Discrete Skew Orthogonal Polymonials Related to Discrete Symplectic Ensemble

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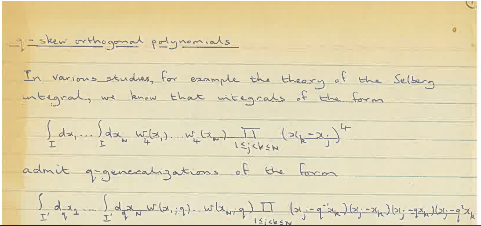
Content (arXiv: 1902.09042)

- Motivations
- Review of the continuous case
- Skew orthogonal polynomials related to the discrete symplectic ensemble on a linear lattice

Motivations

- It has been shown that the correlation kernel of orthogonal and symplectic ensemble on a linear lattice is a rank one perturbation of the correlation kernel of discrete unitary ensemble.
    - See Borodin and Strahov, *Correlation kernels for discrete symplectic and orthogonal ensembles*, Comm. Math. Phys., 2009.
- Motivated by the q-Selberg integral, Prof Forrester conjectured there should be some relations between q-skew orthogonal polynomials and q-orthogonal polynomials with Al-Saham & Carlitz weight 20 years ago.



q-skew orthogonal polynomials

In various studies, for example the theory of the Selberg integral, we knew that integrals of the form

$$\int_I dx_1 \cdots \int_I dx_N \, w_4(x_1) \cdots w_4(x_N) \prod_{1\le j<k\le N} (x_k - x_j)^4$$

admit q-generalizations of the form

$$\int_{I'} d_q x_1 \cdots \int_{I'} d_q x_N \, w(x_1;q) \cdots w(x_N;q) \prod_{1\le j<k\le N} (x_j - q^{-1}x_k)(x_j - x_k)(x_j - qx_k)(x_j - q^2 x_k)$$

Questions

1. Whether there are some discrete skew orthogonal polynomials behind discrete orthogonal ensemble and discrete symplectic ensemble?
2. What's the relationship between the discrete skew orthogonal polynomials and discrete orthogonal polynomials?
3. In terms of the relation, can we formulate the Christoffel-Darboux kernel of the discrete OE and SE as a rank one permutation of the kernel of discrete UE?

Proposition

Assume the logarithmic derivative of the weight function $\omega(x)$ *is a rational function, i.e.* $\frac{d}{dx}\log\omega(x) = -\frac{g(x)}{f(x)}$, *where deg* $f(x) \le 2$, *deg* $g(x) \le 1$, *then there exists an operator* $\mathcal{A} = f(x)\frac{d}{dx} + \frac{f'(x)-g(x)}{2}$, *such that* $\langle\phi, \mathcal{A}\psi\rangle_{2,\omega(x)} = -\langle\mathcal{A}\phi, \psi\rangle_{2,\omega(x)}$, $\langle\phi, \mathcal{A}\psi\rangle_{2,\omega(x)} = \langle\phi, \psi\rangle_{4,f(x)\omega(x)}$.

- Adler, Forrester, Nagao and van Moerbeke, *Classical skew orthogonal polynomials and random matrices*, J. Stat. Phys., 2000.
- $\langle\phi, \psi\rangle_{2,\omega} = \int_{\mathbb{R}} \phi(x)\psi(x)\omega(x)dx$

  $\langle\phi, \psi\rangle_{4,\omega} = \int_{\mathbb{R}} (\phi(x)\psi'(x) - \phi'(x)\psi(x))\omega(x)dx.$

The discrete symplectic ensemble on a linear lattice

Model (09' Borodin & Strahov):

- p.d.f:

  $\frac{1}{Z_N}\prod_{1\le i<j\le N}(x_i-x_j)^2(x_i-x_j+1)(x_i-x_j-1)\prod_{i=1}^N\omega(x_i)$

- configuration space:

  $\xi_N=\{(x_1,\cdots,x_N)|x_1<\cdots<x_N,\,x_i\in\mathbb{Z}\}$

- partition function:

  $Z_N=\sum_{\xi_N}\prod_{1\le i<j\le N}(x_i-x_j)^2(x_i-x_j+1)(x_i-x_j-1)\prod_{i=1}^N\omega(x_i)$

Discrete skew symmetric inner product

One can show:

$Z_N = Pf(A_{i,j})_{i,j=0}^{2N-1}$ with $A_{i,j} = \sum_{x\in\mathbb{Z}}[\pi_i(x)\pi_j(x+1) - \pi_i(x+1)\pi_j(x)]\omega(x)$,

where π_i is a monic polynomial of order i.

Discrete skew symmetric inner product: $\langle\cdot,\cdot\rangle_{s,\omega} : \mathbb{R}[x]\times\mathbb{R}[x]\to\mathbb{R}$

$$\begin{aligned}\langle\phi(x),\psi(x)\rangle_{s,\omega} &= \sum_{x\in\mathbb{Z}}[\phi(x)\psi(x+1) - \phi(x+1)\psi(x)]\omega(x)\\ &= \sum_{x\in\mathbb{Z}}[\phi(x)\Delta\psi(x) - \Delta\phi(x)\psi(x)]\omega(x).\end{aligned}$$

Moment matrix and discrete skew orthogonal polynomials

Consider the moment matrix $M=(m_{i,j})$, $m_{i,j}=\langle x^i,x^j\rangle_{s,\omega}$ and from the skew symmetric decomposition

$$\left[\begin{array}{cc} M_1 & -C^\top \\ C & M_2 \end{array}\right]=\left[\begin{array}{cc} I & \\ CM_1^{-1} & I \end{array}\right]\left[\begin{array}{cc} M_1 & \\ & M_2+CM_1^{-1}C^\top \end{array}\right]\left[\begin{array}{cc} I & -M_1^{-1}C^\top \\ & I \end{array}\right],$$

one can find a strictly lower triangular matrix S and a diagonal matrix (in 2×2 sense) $J=\mathrm{diag}\left\{\begin{pmatrix} 0 & u_0 \\ -u_0 & 0 \end{pmatrix},\begin{pmatrix} 0 & u_1 \\ -u_1 & 0 \end{pmatrix},\cdots\right\}$ such that $M=S^{-1}JS^{-\top}$.

Moment matrix and discrete skew orthogonal polynomials

Therefore, by denoting $\chi(x) = (1, x, x^2, \cdots)^\top$, one can define a family of skew orthogonal polynomials by $Q(x) = S\chi(x)$ and show $\langle S\chi(x), (S\chi(x))^\top \rangle_{s,\omega} = J$.

Moreover, the corresponding correlation kernel of the symplectic ensemble can be written as

$$\tilde{S}_N(x,y) = \begin{pmatrix} S_N(x,y) & \Delta_x S_N(x,y) \\ \Delta_y S_N(x,y) & \Delta_x \Delta_y S_N(x,y) \end{pmatrix}$$

where $S_N(x,y) = \sum_{i=0}^{N-1} \frac{1}{u_i} (Q_{2i}(x) Q_{2i+1}(y) - Q_{2i+1}(x) Q_{2i}(y))$.

Relationship between dSOPs and dOPs

- Key problem:

  For the symmetric inner product $\langle \phi(x), \psi(x) \rangle = \sum_{x\in\mathbb{Z}} \phi(x)\psi(x)\rho(x)$, how to seek an operator $\mathcal{A}_l$, such that $\langle \mathcal{A}_l\phi(x), \psi(x)\rangle = -\langle \phi(x), \mathcal{A}_l\psi(x)\rangle$?

- Solution:

  Discrete version of the 'Pearson-type' equation (See Nikiforov and Suslov, LMP, 1986): $\Delta[f(x_i)\rho(x_i)] = g(x_i)\rho(x_i)\Delta x_{i-1/2}$, where $x_i = x(i)$.

Proposition

There exists a first order difference operator $\mathcal{A}_l = g(x)T + f(x)(\Delta + \nabla)$, *such that*

$$\langle \mathcal{A}_l \phi(x), \psi(x) \rangle = -\langle \phi(x), \mathcal{A}_l \psi(x) \rangle;$$

$$\langle \phi(x), \mathcal{A}_l \psi(x) \rangle = \langle \phi(x), \psi(x) \rangle_{s,\rho(x+1)f(x+1)}.$$

Relationship between dSOPs and dOPs

From this relation between discrete symmetric inner product and skew symmetric inner product, one can find the relationship between discrete orthogonal polynomials and discrete skew orthogonal polynomials as

$$P_{2n+1}(x) = Q_{2n+1}(x), \quad P_{2n}(x) = Q_{2n}(x) = \frac{c_{2n-1}}{c_{2n-2}} Q_{2n-2}(x),$$

where c_{2n-1} depends on the Pearson pair (f, g) and index n, $c_{2n} = u_n$.

Example 1: Meixner case

- weight function: $\rho(x) = \frac{(\beta)_x}{x!} a^x$, $(\beta)_x = \beta \cdots (\beta - x + 1)$ with $\beta \in \mathbb{R}$ and $0 < a < 1$.
- Pearson pair: $(f, g) = (x, (a-1)x + a\beta)$.
- coefficients: $c_n = (1-a)h_{n+1}$, where h_n is the normalisation constant of monic Meixner polynomials.

Example 2: Hahn case

- weight function: $\rho(x) = \binom{\alpha+x}{x}\binom{N+\beta-x}{N-x}$, $\alpha > 0$, $\beta > 0$.
- Pearson pair: $(f, g) = \left(-x^2 + (N+\beta+1)x, -(\alpha+\beta+2)x + N(\alpha+1)\right)$.
- coefficients: $c_n = (n+\alpha+\beta+2)h_{n+1}$.

Thanks!