Random matrices and our recent results

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Overview

1. Extreme gaps

2. Normality of CβE

1 Extreme gaps

2 Normality of CβE

Poisson and CUE

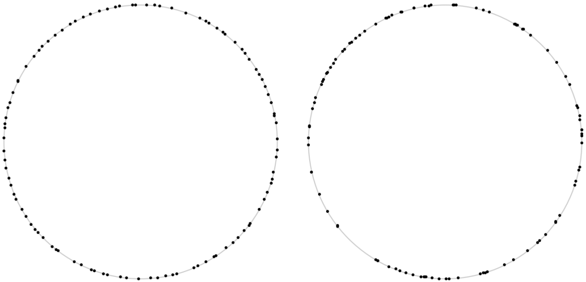

Figure: CUE eigenvalues (left) and Poisson (right) ($N = 100$)

Q: given $\theta_1 < \theta_2 \cdots < \theta_n$, what are the distributions of extreme gaps

$$\min_i |\theta_{i+1} - \theta_i|, \quad \max_i |\theta_{i+1} - \theta_i|?$$

Smallest gaps for CUE

Consider two-dimensional process

$$\chi_n = \sum_{i=1}^{n} \delta_{(n^{4/3}(\theta_{i+1}-\theta_i),\theta_i)}.$$

Theorem (Vinson, Soshnikov, Ben Arous-Bourgade)

χ_n tends to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A \times I) = \left(\frac{1}{24\pi}\int_A u^2 du\right)\left(\int_I \frac{du}{2\pi}\right).$$

Let $t_1^n < t_2^n \cdots < t_k^n$ be the first k smallest eigenangles gaps, denote $\tau_k^n = (72\pi)^{-1/3} t_k^n$, then as a consequence,

$$\lim_{n\to\infty} \mathbb{P}(\tau_k^n \in dx) = \frac{3}{(k-1)!} x^{3k-1} e^{-x^3} dx.$$

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Smallest gaps for CβE

When β is an positive integer, consider the process of eigenangle of CβE

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Theorem [F-Wei, arXiv:1806.01555]

χ_n tends to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A \times I) = \frac{A_\beta |I|}{2\pi} \int_A u^\beta du,$$

where $A_\beta = (2\pi)^{-1} \frac{(\beta/2)^\beta (\Gamma(\beta/2+1))^3}{\Gamma(3\beta/2+1)\Gamma(\beta+1)}$. In particular, the result holds for COE, CUE and CSE with

$$A_1 = \frac{1}{24},\ A_2 = \frac{1}{24\pi},\ A_4 = \frac{1}{270\pi}$$

respectively.

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Smallest gaps for CβE

Corollary

Let's denote t_k^n as the k-th smallest gap, and

$$\tau_k^n = n^{(\beta+2)/(\beta+1)} \times (A_\beta/(\beta+1))^{1/(\beta+1)} t_k^n,$$

then for any bounded interval $A \subset \mathbb{R}_+$, we have

$$\lim_{n\to+\infty} \mathbb{P}(\tau_k^n \in dx) = \frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}} dx.$$

- No determinantal point process structure can be used as CUE (which is used by Vinson, Soshinikov and Ben Arous-Bourgade), we have to start from the Selberg integral
- Conjecture: The result must be true for any $\beta > 0$, but our method does not work other than integer β.

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Smallest gaps for GUE

Consider the 2-dimensional process of (interior) eigenvalues of GUE

$$\chi_n = \sum_{i=1}^{n} \delta_{(n^{\frac{4}{3}}(\lambda_{i+1}-\lambda_i),\lambda_i)} 1_{|\lambda_i|<2-\eta}$$

Theorem (Vinson, Ben Arous-Bourgade)

χ_n tends to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A \times I) = (\frac{1}{48\pi^2}\int_A u^2 du)(\int_I (4-x^2)^2 dx),$$

where $A \subset \mathbb{R}_+$ and $I \subset (-2+\eta, 2-\eta)$.

Thus the rescaling k-th smallest gaps $\tau_k^n = (\int_I (4-x^2)^2 dx/144\pi^2)^{1/3} t_k^n$ has the limiting density $\frac{3}{(k-1)!}x^{3k-1}e^{-x^3}$ which is the same as CUE.

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Smallest gaps for GOE

Consider the 1-dimensional process of eigenvalues of GOE

$$\chi^{(n)} = \sum_{i=1}^{n-1} \delta_{n^{3/2}(\lambda_{(i+1)} - \lambda_{(i)})}$$

Theorem [F-Tian-Wei, to appear in GAFA]

$\chi^{(n)}$ converges to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A) = \frac{1}{4}\int_A u du.$$

Let's denote m_k as the k-th smallest gaps, and $\tau_k = 2^{-3/2} n^{3/2} m_k$, then the limiting density is

$$\frac{2}{(k-1)!} x^{2k-1} e^{-x^2},$$

which is same as COE.

Conjectures

We conjecture that the local statistics of GβE and CβE are the same, i.e., there exists c_β such that $\tau_k^n = c_\beta n^{(\beta+2)/(\beta+1)} t_k$ has the limiting density

$$\frac{\beta+1}{(k-1)!} x^{k(\beta+1)-1} e^{-x^{\beta+1}}.$$

The conjecture should be true for more general universal ensembles,

$$\frac{1}{Z_{n,\beta,V}} e^{-n\beta \sum_{i=1}^{n} V(\lambda_i)} \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta.$$

Heuristically, $\prod_{1\le i<j\le n} |\lambda_j - \lambda_i|^\beta$ indicates the local behavior of the smallest gaps.

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Heuristically, $\prod_{1\le i<j\le n} |\lambda_j - \lambda_i|^\beta$ indicates the local behavior of the smallest gaps.

Order of largest gaps

Let $m_1 > m_2 > \cdots$ be the largest gaps between successive eigenangles of CUE,

Theorem (Ben Arous-Bourgade, AOP 2013)

For any $p > 0$ and $l_n = n^{o(1)}$, one has

$$\frac{nm_{l_n}}{\sqrt{32 \ln n}} \xrightarrow{L^p} 1.$$

Fluctuation of largest gaps

Theorem (F-Wei [arXiv:1807.02149])

Let's denote m_k as the k-th largest gap of CUE, and

$$\tau_k = (2\ln n)^{\frac{1}{2}}(nm_k - (32\ln n)^{\frac{1}{2}})/4 - (3/8)\ln(2\ln n),$$

then $\{\tau_k\}$ will tend to a Poisson distribution and we have the limit of the Gumbel distribution,

$$\lim_{n\to+\infty}\mathbb{P}(\tau_k \in I) = \int_I \frac{e^{k(c_1-x)}}{(k-1)!}e^{-e^{c_1-x}}dx.$$

Here, $c_1 = c_0 + \ln\frac{\pi}{2}$, $c_0 = \frac{1}{12}\ln 2 + 3\zeta'(-1)$. In particular, the limiting density for the largest gap τ_1 is,

$$e^{c_1-x}e^{-e^{c_1-x}}.$$

Fuctuation of largest gaps

Theorem (F-Wei)

Let's denote m_k^* as the k-th largest gap of GUE, $S(I) = \inf_I \sqrt{4-x^2}$ and

$$\tau_k^* = (2\ln n)^{\frac{1}{2}}(nS(I)m_k^* - (32\ln n)^{\frac{1}{2}})/4 + (5/8)\ln(2\ln n),$$

$\{\tau_k^*\}$ will tend to a Poisson distribution we have the limit of the Gumbel distribution,

$$\lim_{n\to+\infty} \mathbb{P}(\tau_k^* \in I_1) = \int_{I_1} \frac{e^{k(c_2-x)}}{(k-1)!} e^{-e^{c_2-x}} dx.$$

Here, $c_2 = c_0 + M_0(I)$ depending on I, where
$M_0(I) = (3/2)\ln(4-a^2) - \ln(4|a|)$ if $a+b<0$,
$M_0(I) = (3/2)\ln(4-b^2) - \ln(4|b|)$ if $a+b>0$,
$M_0(I) = (3/2)\ln(4-a^2) - \ln(2|a|)$ if $a+b=0$.

Fluctuation of largest gaps

- In both proofs, one of the essential parts is to show that the rescaling largest gaps are asymptotic to some Poisson processes, i.e., they are asymptotically independent.
- Another essential part is the correct rescaling factors.
- We do not know how to work for COE/GOE, CSE/GSE.
- Universality:
  - P. Bourgade, Extreme gaps between eigenvalues of Wigner matrices.
  - B. Landon, P. Lopatto, J. Marcinek, Comparison theorem for some extremal eigenvalue statistics.

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Previous results

- For fixed θ, we have macroscopic CLT (Killip, 2006')

$$\sqrt{\frac{\pi^2\beta}{2\ln(2+n\theta)}}[N_n(0,\theta)-\frac{n\theta}{2\pi}]\to N(0,1)$$

where $N_n(0,\theta)$ is the number of eigenangles falling in $[0,\theta]$.

- For all θ that may depend on n, one has uniform upper bound (Najnudel-Virág, 2019)

$$\mathbb{E}[|N_n(0,\theta)-n\theta/(2\pi)|^2]\le C_\beta\ln(2+n\theta).$$

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Berry-Esseen theorem

Theorem (F-Tian-Wei, arXiv:1905.09448)

Let $\theta \in (0, \pi]$, we have

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\sqrt{\frac{\pi^2\beta}{2\ln(2+n\theta)}} \left[N_n(0,\theta) - \frac{n\theta}{2\pi} \right] \leq x \right] - \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \right|$$
$$\leq \frac{C}{(\ln(2+n\theta))^{\frac{1}{2}}},$$

here $C > 0$ is a constant depending only on β.

As a direct consequence, we have

Corollary

Let $\theta_n \in (0, \pi]$, $n\theta_n \to +\infty$, then

$$\sqrt{\frac{\pi^2 \beta}{2 \ln(2 + n\theta_n)}}[N_n(0, \theta_n) - \frac{n\theta_n}{2\pi}] \to N(0, 1).$$

Uniform upper and lower bounds

Proposition

The characteristic function of the counting function has estimate

$$|\mathbb{E}[e^{i\lambda(N_n(0,\theta)-n\theta/(2\pi))} - e^{-\lambda^2/(\beta\pi^2)\cdot\ln(2+n\theta)}| \le C\lambda^2$$

and we have the uniform bounds

$$\left|\mathbb{E}\left[\left|N_n(0,\theta)-\frac{n\theta}{2\pi}\right|^2\right]-\frac{2\ln(2+n\theta)}{\pi^2\beta}\right|\le C.$$

Sine$_\beta$ process

Corollary

Let L be the Sine$_\beta$ point process,

$$|\mathbb{E}[(\mathrm{Card}(L\cap[0,x])-x/(2\pi))^2]-2/(\beta\pi^2)\cdot\ln(2+x)|\leq C,$$

and

$$\sup_{y\in\mathbb{R}}\left|\mathbb{P}\left[\sqrt{\frac{\pi^2\beta}{2\ln(2+x)}}\left[\mathrm{Card}(L\cap[0,x])-\frac{x}{2\pi}\right]\leq y\right]-\int_{-\infty}^{y}\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt\right|$$

$$\leq C(\ln(2+x))^{-\frac{1}{2}}.$$

Thank you for attention!