1904.01468. – 2019.
- Korotyaev E., Saburova N. Schrödinger operators on periodic discrete graphs //Journal of Mathematical Analysis and Applications. – 2014. – V. 420.1. – P. 576-611.
- Korotyaev E., Saburova N. Schrödinger operators with guided potentials on periodic graphs //Proceedings of the American Mathematical Society. – 2017. – V. 145.11. – P. 4869-4883.

References

- Higuchi Y., Shirai T. Weak Bloch property for discrete magnetic Schrödinger operators //Nagoya Mathematical Journal. – 2001. – V. 161. – P. 127-154.
- Rytova A., Yarovaya E. Survival Analysis of Particle Populations in Branching Random Walks //arXiv preprint arXiv:1812.09909. – 2018.
- Sy P. W., Sunada T. Discrete Schrödinger operators on a graph //Nagoya Mathematical Journal. – 1992. – V. 125. – P. 141-150.
- Vatutin V. A., Topchii V. A. Limit theorem for critical catalytic branching random walks //Theory of Probability and Its Applications. – 2005. – V.3. – P. 498-518.
- Yarovaya E. B. A limit theorem for critical branching of a random walk on with a single source //Russian Mathematical Surveys. – 2005. – V. 60.1. – P. 173.

Thank you for your attention!