Low temperature condensation and scattering data

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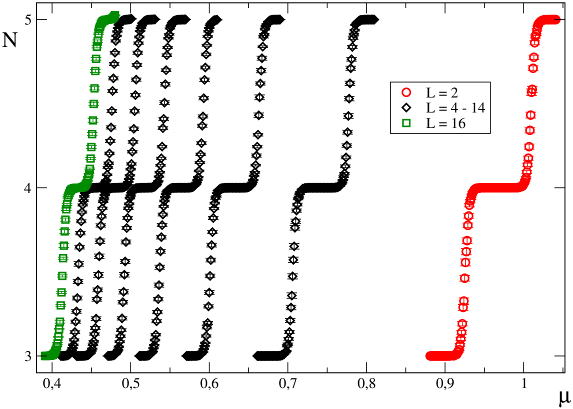
C. Gattringer, M. Giuliani, O. Orasch, PRL 120, 241601 (2018)

Motivation: Particle Number vs. μ

- Typical scenario at low T (fixed) and finite V $(= L^{d-1})$:



- Nature of condensation steps?
- Condensation thresholds $\mu_c^i \Leftrightarrow$ low energy parameters

Our approach

- We study the low temperature regime of a QFT at finite density
- For small spatial lattices particle sectors are separated by finite energy steps $\Rightarrow$ particle condensation
- Critical chemical potentials μ_c^i of these transitions are related to the minimal N-particle energies W_N
- At finite volume scattering information is encoded in W_N
- We measure the μ_c^i with worldline simulations and try to extract low energy scattering parameters

The framework

- Model: complex field with ϕ^4 interaction in 2D and 4D

$$S = \sum_{n \in \Lambda} \left(\eta |\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\nu=1}^{d} \left[\mathrm{e}^{\mu \delta_{\nu,d}} \phi_n^* \phi_{n+\hat{\nu}} + \mathrm{e}^{-\mu \delta_{\nu,d}} \phi_n^* \phi_{n-\hat{\nu}} \right] \right)$$

with $\eta \equiv 2d + m_0^2$ and $\phi_n \in \mathbb{C}$

- Sign problem for $\mu \neq 0 \Rightarrow$ Monte Carlo not possible

- Worldline rep. with real and positive weights solves sign problem

Worldline representation for the complex ϕ^4 field

- In the worldline approach the grand canonical partition sum is exactly rewritten in terms of dual link variables $k_{n,\nu} \in \mathbb{Z}$

$$Z = \sum_{\{k\}} \mathrm{e}^{\mu\beta W_t[k]} \; B[k] \; C[k]$$

- $W_t[k]$ = temporal winding number of the worldlines

- Real and positive weight factors $B[k]$

- Constraints $C[k] = \prod_n \delta(\vec{\nabla} \cdot \vec{k}_n)$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{k}_n = \sum_{\nu} (k_{n,\nu} - k_{n-\hat{\nu},\nu}) = 0 \;\; \forall n$$

C. Gattringer, T. Kloiber, Nucl.Phys. B869, 56 (2013)

Configurations and worldline simulations

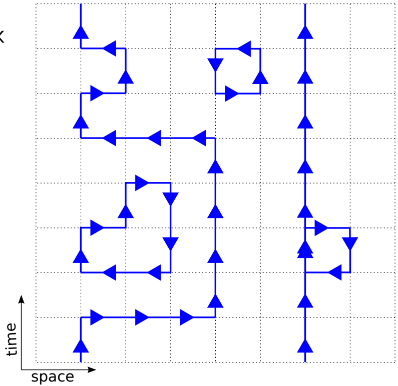
Constraints $C[k]$ yield vanishing divergence condition:

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{k}_n = \sum_{\nu} (k_{n,\nu} - k_{n-\hat{\nu},\nu}) = 0 \;\; \forall n$$

- Admissible configuration for the k-flux are closed and oriented loops

- Observable: $\langle N \rangle(\mu) = \langle W_t[k] \rangle$

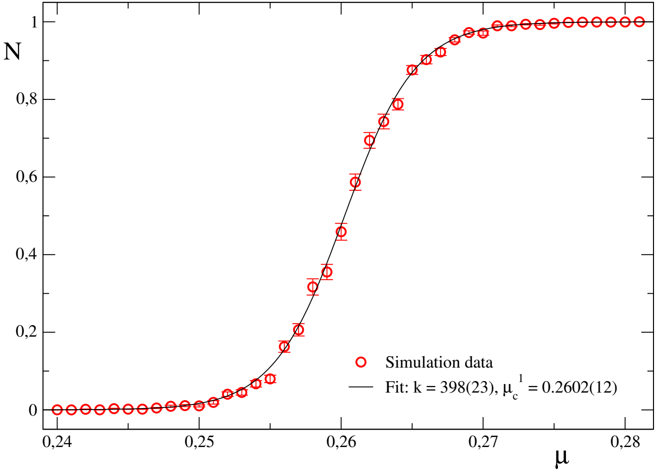
- MC with adapted worm algorithm
  M. Giuliani, C. Gattringer, arXiv:1702.04771



Measuring the condensation thresholds (2D case)

Fit the steps to a logistic function $\Rightarrow$ μ_c^i is inflection point

$$N(\mu) = (i-1) + \left[1 + e^{-k(\mu-\mu_c^i)}\right]^{-1}$$



Interpretation of the condensation steps

Grand canonical partition sum

$$Z = \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{Q})} = e^{-\beta\Omega(\mu)}$$

Low T: Z will be governed by the minimal grand potential $\Omega(\mu)$ in each particle sector

$$\Omega(\mu) \xrightarrow{T\to 0} \begin{cases} \Omega_{min}^{N=0} = 0, & \mu \in [0, \mu_c^1) \\ \Omega_{min}^{N=1} = W_1 - 1\mu, & \mu \in (\mu_c^1, \mu_c^2) \\ \Omega_{min}^{N=2} = W_2 - 2\mu, & \mu \in (\mu_c^2, \mu_c^3) \\ \ldots, \end{cases}$$

with renormalized mass $W_1 \equiv m$, minimal 2-particle energy W_2, ...

Bruckmann *et al.*, PRL 115, 231601 (2015)

Interpretation of the condensation steps

- Condensation thresholds are related to the minimal N-particle energies:

$$
\begin{aligned}
W_1 &= \mu_c^1 \equiv m \\
W_2 &= \mu_c^1 + \mu_c^2 \\
W_3 &= \mu_c^1 + \mu_c^2 + \mu_c^3 \\
&\vdots \\
W_N &= \sum_{i=1}^{N} \mu_c^i
\end{aligned}
$$

- W_N depend on low energy parameters (LEP)

- Describe condensation thresholds in terms of LEP

Important cross-check with conventional spectroscopy

- $\mu = 0 \Rightarrow$ Monte Carlo in conventional representation
- We compute the connected $2N$-point functions

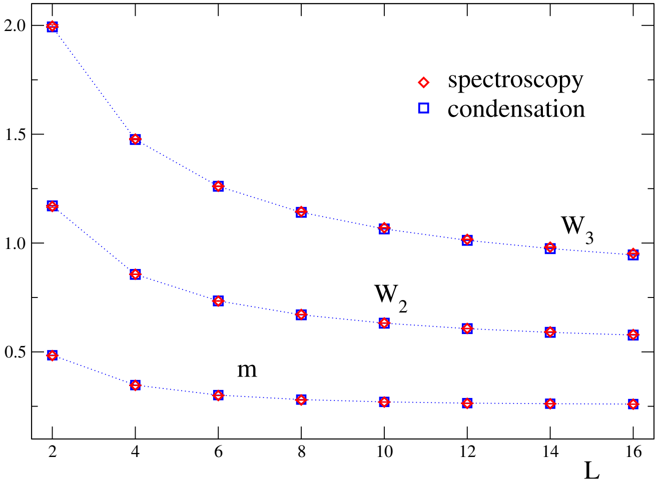
$$\left\langle (\tilde{\phi}_t)^N (\tilde{\phi}_0^*)^N \right\rangle_c \propto e^{-tE_N}$$

with $E_1 = m$, $E_2 = W_2$, $E_3 = W_3$, ...

$\Rightarrow$ fields are projected to zero momentum

- Extract the N-particle energies from the exponential decay of correlators

Spectroscopy versus WL simulations (2D case)



$\Rightarrow$ Interpretation of condensation steps as m, W_2, W_3 confirmed!

L-dependence of N-particle energies (4D case)

$$m = m_\infty + \frac{A}{L^{\frac{3}{2}}} e^{-Lm_\infty}$$

Rummukainen & Gottlieb 1995

$$W_2 = 2m + \frac{4\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 - \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right]$$

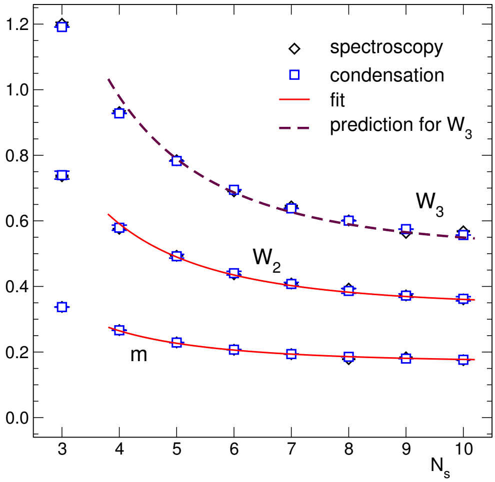
Huang & Yang 1957, Lüscher 1986

$$W_3 = 3m + \frac{12\pi a}{mL^3}\left[1 - \frac{a}{L}\frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 + \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3\right]$$

Beane *et al.* 2007, Hansen & Sharpe 2014, 15, 16, Sharpe 2017

- Infinite-volume mass m_∞
- Scattering length a $\left[\delta(k) = \delta(0) - ak + \mathcal{O}(k^2)\right]$
- Numerical constants $\mathcal{I} = -8.914$ and $\mathcal{J} = 16.532$

Results for 4D



- $m_\infty = 0.168(1)$ and $a = -0.078(7)$
- Good "prediction" of W_3 except for very small L ($\equiv N_s$)

C. Gattringer, M. Giuliani, O. Orasch, PRL 120, 241601 (2018)

Scattering data in 2D

In 2D the full scattering phase shift can be determined from the periodic boundary condition: M. Lüscher, U. Wolff, Nucl. Phys. B 339, 222 (1990)

$$e^{2i\delta(k)} = e^{-ikL}$$

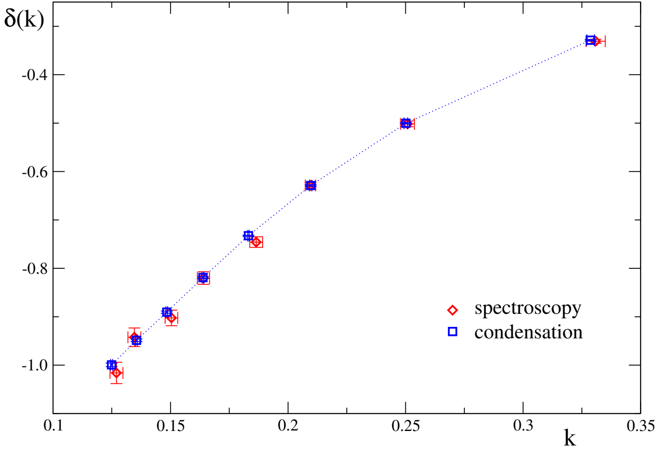
We assume a short-range interaction

$$W_2 = 2\sqrt{m^2 + k^2} \quad \Rightarrow \quad k(L) = \sqrt{\left(\frac{W_2(L)}{2}\right)^2 - m(L)^2}$$

For fixed m_0 and λ the momentum depends only on the lattice size L

$$\delta(L) = \delta(k(L)) = -\frac{k(L)L}{2}$$

Scattering phase shift in 2D



- Very good agreement of spectroscopy and worldline results

3-particle energy in 2D

In 4D: W_3 only depends on $m(L)$ and the scattering length a

We repeat the discussion as in the W_2 case

$$W_3 = \sum_{k=1}^{3} \sqrt{p_k^2 + m^2} \ \text{with} \ \ p_3 = -p_1 - p_2$$

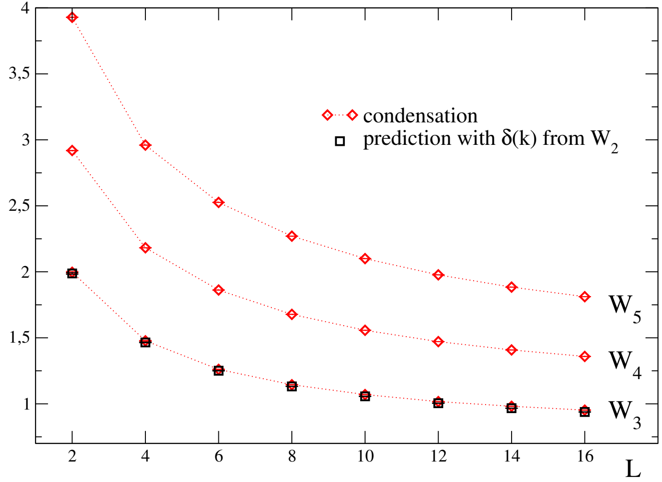
For p_1 and p_2 we employ the quantization conditions

$$e^{-ip_k L} = e^{2i\delta(L)}$$

Then W_3 is completely determined by $m(L)$ and $\delta(L)$

$$W_3(L) = 2\sqrt{\left(\frac{2\delta(L)}{L}\right)^2 + m(L)^2} + \sqrt{\left(\frac{4\delta(L)}{L}\right)^2 + m(L)^2}$$

N-particle energies in 2D



- Good prediction of W_3 with $\delta(k)$ from W_2
- Second and third condensation steps are understood in terms of the scattering phase shift $\delta(k)$
- Higher N-particle energies: work in progress

Summary

- Low temperature study of the ϕ^4 model at finite density with a worldline representation
- Small V: particle number exhibits step-like behaviour
- The N-particle energy is the sum of the μ_c^i, $i = 1, \ldots, N$
- Scattering parameters can be extracted from W_N
  - 4D: scattering length a
  - 2D: scattering phase shift $\delta(k)$
- Leading condensation thresholds are determined by scattering data

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- Leading condensation thresholds are determined by scattering data

Future work

- Better understanding of higher N-particle energies in 2D (in terms of low energy parameters)
- Large volume limit

Thank you for listening!

Backup slides

Simulation parameters

- Parameters for 4D simulations:
  - $\eta = 8 + m_0^2 = 7.44,\ \lambda = 1.0,\ N_T = 320, 640,\ N_s = 3,\ 4,\ \ldots,\ 10$
- Parameters for 2D simulations:
  - $\eta = 4 + m_0^2 = 2.6,\ \lambda = 1.0,\ N_T = 400,\ N_s = 2,\ 4,\ \ldots,\ 16$

Worldline representation of the complex ϕ^4 field

Full partition sum in terms of the dual degrees of freedom, i.e., the link variables $k_{x,\nu} \in \mathbb{Z}$ and $a_{x,\nu} \in \mathbb{N}$

$$Z = \sum_{\{k,a\}} e^{\mu\beta \sum_x k_{x,d}} \left(\prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + a_{x,\nu})! a_{x,\nu}!} \right) \left(\prod_x \delta(\vec{\nabla} \cdot \vec{k}_x) \right) \times \left(\prod_x I\left(\sum_\nu \left[|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(a_{x,\nu} + a_{x-\hat{\nu},\nu}) \right] \right) \right)$$

$$\text{with } I(n) = \int_0^\infty \mathrm{d}r \; r^{n+1} e^{-\eta r^2 - \lambda r^4}$$

C. Gattringer, T. Kloiber, Nucl.Phys. B869, 56 (2013)