The Integrated Density of States for the Almost Mathieu Operator

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joint work with A. Fedotov

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Almost Mathieu Operator (Harper Operator, Hofstadter Model)

$$(H\psi)_k = \psi_{k+1} + \psi_{k-1} + 2\cos(2\pi(\omega k + \theta))\psi_k, \quad \psi \in l^2(\mathbb{Z}),$$

$\omega \in (0,1) \setminus \mathbb{Q}$ *frequency*, $\theta \in [0,1)$ *ergodic* parameter.

Integrated Density of States (IDS)

$$N(E) = \lim_{L\to\infty} \frac{1}{2L+1}(\text{number of the eigenvalues of } H_L \leq E),$$

H_L is H restricted to $[-L, L]$ with $\psi(-L) = \psi(L) = 0$.

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- N exists and doesn't depend on θ almost surely;
- N is continuous and non-decreasing;
- N is constant on the resolvent set (outside of the spectrum);
- $N = 0$ (1) to the left (right) of the *spectrum*.

Spectrum

$Spec(H) := \{E \in \mathbb{C} : (H - E \cdot \mathbb{1})^{-1} \nexists\}$, $Spec(H) \subset [-4, 4]$.

Theorem ("Ten Martini Problem", Avila, Jitomirskaya '09)

The spectrum of the almost Mathieu operator is a cantor set (closed, nowhere dense, no isolated points).

Problem 1 Constructive geometry of the spectrum.

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Theorem ("Gap Labeling Theorem")

The values of the IDS on the gaps (intervals of the resolvent set) belong to

$$\{k\omega \mod 1, k \in \mathbb{Z}\}.$$

Problem 2 ("Dry Ten Martini Problem") To prove that all the gaps predicted by the Gap Labeling Theorem are open, i.e., all the values of the IDS from the set $\{k\omega \mod 1, k \in \mathbb{Z}\}$ are realised.

Semiclassical approximation

$$\omega = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \dots}}}, \quad \alpha_k \in \mathbb{N}, \quad \alpha_k > N \gg 1.$$

Helffer, Sjöstrand, Buslaev, Fedotov.

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Theorem (Fedotov, S.,'19)

Spec(H) is asymptotically equivalent to the set of $E \in [-4, 4]$ such that the trajectory of the model dynamical system starting at E remains in $[-4, 4]$.

Model Dynamical System

$$\begin{cases} \omega^{(j)} = \left(1/\omega^{(j-1)}\right) \mod 1, \\ E^{(j)} = \mathcal{E}(\omega^{(j-1)}, |E^{(j-1)}|), \end{cases} \qquad (\omega^{(0)}, E^{(0)}) := (\omega, E).$$

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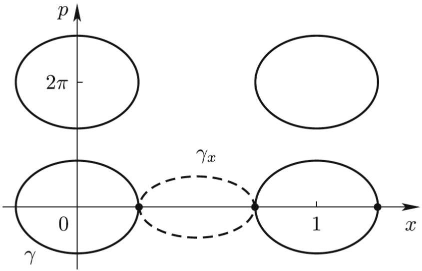


Isoenergy curve

$2\cos p + 2\cos(2\pi x) = E,\ E \in (0,4).$

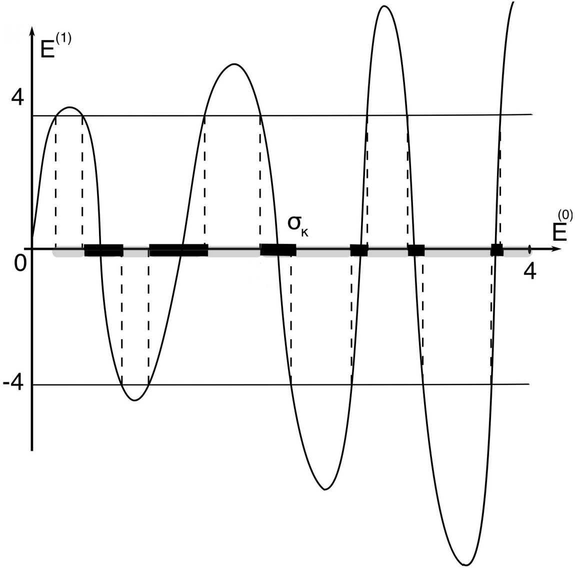
$$\mathcal{E}(\omega, E) = 4\cos\left(\frac{\Phi(E)}{\omega}\right)\cosh\left(\frac{S(E)}{\omega}\right)$$

Phase $\Phi(E) = \frac{1}{2}\oint_{\gamma} p\,dx,$

Action $S(E) = \frac{i}{2}\oint_{\gamma_x} p\,dx.$

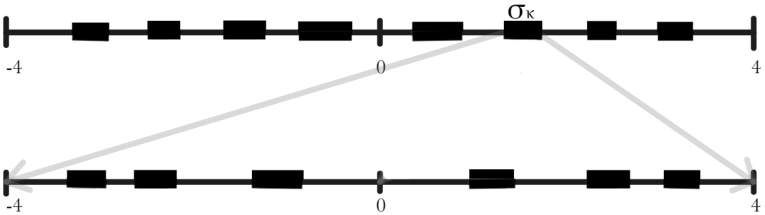
First Step of Model Dynamical System

$$Spec(H) \asymp \{E^{(0)} \in [-4,4] : |E^{(k)}| \leq 4 \ \forall k\}$$

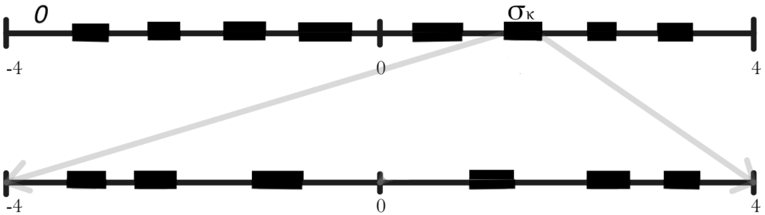


Second Step of Model Dynamical System

$|E^{(1)}| \leq 4,$
$E^{(1)} : \sigma_\kappa \to [-4, 4],$
$|E^{(2)}| \leq 4,$
$\ldots$



Values of Integrated Density of States



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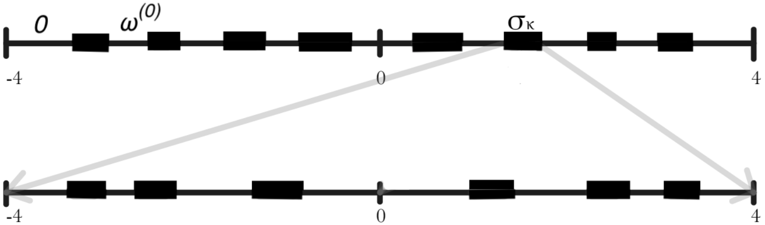

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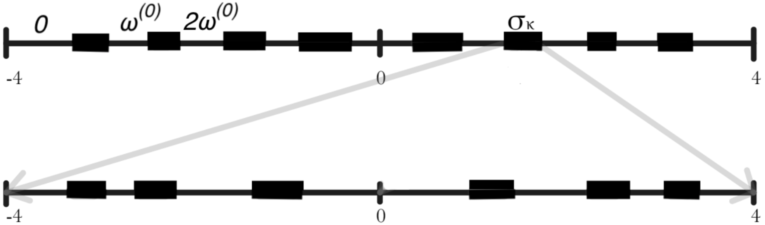

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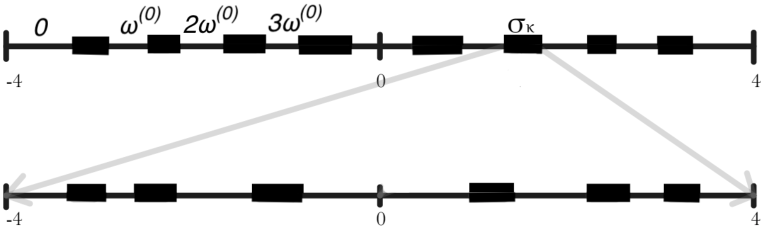

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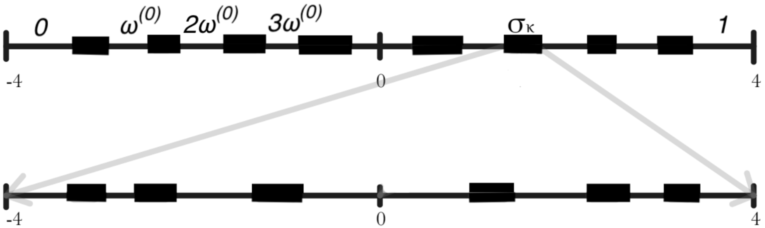

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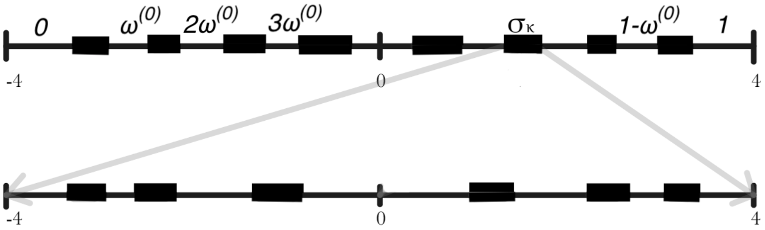

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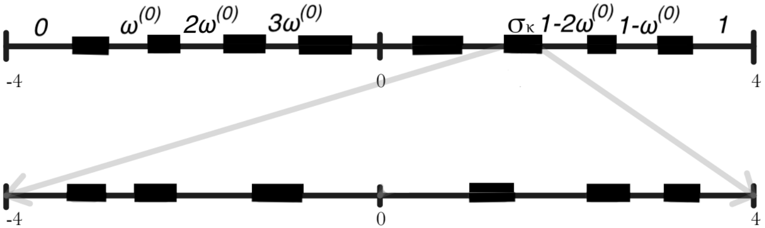

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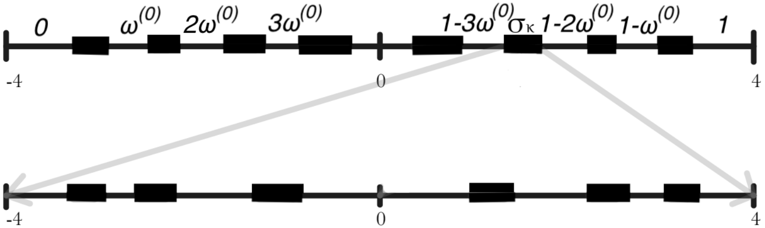

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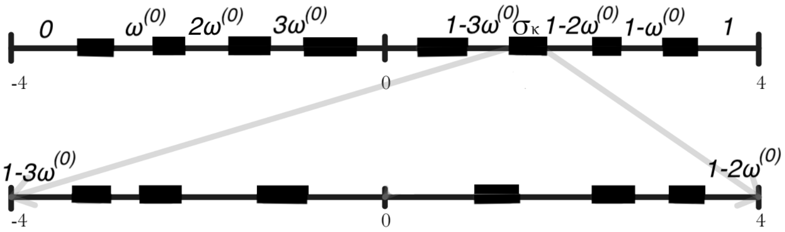

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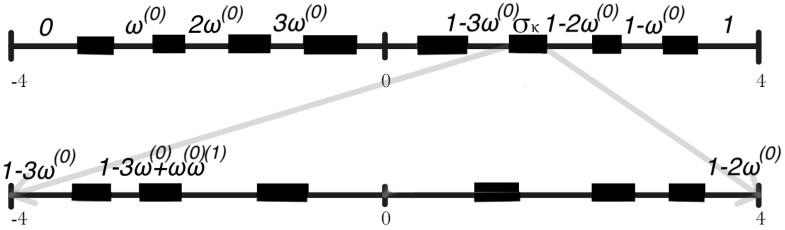

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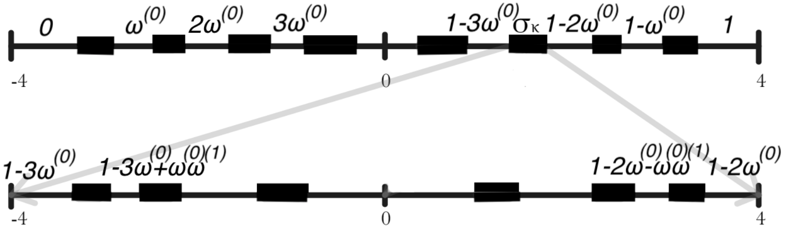

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Theorem (Fedotov, S., '19)

Let $q < 1/2$ *and* $N = k\omega \bmod 1, k \in \mathbb{Z}$. *Then on the first step*

$$0 < |k| \leq q\alpha_1;$$

on the second step

$$(1-q)\alpha_1 < |k| \leq q\alpha_1\alpha_2;$$

on the j-th step

$$(1-q)\alpha_1\alpha_2 \dots \alpha_{j-2} < |k| \leq q\alpha_1\alpha_2 \dots \alpha_{j-1}.$$