Dual formulations of Polyakov loop models and pure gauge LGT

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1. Duals of lattice models and sign problem
2. Polyakov loop models and their duals
3. Abelian models with static quarks
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I. Duals of lattice models and sign problem

Dual transformations are powerful analytical tool which allows to investigate many non-perturbative features of various lattice models. In particular, they may be useful in constructing positive definite Boltzmann weight for theories at finite chemical potential and/or in the presence of a theta term.

- $O(N)$ sigma models
- Principal chiral models
- Lattice gauge theory at finite baryon density

I review one particular approach to the duality transformations for LGT at non-vanishing chemical potential.

II. Polyakov loop models and their duals

Let $U_l \equiv U_\mu(x) \in Z(N), U(N), SU(N)$. LGT in $(d+1)$-dimensions on anisotropic periodic lattice $\Lambda = N_t \times L^d$ with N_f quark flavours

$$Z_\Lambda = \int \prod_l dU_l \ \exp \sum_{x;\mu<\nu} \beta_{\mu\nu} \, \mathrm{Re}\, \chi_f(U_p) \ \prod_{f=1}^{N_f} \ \mathrm{Det}\, Q(m_f, \mu_f)$$

$$U_p \ = \ U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x) \ .$$

$\chi_f(U)$ is the character of the fundamental representation. Staggered or Wilson fermion matrix (colour indices omitted and $\xi = a_t/a_s$)

$$Q(m_f, \mu_f) = V_{tt'}(x)\delta_{xx'} + \xi \sum_{n=-d}^{d} \Gamma_n \, U_n(x,t)\delta_{x+n,x'}\delta_{tt'} \ ,$$

$$V_{tt'}(x) = 2a_t m_f \delta_{tt'} + e^{a_t\mu_f} U_0(x,t)\delta_{t,t'-1} - e^{-a_t\mu_f} U_0^\dagger(x,t')\delta_{t,t'+1}$$

Polyakov loop

1. Definition of (traced) Polyakov loop:

$$W(x) = \mathrm{Tr} \prod_{t=1}^{N_t} U_0(x,t) \ .$$

2. Physical meaning (F_q - free energy of heavy quark):

$$\langle W(x) \rangle = e^{-\beta F_q} \ .$$

$\langle W(x) \rangle$ serves as an exact (approximate) order parameter of a deconfinement phase transition in pure (with dynamical quarks) gauge theory. It probes a realization of a global $Z(N)$ symmetry of the theory at finite temperature.

3. Two-point correlation: potential between quark–anti-quark pair.

4. N-point correlation: N-quark (baryonic) potential.

Strategy

- Integration over fermion fields.

- Integration over spatial gauge fields (usually requires some approximation for fermion determinant and/or Wilson gauge action). Resulting theory is an effective d-dimensional spin model of interacting Polyakov loops in the external complex (if $\mu \neq 0$) field.

- Construction of a dual representation for the effective spin model.

- If the dual Boltzmann weight is positive one uses the conventional MC to study the model.

Finite-temperature effective models

1. $\beta_s = 0$, $\beta_t < 1$, large quark masses

$$S = \beta \sum_{x,n} \mathrm{ReTr} W(x) \mathrm{Tr} W^{\dagger}(x+e_n) + \sum_x \left(h_r \mathrm{Tr} W(x) + h_i \mathrm{Tr} W^{\dagger}(x) \right)$$

- $\beta = \beta(\beta_t)$; $h_r = h_r(m_q, \mu)$ and $h_i = h_i(m_q, \mu)$ are functions of quark mass m_f and baryon chemical potential μ_f

$$h_r = \sum_f h(m_f)\, e^{\mu_f} \;,\; h_i = \sum_f h(m_f)\, e^{-\mu_f}$$

- Global symmetries when $h_r = h_i = 0$: $U(1)$ for $U(N)$ and $Z(N)$ for $SU(N)$

- The effective action is complex if $h_r \neq h_i$ ($\mu_f \neq 0$)

2. $\beta_s = 0$, arbitrary β_t, static quarks

$$Z = \int \prod_x dW(x) \prod_{x,n} \left[\sum_\lambda D_\lambda^{N_t}(\beta_t) \chi_\lambda(W(x)) \chi_\lambda(W^\dagger(x+e_n)) \right]$$
$$\times \prod_{f=1}^{N_f} \prod_x \mathrm{Det}\, \mathrm{V}_{\mathrm{tt'}}(\mathrm{x}) .$$

$D_\lambda(\beta)$ - coefficients of the character expansion.

3. Non-local Polyakov loop model by J. Greensite, R. Hollwieser

$$S = \beta \sum_{x,y} \mathrm{Re}\ \mathrm{Tr} W(x)\ K(x-y)\ \mathrm{Tr} W^\dagger(y) + S_q .$$

Quark contribution, S_q, can be taken as outlined above.

Dual formulation of the model 1

$$Z \;=\; \int \prod_x dW(x)\;\exp\left[S[\{W\}]\right]\;.$$

(for $SU(3)$ see: C. Gattringer, Nucl.Phys. B850 2011)

- Taylor expansion of the Boltzmann weight
- Group integration: $Q_N(s,\bar{s}) =$

$$\int dW\;(\mathrm{Tr}W)^s\;(\mathrm{Tr}W^*)^{\bar{s}} = \begin{cases}\delta_{s,\bar{s}}\sum_\lambda d^2(\lambda)\;,\;U(N)\\ \sum_k \delta_{\bar{s}-s,kN}\sum_\lambda\; d(\lambda)d(\lambda+k)\;,\;SU(N)\end{cases}$$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)$ is a partition of s, *i.e.* $\sum_{i=1}^N \lambda_i = s$.
$d(\lambda)$ - dimension of the permutation group S_s

$$d(\lambda) \;=\; s!\;\frac{\prod_{1\leq i<j\leq l(\lambda)}(\lambda_i-\lambda_j+j-i)}{\prod_{i=1}^{l(\lambda)}(\lambda_i+l(\lambda)-i)!}\;.$$

- For $Z(N)$: $Q_N(s,\bar{s}) = \sum_k \delta_{\bar{s}-s,kN}$.

- Dual transformations (exactly like in $U(1)$ and $Z(N)$ models)

- All above in the presence of sources for n-point correlations of the Polyakov loops

- After only few pages of manipulating with formulae one ends up with

The dual form of $SU(N)$ spin model (flux representation)

$$Z = \prod_l \left[\sum_{p(l)=-\infty}^{\infty} \sum_{q(l)=0}^{\infty} \left(\frac{\beta}{2}\right)^{|p(l)|+2q(l)} \frac{1}{(q(l)+|p(l)|)!q(l)!} \right]$$

$$\prod_x \left[\sum_{k(x)=-\infty}^{\infty} \sum_{t(x)=0}^{\infty} \frac{(h_r h_i)^{t(x)+\frac{1}{2}|m(x)|}}{t(x)!\,(t(x)+|m(x)|)!} \left(\frac{h_r}{h_i}\right)^{\frac{1}{2}m(x)} Q_N(s(x)) \right] ,$$

$$s(x) = \sum_{i=1}^{2d} \left(q(l_i) + \frac{1}{2}|p(l_i)| \right) + t(x) + \frac{1}{2} \sum_{n=1}^{d} (p_n(x) - p_n(x-e_n)) + \frac{1}{2}|m(x)| + \frac{1}{2} m(x) + \eta(x) ,$$

$$m(x) = \sum_{n=1}^{d} (p_n(x-n) - p_n(x)) - k(x) N + \bar{\eta}(x) - \eta(x) .$$

$U(N)$: only term $k(x) = 0$ contributes. Dependence on μ is cancelled from partition function and all invariant observables. Non-invariant ones depend on μ as $e^{-\mu \sum_x (\eta(x) - \bar{\eta}(x))}$. In $SU(N)$ invariants depend on μ as $e^{-\mu N \sum_x k(x)}$.

When external fields are vanishing $h_r = h_i = 0$, one gets $t(x) = m(x) = 0$. One performs duality transformations in any number of dimensions. *E.g.*, in $d = 2$ one finds the following expression on the dual lattice

$$Z = \sum_{r(x)=0}^{N-1} \sum_{k(l)=-\infty}^{\infty} \sum_{q(l)=0}^{\infty} \prod_p Q_N(s(p))$$
$$\prod_l \left[\frac{(\beta/2)^{|r(x)-r(x+n)+k(l)N|+2q(l)}}{(q(l)+|r(x)-r(x+n)+k(l)N|)!q(l)!} \right]$$

If $Q_N(s) = 1$, the remaining part is nothing but the dual of $Z(N)$ vector model. In three dimensions the original spin model is dual to an integer-valued gauge model.

The complex action problem is solved for all $U(N)$ and $SU(N)$ models in any dimension if the product $h_r h_i$ and the ratio h_r/h_i is non-negative.

Dual formulation of the model 2

- Sum over representations

$$\sum_{\lambda} F(\lambda) = \sum_{r=0}^{\infty} \sum_{\lambda \vdash r} F(\lambda) \ , \ \sum_{i=1}^{N} \lambda_i = r$$

- Static quark determinant (staggered fermions)

$$\mathrm{Det}\, \mathrm{V}_{\mathrm{tt'}}(\mathrm{x}) \sim \mathrm{Det}_{\mathrm{C}} \left(1 + \kappa_{+}^{\mathrm{f}} \prod_{\mathrm{t}=1}^{\mathrm{N_t}} \mathrm{U}_0(\mathrm{x},\mathrm{t}) \right) \left(1 + \kappa_{-}^{\mathrm{f}} \prod_{\mathrm{t}=1}^{\mathrm{N_t}} \mathrm{U}_0^{\dagger}(\mathrm{x},\mathrm{t}) \right)$$

$\kappa_{\pm} = \exp(-N_t \sinh^{-1} a_t m \pm \beta\mu)$. By Cauchy identity

$$\prod_{f=1}^{N_f} \mathrm{Det}\, \mathrm{V}_{\mathrm{tt'}}(\mathrm{x}) = \sum_{\mathrm{m,n}} \chi_{\mathrm{m}}(\mathrm{W}(\mathrm{x}))\, \chi_{\mathrm{n}}(\mathrm{W}^{*}(\mathrm{x}))\, \chi_{\mathrm{m'}}(\kappa_{+}^{\mathrm{f}})\, \chi_{\mathrm{n'}}(\kappa_{-}^{\mathrm{f}})$$

For one flavour it gives

$$\sum_{m,m'=0}^{N} \kappa_{+}^{m} \kappa_{-}^{m'} \chi_{1^m}(W(x))\, \chi_{1^{m'}}(W^{*}(x))$$

- Group integration

$$\int dW \prod_{n=1}^{d} \chi_{\lambda(x,n)}(W)\chi_{\lambda(x-e_n,n)}(W^*)\ \chi_{\mathbf{1}^{m(x)}}(W)\ \chi_{\mathbf{1}^{m'(x)}}(W^*)$$

can be done by expanding integrand into the Littlewood-Richardson (LR) coefficients

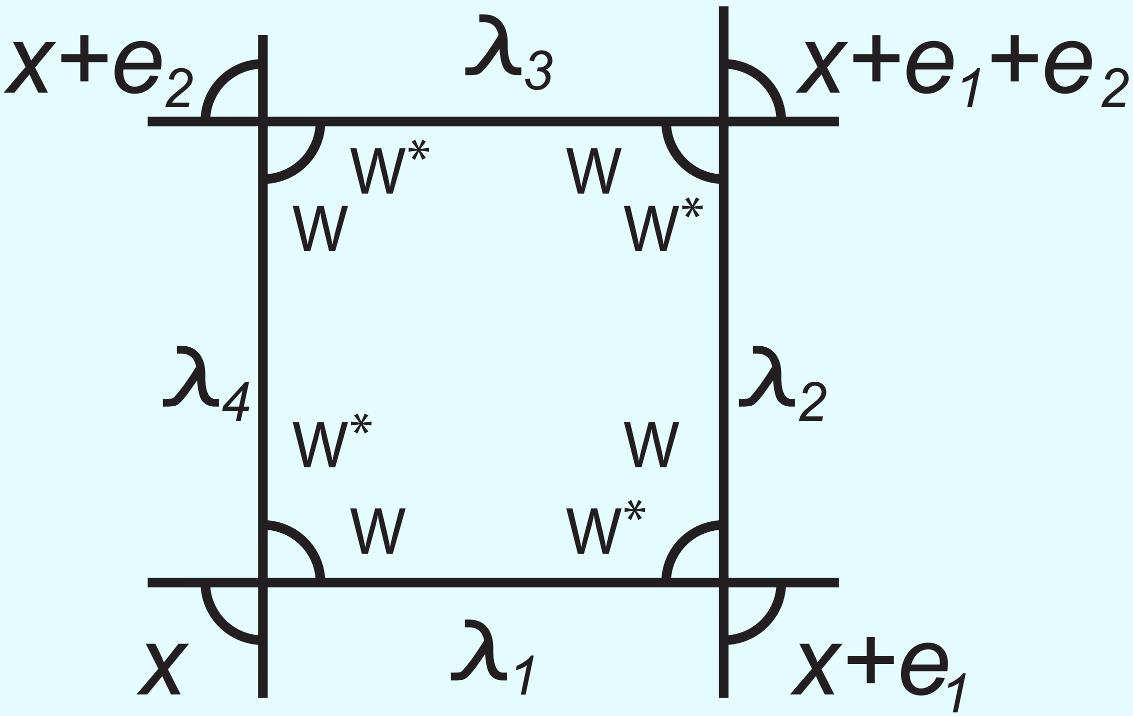
$$\chi_{\lambda_1}\ \chi_{\lambda_2} = \sum_{\lambda} C^{\lambda}_{\lambda_1\ \lambda_2}\ \chi_{\lambda}$$

LR coefficients are positive integers (multiplicities) given by

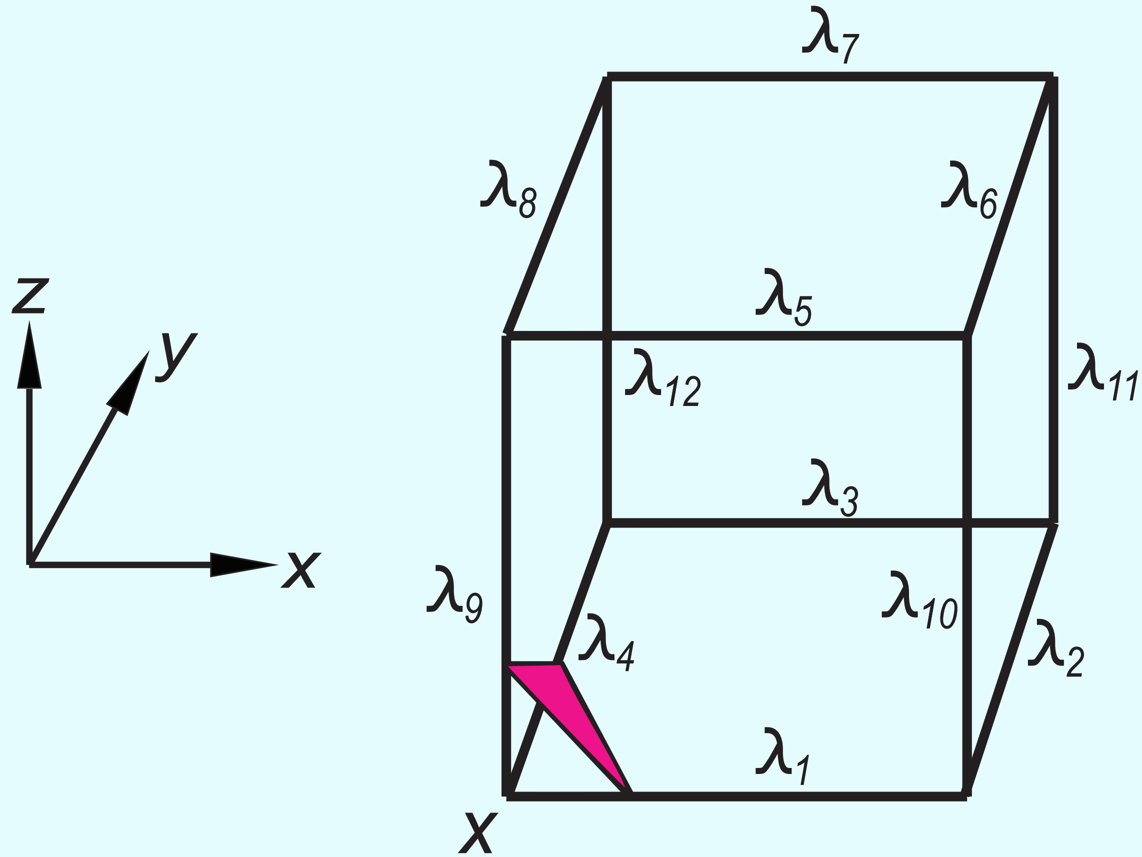
$$C^{\lambda}_{\lambda_1\ \lambda_2} = \int dW \chi_{\lambda_1}(W)\ \chi_{\lambda_2}(W)\chi_{\lambda}(W^*)$$

For $SU(N = 2, 3)$ exact formulae are known.

- Special coupling of characters as shown below for $d = 2, 3$



Multiplication of characters in the group integrals can be done such that all summations over representations are closed inside even plaquettes, $d = 2$



Multiplication of characters in the group integrals can be done such that all summations over representations are closed inside even cubes, $d = 3$

Partition function with positive weight

$$Z = \sum_{m(x)=0}^{N} \sum_{m'(x)=0}^{N} \sum_{r_l=0}^{\infty} \sum_{\rho(x)} \prod_{x} \sum_{k=-\infty}^{\infty} \delta(G(x) - kN)$$
$$\times \prod_{x} \kappa_{+}^{m(x)} \kappa_{-}^{m'(x)} \prod_{p,even} B_p$$

$$B_p = \sum_{\lambda_1 \vdash r_1} D_{\lambda_1}^{N_t}(\beta) ... \sum_{\lambda_4 \vdash r_4} D_{\lambda_4}^{N_t}(\beta) \sum_{\mu_1} ... \sum_{\mu_4} C^{\mu_1}_{\lambda_1 \lambda_4} C^{\mu_2}_{\lambda'_1 \lambda_2} C^{\mu_3}_{\lambda'_2 \lambda'_3} C^{\mu_4}_{\lambda_3 \lambda'_4}$$
$$\times C^{\rho(x)}_{\mu_1 1^{m(x)}} C^{\rho(x+e_1)}_{\mu_2 1^{m'(x+e_1)}} C^{\rho(x+e_1+e_2)}_{\mu_3 1^{m'(x+e_1+e_2)}} C^{\rho(x+e_2)}_{\mu_4 1^{m(x+e_2)}}$$

$$G(x) = \sum_n (r_n(x) - r_n(x - e_n)) + m(x) - m'(x)$$

For $SU(N)$ the dependence on chemical potential appears as $e^{\beta \mu N \sum_x k(x)}$.
For $U(N)$ the dependence drops out.

III. Abelian models with static quarks

Can we avoid approximations used so far: 1) $\beta_s = 0$ and 2) static quarks? For abelian models we do not need $\beta_s = 0$ and can use the full Wilson or any other local action

$$\sum_{r(p)=-\infty}^{\infty} D_{r(p)} \, e^{ir(p)\phi(p)}$$

The underlying reason for that is we know exact and positive dual form of any abelian $U(1)$ and $Z(N)$ pure gauge theory in any dimension. Adding static fermions does not violate positivity of the dual Boltzmann weight even at finite density. Fermion determinant for abelian models

$$\sum_{m(x)=0}^{1} \ \sum_{m'(x)=0}^{1} \ \prod_x \kappa_+^{m(x)} \ \kappa_-^{m'(x)} \ W^{m(x)-m'(x)}(x)$$

Partition function is expressed in terms of fermion numbers $m(x), m'(x)$ and plaquette occupation numbers (their number can be reduced by solving constraints on all plaquettes but p_t in one fixed time slice)

$$Z = \sum_{m(x)=0}^{1} \sum_{m'(x)=0}^{1} \prod_x \kappa_+^{m(x)} \kappa_-^{m'(x)} \sum_{r(p)} \prod_{p_s} D_{r(p_s)}(\beta_s) \prod_{p_t} D_{r(p_t)}(\beta_t)$$

$$\prod_{l_s} \delta\left(\sum_{p\in l_s} r(p)\right) \prod_{l_t} \delta\left(\sum_{p\in l_t} r(p) + m(x) - m'(x)\right)$$

- The final form depends on the model, $Z(N)$ or $U(1)$, and on the space dimension (solution of constraints). Dual weight is always positive.
- For $Z(N)$ the dependence on chemical potential appears as $e^{\beta\mu N \sum_x k(x)}$. For $U(1)$ the dependence drops out.
- Can be trivially extended to any number of flavours and to Wilson fermions.

IV. Pure LGT and two-dimensional QCD

- Dual representations based on the plaquette formulation. Dual variables are introduced as variables conjugate to local Bianchi identities. The dual model is non-local due to the presence of connectors

- Dual representations based on 1) the character expansion of the Boltzmann weight and 2) the integration over link variables using Clebsch-Gordan expansion

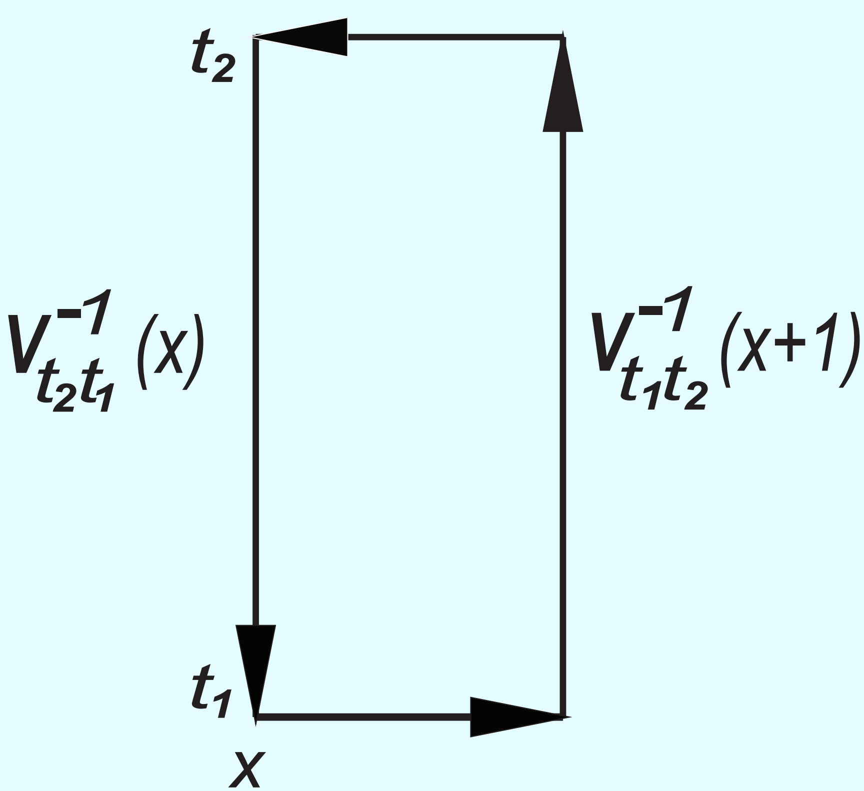
$$Z = \sum_{r_p, r_l} \prod_p C_{r_p}(\beta_{\mu\nu}) \prod_x (6j \text{ links}) \prod_{\text{C}} (\text{6j cubes})$$

- Other approaches: n-link action, abelian colour cycles, Weingarten functions, $\cdots$

Corrections to the static quarks can be systematically computed by expanding the fermion determinant as

$$\mathrm{Det}\ \mathrm{Q}(\mathrm{m},\mu) = \mathrm{Det}\mathrm{V} \sum_{\mathrm{s}=0}^{\mathrm{L}\mathrm{N}_\mathrm{t}\mathrm{N}/2} \frac{\xi^{2\mathrm{s}}}{(2\mathrm{s})!} \sum_{\sigma\in \mathrm{S}_{2\mathrm{s}}} \mathrm{sgn}(\sigma)\mathrm{P}_\sigma(\mathrm{V}^{-1}\mathrm{H})\ .$$

$P_\sigma(X)$ is power sum symmetric function of the matrix argument. E.g., the leading correction is sum of all graphs of the form



Some results for two-dimensional theory

- Leading order corrections, ξ^2, are all positive both for abelian and non-abelian models.

- $U(1)$: Loops containing odd number of sites at ξ^4 order give negative contribution.

- Sum of all loop configurations give presumably positive contribution (so far proven only for restangular loops).

If last property holds for loops of all forms, one can extend result by Gattringer et.al. for the massless Schwinger model to the massice fermions.

V. Applications of the dual formulation

1. Exact large-N solution

2. Polyakov loop correlations: oscillatory behaviour

Large-N solution

- Exact solution of the two-dimensional $U(N)$ LGT in the large-N limit (D. Gross, E. Witten, Phys.Rev. D21 1980; S. Wadia, Phys.Lett. B93 1980) can be constructed if $\lambda = \beta/N = (g^2 N)^{-1}$ is fixed ('t Hooft limit)

$$Z_{2d} = \int dU \ \exp\left[\frac{\beta}{2}(\mathrm{Tr}U + \mathrm{Tr}U^{\dagger})\right] \ .$$

- Mean-field solution of $U(N)$ Polyakov loop model in the large-N limit at $\mu = 0$ (P. Damgaard, A. Patkos, Phys.Lett. B172, 1986) and at $\mu \neq 0$ (C. H. Christensen, Phys.Lett. B714 2012).

Third order phase transition has been found in both cases.

The following relations hold between one-link integral and function $Q_N(s)$

$$\sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s} \frac{1}{(s!)^2} Q_N(s) = \mathrm{det} I_{i-j}(x)$$

for $U(N)$

$$\sum_{s,\bar{s}=0}^{\infty} \left(\frac{x}{2}\right)^{s+\bar{s}} e^{(s-\bar{s})\mu} \frac{1}{s!\bar{s}!} Q_N(s,\bar{s}) = \sum_{k=-\infty}^{\infty} e^{-kN\mu} \mathrm{det} I_{i-j+k}(x)$$

for $SU(N)$. They can be used to derive various expansions of $Q_N(s)$ functions at large N and/or s.

$U(N)$: for any N and $s \leq N$

$$Q_N(s) \; = \; s! \; .$$

$SU(N)$: for any N and $s \leq N$, $\bar{s} = s + kN, k \geq 0$

$$Q_N(s, \bar{s}) \; \approx \; \frac{(s + kN)!}{(N!)^k} \; .$$

$SU(N)$: for large s and $\bar{s} = s + kN$, $k \geq 0$

$$Q_N(s, \bar{s}) \asymp \frac{G(N+1)}{(2\pi)^{(N-1)/2}} \frac{N^{2s+kN+N^2/2}}{\Gamma\left((N^2-1)/2\right)} B\left(2s + kN + 3/2, (N^2-1)/2\right).$$

$k = 0$ gives leading term for $U(N)$. $G(m)$ is the Barnes function, $B(a, b)$ is the Beta-function.

A. Strong-coupling phase: $d\beta < 1$

In this region the spin model can be exactly mapped onto the following Gaussian-like partition function ($h_{\pm} = h_r \pm h_i$)

$$Z = \int_{-\infty}^{\infty} \prod_x d\alpha_x d\sigma_x \ \exp\left[-\alpha_x G^{-1}_{xy}\alpha_y - \sigma_x G^{-1}_{xy}\sigma_y + \sum_x (h_+\alpha_x - ih_-\sigma_x)\right]$$
$$\times \prod_x \left[1 + \frac{2}{N!}\,\mathrm{Re}\,(\alpha_x + i\sigma_x)^N\right]$$

with the Green function (in thermodynamic limit)

$$G_{xy} = \int_0^{2\pi} \left(\frac{d\phi}{2\pi}\right)^d \frac{e^{i\phi_n (x-y)_n}}{1 - \beta\sum_{n=1}^d \cos\phi_n}$$

The 1st line is the large-N limit of $U(N)$, the 2nd line presents first non-trivial correction from $SU(N)$.

Two different large-N limits can be constructed in this region.

Let $\tilde{h}_r = h_r/N$ and $\tilde{h}_i = h_i/N$ be fixed. Then, the free energy is calculated as

$$F = \lim_{N\to\infty} \lim_{L\to\infty} \frac{1}{N^2 L^d} \log Z = \frac{\tilde{h}_r \tilde{h}_i}{1 - d\beta}$$

This solution coincides with the mean-field solution for the $U(N)$ spin model.

$$F = \lim_{N\to\infty} \lim_{L\to\infty} \frac{1}{L^d} \log Z \ = \ \int_0^{2\pi} \left(\frac{d\phi}{2\pi}\right)^d \log[1 - \beta \sum_{n=1}^{d} \cos\phi_n]$$

$$+ \ \frac{h_r h_i}{1 - d\beta} + \frac{1}{N!} V_{SU(N)} \ ,$$

$$V_{SU(N)} \ = \ \frac{h_r^N + h_i^N}{(1 - d\beta)^N} \ .$$

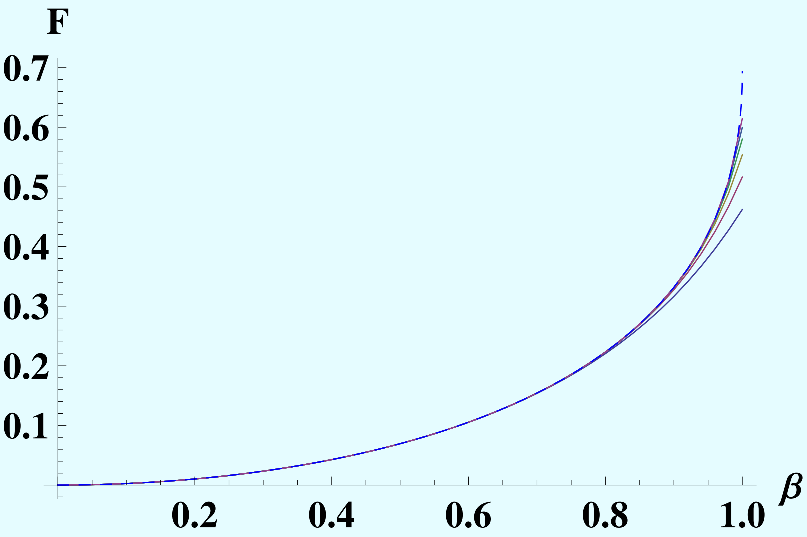
Here is a deconfinement phase transition at $d\beta = 1$ characterised by:

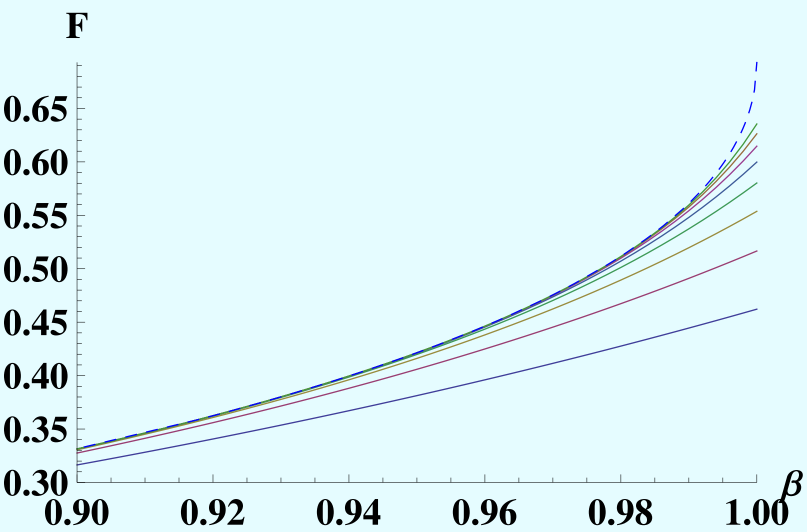
1. Vanishing mass gap

$$m(\beta) = (1 - d\beta)^\nu \ , \ \ \nu = \frac{1}{2}$$

2. Growing baryon density

$$B \ = \ \frac{\partial}{\partial\mu} F \ = \ \frac{1}{(N-1)!} \left(\frac{h}{1 - d\beta}\right)^N \sinh\mu N \ .$$





Exact $N = \infty$ solution *vs* finite $N = 3 - 10$ solutions for one-dimensional theory obtained via transfer matrix method. $\beta < 1$, $h_r = h_i = 0$

B. Weak-coupling phase: $d\beta > 1$

$$Z_N(\beta, h, \mu) \;=\; \prod_x \left[\frac{G(N+1)N^{N^2/2}}{\Gamma\Big((N^2-1)/2\Big)}\right] \sum_{k_x=-\infty}^{\infty} e^{-N\mu k_x}$$
$$\times \;\; \int_0^1 \prod_x dt_x \int_0^{2\pi} \prod_x \frac{d\phi_x}{2\pi} \; e^{N^2 \, S_{eff} + iNk_x\phi_x}$$

$$S_{eff} = \beta \sum_{x,n} t_x t_{x+n} \cos(\phi_x - \phi_{x+n}) + \sum_x \Big(h t_x \cos\phi_x + \frac{1}{2}\log(1 - t_x) \Big)$$

- $U(N)$: S_{eff} is an action of the d-dimensional XY model with fluctuating positive coupling

- $SU(N)$: S_{eff} is an action of the d-dimensional $Z(N)$ vector model with fluctuating positive coupling

Free energy

$$F_N(\beta,h,\mu) = \lim_{L\to\infty} \frac{1}{L^d N^2} \log Z_N(\beta,h,\mu) = F_0 + \frac{1}{N^2} F_1 + F_{SU(N)}$$

Large-N limit is given by (t_s is a saddle-point solution)

$$F_0 = \beta d t_s^2 + h t_s + \frac{1}{2}\log(1-t_s) - \frac{1}{4} + \frac{1}{2}\log 2$$

Corrections due to fluctuations

$$F_1 = \frac{1}{2}(\log \mathrm{Det} M_{xy} \ + \log \mathrm{Det} G_{xy})$$

Corrections from $SU(N)$ (depend on chemical potential)

$$F_{SU(N)} = \frac{1}{L^d N^2} \log \left[\sum_{k_x} \ e^{-\frac{1}{4}k_x G_{xy} k_y - N\mu \sum_x k_x} \right] \sim \frac{h t_s \mu^2}{2}, \beta t_s >> 1$$

$$G_{xy} \ = \ \int_0^{2\pi} \ \left(\frac{d\phi}{2\pi}\right)^d \frac{e^{i\phi_n (x-y)_n}}{h t_s/2 + \beta t_s^2 (d - \sum_{n=1}^d \cos\phi_n)}$$

When $\beta=\mu=0$, F_0 coincides with Gross-Witten and Wadia solution for the one-plaquette integral in the large-N limit.

Two-point correlation function

$$\Gamma_N(\beta,h,\mu)=t_s^2\,\mathrm{exp}\left[\frac{1}{2t_s^2N^2}(M_0+M_R)-\frac{1}{2N^2}(G_0-G_R)\right]$$

The large-N limit is trivial: $\Gamma = t_s^2$. At large but finite N the properties of Γ depend on the dimension and presence/absence of external field h.

- M_R decays exponentially in any dimension: $M_R \sim e^{-mR}$
- $D_R = G_0 - G_R$ decays exponentially in any dimension if $h \neq 0$ and is bounded from above for $d \geq 3$ if $h = 0$. In these two cases the quark-antiquark potential is screened.
- If $d = 2$ and $h = 0$, $D(R) \sim \log R$. The whole correlation decays algebraically. This property hints on a BKT phase transition in the system.

Solution of the Greensite-Hollwieser model

The same strategy can be applied to solve ANY Polyakov loop model if its action depends only on the fundamental character of the group. *E.g.*,

$$S = \beta \sum_{x,y} \text{Re Tr}U(x)\, K(x-y)\, \text{Tr}U^\dagger(y) + \sum_x \Big(h_r \text{Tr}U(x) + h_i \text{Tr}U^\dagger(x)\Big)$$

1. $U(N)$, $\beta K(0) < 1$:

$$F = -\frac{1}{L^d} \log \text{Det}(\delta_{x,y} - K(x-y)) + h_r h_i \sum_x (\delta_{x,0} - K^{-1}(x))^{-1}$$

2. $U(N)$, $\beta K(0) > 1$:

$$S_{eff} = \beta \sum_{x,y} t_x t_y \cos(\phi_x - \phi_y)\, K(x-y) + \sum_x \left(h t_x \cos\phi_x + \frac{1}{2}\log(1-t_x)\right)$$

Polyakov loop correlations: search for oscillatory behaviour

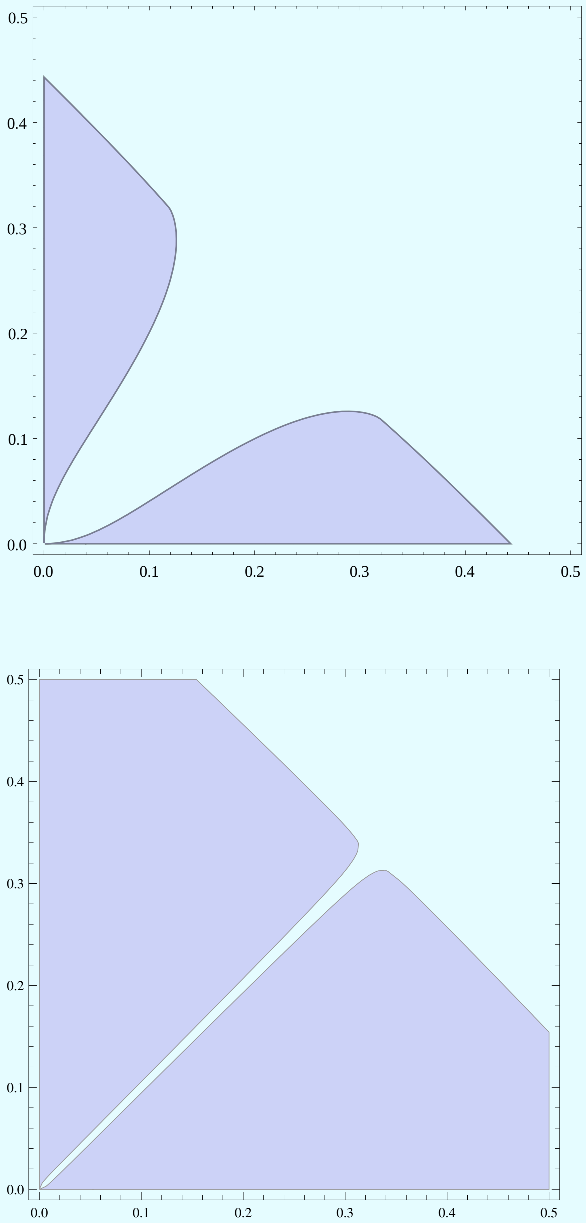
$$\langle W(x)W^*(y)\rangle \sim e^{-m_r R} \cos m_i R$$

- Theoretical arguments advanced by M. Ogilvie et.al (2014,2015).

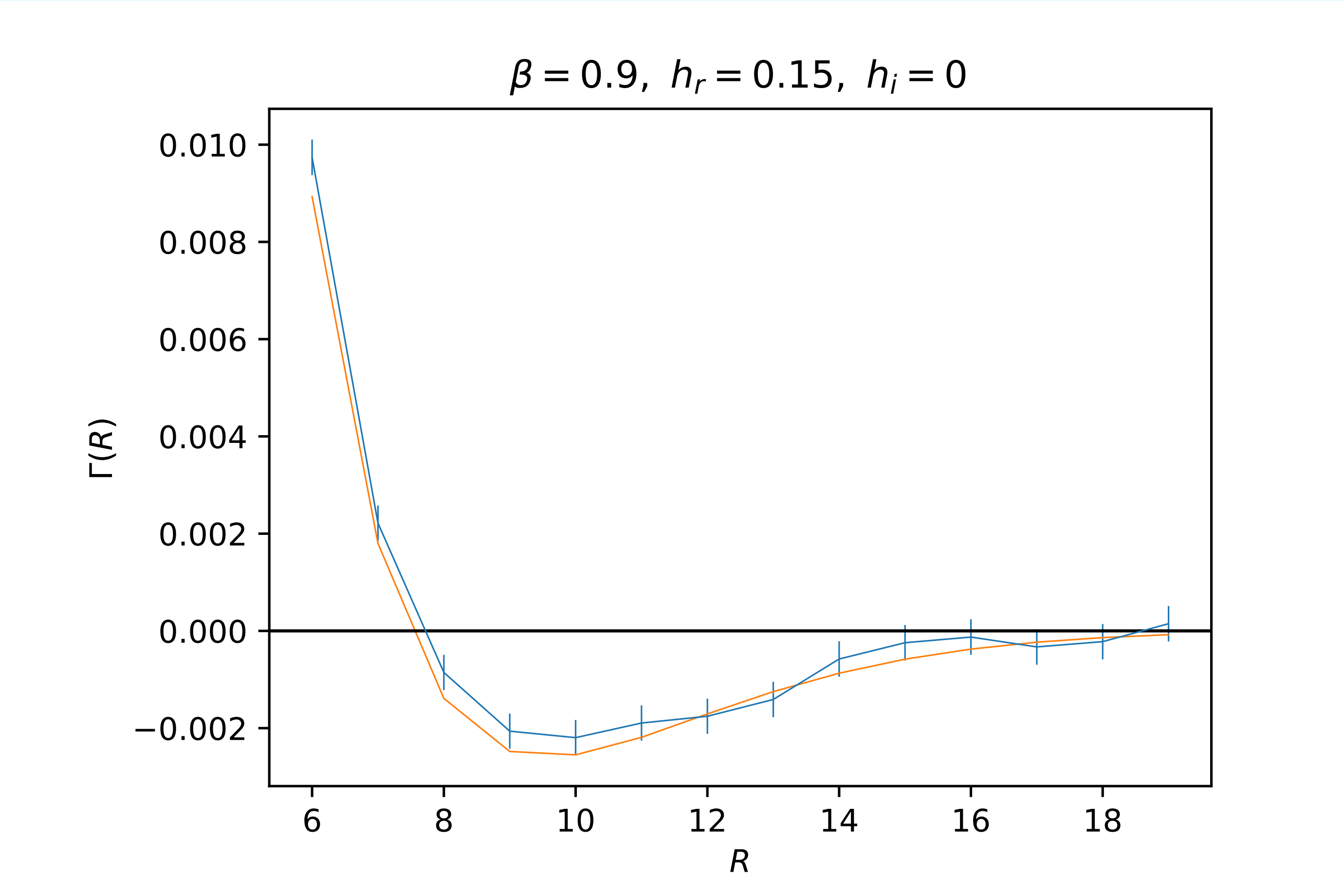
- $Z(3)$ spin model in external complex field studied by Ph. de Forcrand et.al (2016)

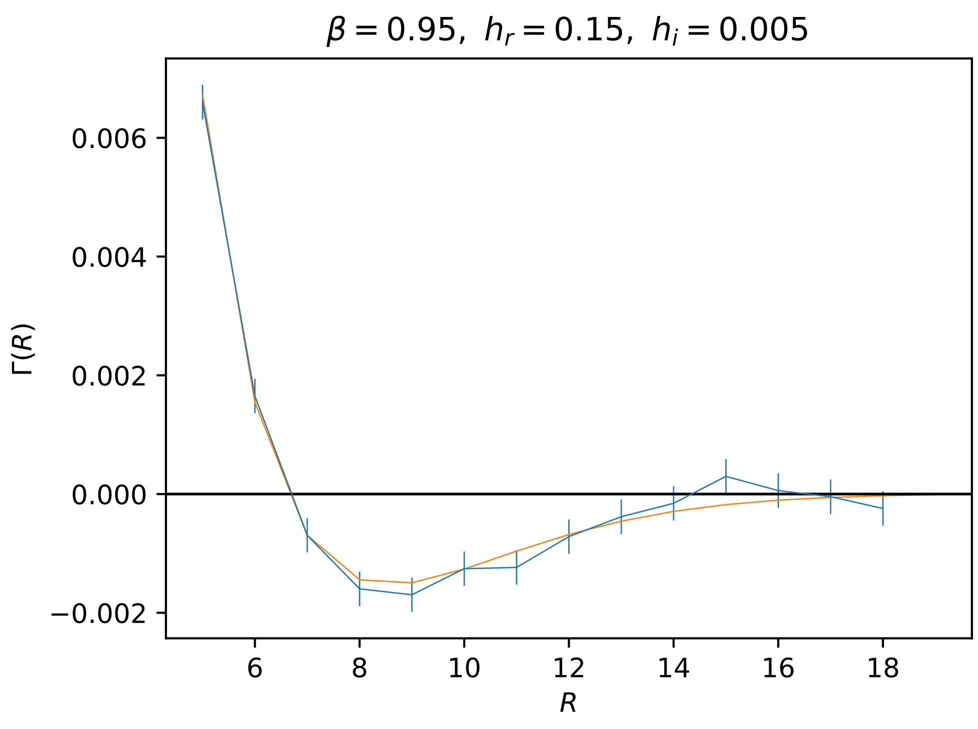
Preliminary results

- Transfer matrix applied to the dual of one-dimensional Polyakov loop model exhibits existence of oscillating phase
- Large-N solution also confirms such behaviour
- Monte-Carlo simulations for three-dimensional model are in progress



Regions of oscillating phase for $SU(3)$, $\beta = 0.9$ (above), $\beta = 1.2$ (below). $h_r = h(m)\,\exp(\mu), h_i = h(m)\,\exp(-\mu)$.


$\beta = 0.9,\ h_r = 0.15,\ h_i = 0$
0.010
0.008
0.006
0.004
0.002
0.000
−0.002
$\Gamma(R)$
6
8
10
12
14
16
18
R



$\beta = 0.95,\ h_r = 0.15,\ h_i = 0.005$
0.006
0.004
0.002
0.000
−0.002
Γ(R)
6
8
10
12
14
16
18
R

VI. Summary

- Dual formulation is constructed for all $U(N)$ and $SU(N)$ Polyakov loop models
- Dual Boltzmann weight is positive in the presence of baryonic chemical potential for all N
- Exact solution is given in the large-N limit
- Numerical simulations: liquid phase at finite-density (oscillatory behaviour of the Polyakov loop correlators)
- Dual formulation of abelian models
- Corrections to the static quark contribution