Thimbles and Lattice Gauge Theories

Felix Ziesché   Christian Schmidt

SIGN 2018, Bielefeld, September 11, 2018

11.09.2018
Let $M$ be a smooth compact $m$-dimensional manifold,

$f : M \to \mathbb{R}$ a at least two times differentiable function.

$p \in M$ is called a non-degenerate critical point of $f$, if $\nabla f(p) = 0$ and $\det \nabla^2 f(p) \neq 0$.

$M^a := \{ p \in M | f(p) \leq a \} = f^{-1}((-\infty, a])$ is compact, since $M$ is compact.

$\Rightarrow M$ has the homotopy type of a CW-complex, where each cell is related to a non-degenerate critical points. Its dimension is the number of positive eigenvalues of $\nabla^2 f$. A CW-complex is a cell-complex, where a $k$-cell is a closed disc

$$D^k = \{ \vec{x} \in \mathbb{R}^k | |\vec{x}| \leq 1 \},$$

which are glued together at the boundaries to form a compact manifold.
An example

\[ f(z) = \text{Re} \left( z^2 - 1 \right), \quad z \in U(1) \]

- \( f \) has four critical points \( z = 1, -1, i, -i \).
- \( \partial_z^2 f(z) > 0 \) for \( z = i, -i \) at \( f = -2 \) and \( \partial_z^2 f(z) < 0 \) for \( z = 1, -1 \) at \( f = 0 \)

\[ \Rightarrow \] We have two 1-cells glued together at two 0-cells.

In complexified space, these cells can be chosen to have constant phase and are then called \textit{Lefschetz thimbles}.
### Some Important theorems

**The Lefschetz theorem**

A complex analytic manifold $M$ of complex dimension $k$, bianalytically embedded as a closed subset of $\mathbb{C}^n$ has the homotopy type of a $k$-dimensional CW-complex.

This means, that every Lefschetz thimble has the same dimension.

**The Monodromy theorem**

- Let $f : \tilde{\Gamma} \rightarrow \mathbb{C}$ be a holomorphic function on $\tilde{\Gamma}$ and
- $\Gamma, \Gamma' \subset \tilde{\Gamma}$ be homotopic submanifolds of $\tilde{\Gamma}$ ($\Gamma \simeq \Gamma'$).

Then

\[ \int_{\Gamma} dz f(z) = \int_{\Gamma'} dz f(z). \]
The model: One flavor $0+1$ d-QCD

One space-time dimension: $F_{\mu\nu} = 0 \Rightarrow S_G = 0$.

$\rightarrow S = S_F$ and the discretized staggered fermion action reads:

$$\hat{S}_F(\mu) = \frac{1}{2} \sum_{n=0}^{N_\tau-1} \bar{\chi}(n) \left( e^\mu U(n) \chi(n + 1) - e^{-\mu} U^{-1}(n - 1) \chi(n - 1) + 2m\chi(n) \right)$$

Integrating out the fermion fields in the partition sum, we have

$$Z(N_\tau, \mu) = \int dU d\bar{\chi} d\chi e^{-\bar{\chi}M[U]\chi} = \int dU \det M[U]$$

This determinant can be reduced to

$$\det(M[U]) = \frac{1}{2^{3N_\tau}} \det \left( 2 \cosh(N_\tau \sinh^{-1}(m)) \mathbb{I} + e^{N_\tau \mu} P + e^{-N_\tau \mu} P^{-1} \right)$$

$$P = \prod_{n=0}^{N_\tau-1} U(n).$$

For $\mu > 0$, this is complex.
The geometric structure of $0+1$d-QCD


- The main critical points obtained are

$$P_\sigma = \mathbb{I}, e^{\pm i\frac{2\pi}{3}} \mathbb{I}.$$ 

These are the center elements of SU(3).

Including zeros of det$[M]$, we have:
The triangulation approach

1. Get the relevant saddle-point/thimble structure. (Done for 1d-QCD)
2. Solve the flow equations for specific directions around the saddle points and record the points with a minimum curvature.
3. With these points, we construct a mesh of $d$-simplices, which approximates the thimbles.
4. Do an ergodic MCMC on this mesh.

Advantages:
- The Monodromy Theorem is fulfilled $\Rightarrow$ No a priori wrong results.
- Sampling on a mesh is fast!
Sampling the points

1. Start at a saddle point.

2. Go an $\epsilon$-step in direction of the Takagi-Vectors defined by

$$H(\omega_\sigma)^* \hat{\omega}_\sigma^* = \lambda \hat{\omega}_\sigma, \; \lambda \in \mathbb{R}$$

3. The steepest ascent equation can be reformulated to

$$\dot{\omega} = \left( \frac{\partial S}{\partial \omega} \right)^* \Rightarrow \dot{\omega}_R = \frac{\partial S_I}{\partial \omega_I} \quad \text{and} \quad \dot{\omega}_I = - \frac{\partial S_I}{\partial \omega_R}.$$  

This can be solved effectively by symplectic methods. We use Verlet-integration.

4. Flow until you hit $|\nabla S| < \epsilon$ or $S \geq S_{\text{CutOff}}$ and record the points.

5. Reduce the number of points depending on a maximal discrete curvature.
The recorded points in $\omega_8$-direction and the corresponding imaginary part of the action.
Building the triangulation

We take two neighboring flowlines consisting of the points \( \{ p^0_1, p^1_1, p^2_1, \ldots \} \) and \( \{ p^0_2, p^1_2, p^2_2, \ldots \} \), where \( p^0_1 = p^0_2 \) is the common critical point. We connect both flowlines for themselves and define \( \tilde{l}^k := l^k_i/L_i \) to be the normalised distance to the critical point.

1. Start with \( m_1 = m_2 = 1 \).
2. Connect \( p^{m_1}_1 \) with \( p^{m_2}_2 \).
3. If \( \sum_{k=1}^{m_2} \tilde{l}^2_k \) is smaller than \( \sum_{j=1}^{m_1} \tilde{l}^1_j \), increment \( m_2 \) or vice versa and go back to step 2.
4. When one of the \( m_i \)'s reaches \( n_i \), leave it there and just increment the other one and connect them until this reaches the end, too.

This procedure is generalisable for higher dimensions!

\[
\text{Triangles} \rightarrow n\text{-simplices}.
\]
Triangulation in $\omega_8$ and $\omega_3$ direction at $N_T = 4$ and $m = 1.0$ at $\mu = 0.5$. 
Sampling the triangulation (RW-Slice Sampling)

1. Select starting point $P_0$ on the triangulation.
2. Pick a random number $u \in [\exp(-S_{\text{CutOff}}), \exp(-S_R(F_t(P_n)))]$ uniformly.
3. Pick $P_{n+1}$ from an isotropic, ergodic distrib. around $P_n$ on the triangulation.
4. Accept $P_{n+1}$, if $\exp(-S_R(\tilde{P}_{n+1})) > u$. Otherwise $P_n = P_{n+1}$ and repeat from 3.

This obviously can be improved by using a more heatbath-like update, but this has at least no ergodicity problems.

The walking on the triangulation is still to be implemented. So we have Results just for the $\omega_8$-direction.
Results (at least for $\omega_8$)

The Polyakov Loop for $N_T = 4, m = 1.0$. 

The Polyakov Loop for $N_T = 4, m = 1.0$. 

Felix Ziesch, Christian Schmidt

Thimbles and Lattice Gauge Theories

11.09.2018 13 / 18
A simple $U(1)$-integral

$$f(U) = 2 \cosh(K_c) + e^K U + e^{-K} U^{-1}, \quad U \in U(1)$$

$$Z = \int_{U(1)} dU f(U) = \int_0^{2\pi} \frac{d\phi}{2\pi} f(e^{i\phi}) = \int_{B_1(0)} \frac{dz}{2\pi i} \frac{f(z)}{z}$$

- Residual theorem: $Z = 2 \cosh(K_c)$ and $< e^{i\phi} > = \frac{e^{-K}}{2\cosh(K_c)}$
- $S = -\log f(z) \longrightarrow z_c = \pm e^{-K}$ and poles at $z = -e^{-K} \pm K_c$.

Now, what if we have more links?
The critical equation goes to
\[ e^{i(\phi_1 + \phi_2)} = \pm e^{-K} \Rightarrow \phi_1^R + \phi_2^R = 0/\pi \mod 2\pi, \quad \phi_1^I + \phi_2^I = K \]
So we have two 2-dimensional critical manifolds.
According to Witten, we have to take a cycle on this critical manifold and then sample from that cycle up to the so-called generalized Lefschetz thimble.
Outlook

- Extending the simulations from 0+1d-QCD to 1+1d-QCD, by using Lattice symmetries.
- Explore the relationship between Lattice symmetries and thimble structure in general (Test case: 1+1d-Quenched QED).
- Find an effective way of dealing with gauge orbits (Promising work in progress ;-) )
- Complete the sampling on triangulations and implement for higher dimensions.

Thank you for your attention!
Projection of $\mathbb{C}^2$ onto $\mathbb{R}^3$

\[
x = (r_1 e^{-\text{Im} (\phi_1)} + r_2 e^{-\text{Im} (\phi_1)} \cos(\text{Re} (\phi_2))) \cos(\text{Re} (\phi_1))
\]
\[
y = (r_1 e^{-\text{Im} (\phi_1)} + r_2 e^{-\text{Im} (\phi_1)} \cos(\text{Re} (\phi_2))) \sin(\text{Re} (\phi_1))
\]
\[
z = r_2 e^{-\text{Im} (\phi_2)} \sin(\text{Re} (\phi_2))
\]