Using the Complex Langevin equation to map out the phase diagram of QCD

Dénes Sexty
Wuppertal University, Jülich JSC

SIGN 2018, Bielefeld 10th of September, 2018

1. Introduction to the sign problem and CLE
2. Review of results so far
3. Full QCD
   - Challenges of a full QCD simulation with CLE
   - Pressure
   - Improved actions
We are interested in a system described with the partition sum:

\[ Z = \text{Tr} \ e^{-\beta (H - \mu N)} = \sum_C W[C] \]

Typically exponentially many configurations, no direct summation possible.

If the Weight is positive, build a Markov chain with the Metropolis alg.

\[ \ldots \rightarrow C_{i-1} \rightarrow C_i \rightarrow C_{i+1} \rightarrow \ldots \]

Probability of visiting \(C\)

\[ p(C) = \frac{1}{N W} W[C] \]

Importance sampling

\[ \langle X \rangle = \frac{1}{Z} \text{Tr} \ X \ e^{-\beta (H - \mu N)} = \frac{1}{N W} \sum_C W[C] X[C] = \frac{1}{N} \sum_i X[C_i] \]

This works if we have \(W[C] \geq 0\)

Otherwise we have a **Sign problem**
Sign problems in high energy physics

Real-time evolution in QFT

“strongest” sign problem $e^i S_m$

Non-zero density (and fermionic systems)

$$Z = \text{Tr} \ e^{-\beta (H - \mu N)} = \int DU \ e^{-S[U]} \ det(M[U])$$

Many systems: Bose gas
XY model
SU(3) spin model
Random matrix theory
QCD

Theta therm

$$S = F_{\mu \nu} F^{\mu \nu} + i \Theta \epsilon^{\mu \nu \theta \rho} F_{\mu \nu} F_{\theta \rho}$$

And everything else with complex action

$$w[C] = e^{-S[C]} \quad w[C] \text{ is positive} \leftrightarrow S[C] \text{ is real}$$
How to solve the sign problem?

Probably no general solution  
- There are sign problems which are NP hard

[ Troyer Wiese (2004) ]

Many solutions for particular models with sign problem exist

Transforming the problem to one with positive weights

\[ Z = \text{Tr} e^{-\beta (H - \mu N)} = \sum_C W[C] = \sum_D W'[D] \quad \text{Dual variables} \]
\[ Z = \text{Tr} e^{-\beta (H - \mu N)} = \sum_n Z_n e^{\beta \mu n} \quad \text{Canonical ensemble} \]
\[ Z = \text{Tr} e^{-\beta (H - \mu N)} = \int dE \rho_{\mu}(E) e^{-\beta E} \quad \text{Density of states} \]
\[ Z = \text{Tr} e^{-\beta (H - \mu N)} = \sum_C W[C] = \sum_S \left( \sum_{C \in S} W[C] \right) \quad \text{Subsets} \]
How to solve the sign problem?

Extrapolation from a positive ensemble

Reweighting

\[ \langle X \rangle_w = \frac{\sum_c W_c X_c}{\sum_c W_c} = \frac{\sum_c W'_c (W_c/W'_c) X_c}{\sum_c W'_c (W_c/W'_c)} = \frac{\langle (W/W') X \rangle_w}{\langle W/W' \rangle_w}. \]

Taylor expansion

\[ Z(\mu) = Z(\mu=0) + \frac{1}{2} \mu^2 \partial^2 \mu Z(\mu=0) + \ldots \]

Analytic continuation from imaginary sources
(chemical potentials, theta angle,..)

Using analyticity (for complexified variables)

Complex Langevin

Complexified variables – enlarged manifolds

Lefschetz thimble
Integration path shifted onto complex plane
Complex Langevin Equation

Given an action $S(x)$

$$\frac{d x}{d \tau} = - \frac{\partial S}{\partial x} + \eta(\tau)$$

Gaussian noise

$$\langle \eta(\tau) \rangle = 0$$
$$\langle \eta(\tau) \eta(\tau') \rangle = \delta(\tau - \tau')$$

Stochastic process for $x$:

Averages are calculated along the trajectories:

$$\langle O \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T O(x(\tau)) d \tau = \frac{\int e^{-S(x)} O(x) dx}{\int e^{-S(x)} dx}$$

The field is complexified

real scalar $\rightarrow$ complex scalar

link variables: $SU(N) \rightarrow SL(N,\mathbb{C})$
compact $\rightarrow$ non-compact

$$\text{det}(U) = 1, \quad U^* \neq U^{-1}$$

Analytically continued observables

$$\frac{1}{Z} \int P_{\text{comp}}(x) O(x) dx = \frac{1}{Z} \int P_{\text{real}}(x, y) O(x + iy) dx dy$$

$$\langle x^2 \rangle_{\text{real}} \rightarrow \langle x^2 - y^2 \rangle_{\text{complexified}}$$
Theoretically

Good understanding of the failure modes (boundary terms, poles)
Monitoring prescriptions allow for independent detection of failure
unitarity norm, eigenspectrum, histograms, boundary terms
Is a cutoff allowed? (Dynamical stabilization)
How to cure problems? – No general answer, hit and miss

In practice

Many lattice models solved, crosschecked with alternative methods
(Bose gas, SU(3) Spin model, HDQCD, kappa exp., cond. mat. systems...)
Some remain unsolved (xy model, Thirring,... )

Full QCD

High temperatures seem to be unproblematic
checks with reweighting, Taylor expansion
Status of low T and near T_c is unclear – more work needed

See below for ongoing work
concerning phase diag, EOS and improved actions
Proof of convergence for CLE results

If there is fast decay \( P(x, y) \to 0 \) as \( x, y \to \infty \)

and a holomorphic action \( S(x) \)

then CLE converges to the correct result

[Aarts, Seiler, Stamatescu (2009)
Aarts, James, Seiler, Stamatescu (2011)]

Loophole 1: Non-holomorphic action for nonzero density

\( S = S_w[U_\mu] + \ln \text{Det } M(\mu) \)

measure has zeros \( \text{(Det } M=0) \)
complex logarithm has a branch cut
meromorphic drift

[Mollgaard, Splittorff (2013), Greensite(2014)]

Drift around a pole:

\[
\rho(x) = (x - z_p)^{n_f} e^{-S(x)}
\]

\[
K(z) = \frac{\partial_z \rho(z)}{\rho(z)} = \frac{n_f}{x-z_p} + K_S(z)
\]
Poles can be inside the distribution

Pole pinches distribution
Acts as a bottleneck
might cause “separation phenomenon”
(potentially) wrong results

outside of the distribution

\[ \rho (x) = (1 + \kappa \cos(x - i \mu))^n e^{-\beta \cos(x)} \]

Langevin time evolved observables get singularities around pole
Zero of the distribution counteracts that
Proof goes through correct results

For HDQCD and full QCD at high temperatures this is satisfied
[Aarts, Seiler, Sexty, Stamatescu ‘17]
Loophole 2: decay not fast enough

\[ \int dx \rho(x) O(x) = \int dx\,dy P(x, y) O(x+iy) \]

What we want

What we get with CLE

Using analyticity and partial integrations

boundary terms can be nonzero

explicit calculation of boundary terms

[Scherzer, Seiler, Sexty, Stamatescu (2018)]

See talk by Stamatescu
Gaussian Example

\[ S[x] = \sigma x^2 + i \lambda x \]

\[ \frac{d}{d \tau} (x + i y) = -2 \sigma (x + iy) - i \lambda + \eta \]

\[ P(x, y) = e^{-a(x-x_0)^2 - b(y-y_0)^2 - c(x-x_0)(y-y_0)} \]

Gaussian distribution around critical point

\[ \left. \frac{\partial S(z)}{\partial z} \right|_{z_0} = 0 \]
QCD sign problem

Euclidean SU(3) gauge theory with fermions:

\[ Z = \int DU \exp(-S_E[U]) \det(M(U)) \]

for \( \det(M(U)) > 0 \)  Importance sampling is possible  

Hadron masses, EOS, ...

Non-zero chemical potential

For nonzero chemical potential, the fermion determinant is complex

\[ \det(M(U, -\mu^*)) = (\det(M(U), \mu))^* \]

Sign problem  Naive Monte-Carlo breaks down
In QCD direct simulation only possible at $\mu = 0$

Taylor extrapolation, Reweighting, continuation from imaginary $\mu$, canonical ens. all break down around

$$\frac{\mu_q}{T} \approx 1 - 1.5 \quad \frac{\mu_B}{T} \approx 3 - 4.5$$

Around the transition temperature

Breakdown at

$$\mu_q \approx 150 - 200 \text{ MeV} \quad \mu_B \approx 450 - 600 \text{ MeV}$$

Results on $N_T = 4, N_F = 4, ma = 0.05$

using

Imaginary mu, Reweighting, Canonical ensemble

Agreement only at $\mu/T < 1$
Some results of CLE so far

Bose Gas at zero temperature

[Aarts '08]

Silver Blaze problem:

At zero temperature, nothing happens until first excited state (=1 particle) contributes

First spectacular success of complex Langevin

Gauge cooling and study of HDQCD

[Seiler, Sexty, Stamatescu '13]

Full QCD with light quarks

[Sexty '14]
Gauge cooling

[Seiler, Sexty, Stamatescu (2012)]

complexified distribution with slow decay $\longrightarrow$ convergence to wrong results

Keep the system from trying to explore the complexified gauge degrees of freedom

Minimize unitarity norm

Distance from SU(N) $\sum_i \text{Tr}(U_i U_i^+ - 1)$

Dynamical steps are interspersed with several gauge cooling steps

Empirical observation:

Cooling is effective for $\beta > \beta_{\text{min}}$

Can we do more?

Dynamical Stabilization soft cutoff in imaginary directions

[Attanasio, Jäger (2018)]

but remember, $\beta \rightarrow \infty$ in cont. limit

$a < a_{\text{max}} \approx 0.1 - 0.2 \, \text{fm}$
Chiral random matrix theory
[Mollgaard, Splittorff '13+'14]

Poles can be problematic

Study of the pole problem
[Nishimura, Shimasaki '15]
[Aarts, Seiler,Sexty, Stamatescu '17]

Distribution at poles (spectrum) should be monitored

Hopping parameter expansion
[Aarts, Seiler, Sexty, Stamatescu ‘15]

Very high orders easily calculated

Investigating Silver Blaze for QCD
[Kogut, Sinclair ‘16]
[Ito, Nishimura ‘16]
[Tsutsui, Ito, Matsufuru, Nishimura, Shimasaki, Tsuchiya ‘18]

Jury still out

0+1 dim Thirring model
[Fujii, Kamata, Kikukawa ‘17]

Reweighting or deformation makes CLE ok

Gauge cooling for eigenvalues
[Nagata, Nishimura, Shimasaki ‘16]

Shifts e.v.s away from origin in RMT

Gauge cooling for Random Matrix models
[Bloch, Glessaen, Verbaarschot, Zafeiropoulos ‘18]

1 hit 1 miss
Exact drift terms with selected inverse  
[Bloch, Schenk ‘17]

Fermionic drift term:  \( \text{Tr} \left( M^{-1} D_{a\mu x} M \right) \)
with sparse Dirac Matrix  \( M \)

Use sparse LU decomposition to calculate inverse

No additional noise from stochastic estimator

Unitarity norm is better controlled

Reweighting complex Langevin trajectories  
[Bloch ‘17]

Reweighting from one non-positive ensemble to another
Equation of state for 1D non-relativistic fermions

\[ Z = \int d\sigma \det M_{\text{up}}(\sigma) \det M_{\text{down}}(\sigma) \]

\[ \sigma = \text{Hubbard-Stratonovich field} \]

Modify action to add an attractive force

\[ S(\sigma) = S_{\text{old}}(\sigma) + \xi \sigma^2 \]

Local interactions

2 parameters: coupling \( \lambda \), chemical pot. \( \mu \)

Attractive – pos. det

Repulsive – non pos. det
Mapping the phase diagram of HDQCD

Hopping parameter expansion of the fermion determinant
Spatial fermionic hoppings are dropped
Full gauge action

\[ \text{Det } M(\mu) = \prod_x \det(1 + C P_x)^2 \det(1 + C' P_x^{-1})^2 \]

Strategy to map \( T - \mu \) plane

fixed \( \beta = 5.8 \rightarrow a \approx 0.15 \text{ fm} \) \quad \text{Unitarity norm is mostly under control}

\( \kappa = 0.04 \)
onset transition at \( \mu = -\ln(2\kappa) \)

\( N_t \ast (6^3, 8^3, 10^3) \) lattice
\( N_t = 2 \ldots 28 \)

Temperature scanning
\( T = 48 \ldots 671 \text{ MeV} \)
Mapping the phase diagram of HDQCD

Onset in fermionic density
Silver blaze phenomenon

Polyakov loop
Transition to deconfined state
Fits of the phase transition line

Deconfinement transition and onset transition meet in the middle
Errors from discretisation scheme
Volume dependence under control

much simpler phase diagram than full QCD
Reweighting

\[
\langle F \rangle_\mu = \frac{\int DU \ e^{-S_E} \det M(\mu) \ F}{\int DU \ e^{-S_E} \det M(\mu)} = \frac{\int DU \ e^{-S_E} \ R \frac{\det M(\mu)}{R} \ F}{\int DU \ e^{-S_E} \ R \frac{\det M(\mu)}{R}}
\]

\[
= \frac{\langle F \det M(\mu)/R \rangle_R}{\langle \det M(\mu)/R \rangle_R} \quad R = \det M(\mu = 0), \ |\det M(\mu)|, \ etc.
\]

\[
\left\langle \frac{\det M(\mu)}{R} \right\rangle_R = \frac{Z(\mu)}{Z_R} = \exp \left( -\frac{V}{T} \Delta \! f(\mu, T) \right)
\]

\[
\Delta \! f(\mu, T) = \text{free energy difference}
\]

Exponentially small as the volume increases \( \langle F \rangle_\mu \to 0/0 \)

Reweighting works for large temperatures and small volumes

Sign problem gets hard at \( \mu/T \approx 1 \)
Comparison with reweighting for full QCD
[Fodor, Katz, Sexty, Török 2015]

Reweighting from ensemble at
\( R = \text{Det} M (\mu = 0) \)
Comparisons as a function of beta

Similarly to HDQCD
Cooling breaks down at small beta

at $N_T=4$ breakdown at $\beta=5.1 - 5.2$

At larger $N_T$?
Comparisons as a function of beta

$N_T = 6$

$N_T = 8$

Breakdown prevents simulations in the confined phase for staggered fermions with $N_T = 4, 6, 8$

Two ensembles:

$m_\pi \approx 4.8 \, T_c$

$m_\pi \approx 2.3 \, T_c$
Ongoing efforts concerning the QCD phase diag
with Manuel Scherzer and Nucu Stamatescu

1. Following phase transition line
   Do we meet a critical point?

2. Onset transition at small temperatures
   \[
   \frac{m_\pi}{2} \quad \text{vs.} \quad \frac{m_N}{3}
   \]

3. Calculating the pressure at high temperatures
   compare with know results

4. Implementing improved actions
   also for fermions
Mapping out the phase transition line

Follow the phase transition line starting from $\mu = 0$

Using Wilson fermions

Can follow the line to quite high $\mu/T$

Compatible with expected behavior at small chemical pot.

See talk by Scherzer
Onset transition in QCD

Low temperature, chemical potential is increased

Nuclear matter onset at $\mu_c = m_N / 3$

“benchmark:” Phasequenched theory (equivalent to isospin chem. pot.)

$\det M(\mu) \rightarrow |\det M(\mu)|$  Simulation with ordinary importance sampling

Pion condensation onset at $\mu_{c,PQ} = m_\pi / 2$

Can we see the difference?

Hard problem:
For large quark masses $m_\pi / 2 \approx m_N / 3$
Low quark masses are expensive

Temperature effects might shift $\mu_c$
Low temperature is expensive

Huge finite size effects
Thermalization potentially slow
Long runs with CLE

Unitarity norm has a tendency to grow slowly (even with gauge cooling)

Runs are cut if it reaches $\sim 0.1$

Thermalization usually fast
  - might be problematic close to critical point or at low $T$
Getting closer to continuum limit

Test with Wilson fermions
Increase $\beta$ by 0.1 – reduces lattice spacing by 30%
change everything else to stay on LCP

behavior of Unitarity norm improves
Pressure of the QCD Plasma at non-zero density

\[
\frac{p}{T^4} = \frac{\ln Z}{V T^3}
\]

Derivatives of the pressure are directly measureable
Integrate from $T=0$

Other strategies:

Measure the Stress-momentum tensor using gradient flow
[Suzuki, Makino (2013-)]

Shifted boundary conditions
[Giusti, Pepe, Meyer (2011-)]

Non-equilibrium quench
[Caselle, Nada, Panero (2018)]

First integrate along the temperature axis, then explore $\mu > 0$

Taylor expansion [Allton et. al. (2002-), ...]

Simulating at imaginary $\mu$ to calculate susceptibilities
[Bud.-Wupp. Group (2018)]
Pressure of the QCD Plasma at non-zero density

\[ \Delta \left( \frac{p}{T^4} \right) = \frac{p}{T^4} (\mu = \mu_q) - \frac{p}{T^4} (\mu = 0) \]

If we want to stay at \( \mu = 0 \)

\[ \Delta \left( \frac{p}{T^4} \right) = \sum_{n>0, \text{even}} c_n(T) \left( \frac{\mu}{T} \right)^n \]

\[ c_2 = \frac{1}{2} \frac{N_T}{N_s^3} \frac{\partial^2 \ln Z}{\partial \mu^2} \]
\[ c_4 = \frac{1}{24} \frac{1}{N_s^3 N_T} \frac{\partial^4 \ln Z}{\partial \mu^4} \]

Measuring the coefficients of the Taylor expansion

\[ \frac{\partial^2 \ln Z}{\partial \mu^2} = N_F^2 \langle T_1^2 \rangle + N_F \langle T_2 \rangle \]
\[ \frac{\partial^4 \ln Z}{\partial \mu^4} = -3 \left( \langle T_2 \rangle + \langle T_1^2 \rangle \right)^2 + 3 \langle T_2^2 \rangle + \langle T_4 \rangle + 4 \langle T_3 T_1 \rangle + 6 \langle T_2 T_2 \rangle \]
\[ \frac{\partial^2 \ln Z}{\partial \mu^2} = \text{Tr} \left( M^{-1} \partial_\mu M \right) \]
\[ T_{i+1} = \partial_\mu T_i \]
\[ T_2/N_F = \text{Tr} \left( M^{-1} \partial_\mu^2 M \right) - \text{Tr} \left( (M^{-1} \partial_\mu M)^2 \right) \]
\[ T_3/N_F = \text{Tr} \left( M^{-1} \partial_\mu^3 M \right) - 3 \text{Tr} \left( M^{-1} \partial_\mu M M^{-1} \partial_\mu^2 M \right) + 2 \text{Tr} \left( (M^{-1} \partial_\mu M)^3 \right) \]
\[ T_4/N_F = \text{Tr} \left( M^{-1} \partial_\mu^4 M \right) - 4 \text{Tr} \left( M^{-1} \partial_\mu M M^{-1} \partial_\mu^3 M \right) - 3 \text{Tr} \left( M^{-1} \partial_\mu^2 M M^{-1} \partial_\mu^2 M \right) - 6 \text{Tr} \left( (M^{-1} \partial_\mu M)^4 \right) + 12 \text{Tr} \left( (M^{-1} \partial_\mu M)^2 M^{-1} \partial_\mu^2 M \right) \]
Pressure of the QCD Plasma using CLE

If we can simulate at $\mu > 0$

$$\Delta \left( \frac{p}{T^4} \right) = \frac{p}{T^4} (\mu = \mu_q) - \frac{p}{T^4} (\mu = 0) = \frac{1}{V T^3} \left( \ln Z(\mu) - \ln Z(0) \right)$$

$$\ln Z(\mu) - \ln Z(0) = \int_0^\mu d\mu \frac{\partial \ln Z(\mu)}{\partial \mu} = \int_0^\mu d\mu \Omega n(\mu)$$

$$n(\mu) = \langle \text{Tr} \left( M^{-1}(\mu) \partial_\mu M(\mu) \right) \rangle$$

Using CLE it’s enough to measure the density – much cheaper
Taylor expansion

Using naive staggered action with $N_F = 4$

Observables very noisy

State of the art calculations barely see a signal at 8th order

Disconnected terms

e.g. $\langle T_1^2 T_2 \rangle$

Contribute most of the noise
Pressure calculated with CLE

Integration performed numerically
Jackknife error estimates

\[ T = 250 \text{ MeV}, \quad T_c \approx 190 \text{ MeV} \]

\[ T = 475 \text{ MeV} \]

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( a ) (fm)</th>
<th>( c_2 ) HMC</th>
<th>( c_4 ) HMC</th>
<th>( c_2 ) CLE</th>
<th>( c_4 ) CLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3</td>
<td>0.099 ± 0.001</td>
<td>1.986 ± 0.042</td>
<td>0.27 ± 0.23</td>
<td>2.117 ± 0.1</td>
<td>0.152 ± 0.05</td>
</tr>
<tr>
<td>5.6</td>
<td>0.052 ± 0.0013</td>
<td>2.351 ± 0.044</td>
<td>0.16 ± 0.12</td>
<td>2.168 ± 0.1</td>
<td>0.200 ± 0.05</td>
</tr>
</tbody>
</table>
Improved actions for lattice QCD

Carrying out continuum extrapolation \( a \to 0 \)

Simulate at multiple lattice spacings

Fitting some observable

\[ O(a) = O_0 + O_1 a + O_2 a^2 + \ldots \]

Change action such that \( O_1 \) is eliminated

Gauge improvement

Include larger loops in action

Symanzik action:

\[ S = -\beta \left( \frac{5}{3} \sum \text{ReTr} \begin{array}{c} \square \end{array} - \frac{1}{12} \sum \text{Re Tr} \begin{array}{c} \square \square \end{array} \right) \]

Straightforwardly implemented in CLE

Analyticity must be preserved:

\[ 2 \text{ReTr} U = \text{Tr} U + \text{Tr} U^* \quad \Rightarrow \quad \text{Tr} U + \text{Tr} U^{-1} \]
**Improved fermion actions**

Changing the Dirac operator

- **Wilson fermions: clover improvement**
  adds a clover term

- **Staggered fermions: naik or p4**
  take into account 3-link terms

**Fat links**

Smear the gauge fields inside the Dirac operator

APE, HYP

\[ V_\mu = (1 - \alpha) U_\mu + \alpha \sum \text{staples} \]

\[ U'_\mu = \text{Proj}_{SU(3)} V_\mu \]

Stout

\[ U'_\mu = e^{iQ_\mu} U_\mu \quad Q_\mu = \rho \sum \text{staples} \]

essentially one step of gradient flow with stepsize \( \rho \)
Stout smearing

\[ U'_\mu = e^{iQ_\mu} U_\mu \quad Q_\mu = \rho \sum \text{staples} \]

Usually multiple steps: \[ U \rightarrow U^{(1)} \rightarrow U^{(2)} \rightarrow \ldots \rightarrow U^{(n)} \]

Replace gauge fields in Dirac matrix \[ \det M(U) \rightarrow \det M(U^{(n)}) \]

For the Langevin eq. we need drift terms: \[ \frac{\partial S_{\text{eff}}}{\partial U} \quad \text{with} \quad S_{\text{eff}} = S_g + \ln \det M(U^{(n)}) \]

Calculated by “going backwards” \[ \frac{\partial S_{\text{eff}}}{\partial U} = \frac{\partial S_{\text{eff}}}{\partial U^{(n)}} \frac{\partial U^{(n)}}{\partial U^{(n-1)}} \ldots \frac{\partial U^{(1)}}{\partial U} \]

One iteration: \[ \frac{\partial U'}{\partial U} = \frac{\partial e^{iQ}}{\partial U} U + e^{iQ} \quad \text{local terms} + \text{nonlocal terms from staples} \]
Stout smearing and complex Langevin

\[ U'_\mu = e^{iQ_\mu} U_\mu \quad Q_\mu = \rho \sum \text{staples} \]

Adjungate is replaced with inverse for links

\( Q^+ \) is not replaced with \( Q^{-1} \) (because its a sum)

\( Q \) is no longer hermitian

Calculation of \( \frac{\partial e^{iQ}}{\partial U} \) becomes trickier

Benchmarking with HMC at \( \mu = 0 \)

\[ a(\beta=3.6) = 0.12 \text{ fm} \quad a(\beta=3.9) = 0.064 \text{ fm} \]
What happens with the configurations?

Real part of gauge fields decay
Unitarity norm slightly rises

<table>
<thead>
<tr>
<th>smearing step</th>
<th>plaqavg</th>
<th>unitarity norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.562948</td>
<td>0.00913145</td>
</tr>
<tr>
<td>1</td>
<td>0.837735</td>
<td>0.0108531</td>
</tr>
<tr>
<td>2</td>
<td>0.929264</td>
<td>0.0118543</td>
</tr>
<tr>
<td>3</td>
<td>0.964314</td>
<td>0.0125011</td>
</tr>
<tr>
<td>4</td>
<td>0.97947</td>
<td>0.0129698</td>
</tr>
<tr>
<td>5</td>
<td>0.986835</td>
<td>0.0133134</td>
</tr>
<tr>
<td>6</td>
<td>0.990828</td>
<td>0.0135655</td>
</tr>
<tr>
<td>7</td>
<td>0.993218</td>
<td>0.0137527</td>
</tr>
<tr>
<td>8</td>
<td>0.994766</td>
<td>0.0139118</td>
</tr>
<tr>
<td>9</td>
<td>0.995831</td>
<td>0.0140539</td>
</tr>
<tr>
<td>10</td>
<td>0.996598</td>
<td>0.0141745</td>
</tr>
</tbody>
</table>
What happens with the drift terms?

\[
\frac{\partial S_{\text{eff}}}{\partial U} = \frac{\partial S_{\text{eff}}}{\partial U^{(n)}} \frac{\partial U^{(n)}}{\partial U^{(n-1)}} \cdots \frac{\partial U^{(1)}}{\partial U}
\]

\[
\frac{\partial S_{\text{eff}}}{\partial U^{(n)}} = F_0, \quad \frac{\partial S_{\text{eff}}}{\partial U^{(n)}} \frac{\partial U^{(n)}}{\partial U^{(n-1)}} = F_1, \quad \ldots \quad \frac{\partial S_{\text{eff}}}{\partial U} = F_n
\]

Average drift term is smaller
Long tail
More prone to runaways
Smaller stepsize needed
Pressure with improved action

\[ C_4 \text{ is measurable with this action at high } T \text{ (with } O(500) \text{ configs.)} \]
Pressure with improved action

Symanzik gauge action
stout smeared staggered fermions

$T = 260$ MeV

$T = 385$ MeV

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a_{(\text{fm})}$</th>
<th>$c_2 \text{ HMC}$</th>
<th>$c_4 \text{ HMC}$</th>
<th>$c_2 \text{ CLE}$</th>
<th>$c_4 \text{ CLE}$</th>
<th>$c_6 \text{ CLE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7</td>
<td>$0.094 \pm 0.001$</td>
<td>$2.127 \pm 0.026$</td>
<td>$0.122 \pm 0.046$</td>
<td>$2.143 \pm 0.07$</td>
<td>$0.151 \pm 0.02$</td>
<td>$0.0014 \pm 0.001$</td>
</tr>
<tr>
<td>3.9</td>
<td>$0.064 \pm 0.001$</td>
<td>$2.302 \pm 0.026$</td>
<td>$0.138 \pm 0.021$</td>
<td>$2.314 \pm 0.04$</td>
<td>$0.143 \pm 0.007$</td>
<td>$0.0018 \pm 0.0003$</td>
</tr>
</tbody>
</table>

Good agreement
CLE calculation is much cheaper
Summary

CLE is a versatile tool to solve sign problems

Potential problems with boundary terms and poles
  Monitoring of the process is required

Promising results for many systems:
  phase diagram of HDQCD mapped out

Ongoing effort for full QCD to get physical results
  Mapping out phase transition line
  Onset transition at small temperatures
  Calculating the pressure
  Using improved actions