UNIVERSITÄT BIELEFELD
Fakultät für Physik

CRC-TR 211

Ward identities and the pion decay constant

Sajid Ali

Fakultät für Physik

Journal Club, 18.12.2020, Universität Bielefeld

Noether's theorem
- Symmetries and conservatin laws
- QCD

Ward identities
- The fermionic action
- The Axial WI
- Gell–Mann–Oakes–Renner

$\mathscr{N}=1$ SYM theory

SUSY WIs
- Renormalization
- Numerical results

WIs analysis
- Methods
- Chiral limit
- Continuum limit

Noether's theorem

Every symmetry of the action has corresponding conservation law if the motion is governed by the principle of least action. (classical physics)

Noether's theorem

Every symmetry of the action has corresponding conservation law if the motion is governed by the principle of least action. (classical physics)

- The conservation of momentum is related to the homogeneity of space.
- Invariance under translation in time means that the law of conservation of energy is valid.
- The conservation of angular momentum results from the isotropy of space.
- Gauge symmetry leads to conservation of electric charge.
- and so on

Noether's theorem

Every symmetry of the action has corresponding conservation law if the motion is governed by the principle of least action. (classical physics)

- The conservation of momentum is related to the homogeneity of space.
- Invariance under translation in time means that the law of conservation of energy is valid.
- The conservation of angular momentum results from the isotropy of space.
- Gauge symmetry leads to conservation of electric charge.
- and so on

What is the quantum counterpart of Noether theorem?

ans: Ward-Takahashi identities

John Clive Ward and Yasushi Takahashi

Symmetries in QCD

Vector and axial vector unitary transformations

$$\psi' = e^{i\alpha T_i}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{-i\alpha T_i}, \text{ (nonsinglet vector)}$$

$$\psi' = e^{i\alpha \mathbb{1}}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{-i\alpha \mathbb{1}}, \text{ (flavour singlet vector)}$$

$$\psi' = e^{i\alpha \gamma_5 T_i}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{i\alpha \gamma_5 T_i}, \text{ (nonsinglet axial)}$$

$$\psi' = e^{i\alpha \gamma_5 \mathbb{1}}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{i\alpha \gamma_5 \mathbb{1}}, \text{ (flavour singlet axial)}$$

$$\psi' = e^{i\alpha \lambda}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{i\alpha \hat{\lambda}}, \text{ (short hand)}$$

Symmetries in QCD

Vector and axial vector unitary transformations

$$\psi' = e^{i\alpha T_i}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{-i\alpha T_i}, \ \text{(nonsinglet vector)}$$
$$\psi' = e^{i\alpha \mathbb{1}}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{-i\alpha \mathbb{1}}, \ \text{(flavour singlet vector)}$$
$$\psi' = e^{i\alpha \gamma_5 T_i}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{i\alpha \gamma_5 T_i}, \ \text{(nonsinglet axial)}$$
$$\psi' = e^{i\alpha \gamma_5 \mathbb{1}}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{i\alpha \gamma_5 \mathbb{1}}, \ \text{(flavour singlet axial)}$$
$$\psi' = e^{i\alpha \lambda}\psi, \qquad \bar{\psi}' = \bar{\psi}\, e^{i\alpha \hat{\lambda}}, \ \text{(short hand)}$$

Taken altogether, the massless action has the symmetry

$$\boxed{SU(N_f)_L x SU(N_f)_R x U(1)_V x U(1)_A.}$$

The partition function

$$Z = \int \mathscr{D}[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]} = \int \mathscr{D}[\psi', \bar{\psi}', U']\, e^{-S[\psi',\bar{\psi}',U']}.$$

The partition function

$$Z = \int \mathscr{D}[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]} = \int \mathscr{D}[\psi', \bar{\psi}', U']\, e^{-S[\psi',\bar{\psi}',U']}.$$

The expectation value of arbitrary O

$$\langle 0|O|0\rangle = \frac{1}{Z}\int \mathscr{D}[\psi, \bar{\psi}, U]\, O[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]}.$$

The partition function

$$Z = \int \mathscr{D}[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]} = \int \mathscr{D}[\psi', \bar{\psi}', U']\, e^{-S[\psi',\bar{\psi}',U']}.$$

The expectation value of arbitrary O

$$\langle 0|O|0\rangle = \frac{1}{Z}\int \mathscr{D}[\psi, \bar{\psi}, U]\, O[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]}.$$

Local, unitary, non-anomalous and infinitesimal symmetry transformations

$$\psi \to \psi + \delta\psi, \quad \bar{\psi} \to \bar{\psi} + \delta\bar{\psi}.$$

The partition function

$$Z = \int \mathscr{D}[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]} = \int \mathscr{D}[\psi', \bar{\psi}', U']\, e^{-S[\psi',\bar{\psi}',U']}.$$

The expectation value of arbitrary O

$$\langle 0|O|0\rangle = \frac{1}{Z}\int \mathscr{D}[\psi, \bar{\psi}, U]\, O[\psi, \bar{\psi}, U]\, e^{-S[\psi,\bar{\psi},U]}.$$

Local, unitary, non-anomalous and infinitesimal symmetry transformations

$$\psi \to \psi + \delta\psi, \quad \bar{\psi} \to \bar{\psi} + \delta\bar{\psi}.$$

$$\langle 0|\delta O|0\rangle - \langle 0|O\delta S|0\rangle = 0.$$

The simplest case $O = 1$, $\langle 0|\delta O|0\rangle$ vanishes

$$\boxed{\langle 0|\delta S|0\rangle = 0.}$$

The fermionic action

$$S = \int d^4x\, \bar{\psi}\Big(\gamma_\mu D_\mu + \mathscr{M}\Big)\psi, \quad D_\mu = \partial_\mu + \mathrm{i}A_\mu$$

The fermionic action

$$S = \int d^4x\, \bar{\psi}\Big(\gamma_\mu D_\mu + \mathscr{M}\Big)\psi, \quad D_\mu = \partial_\mu + \mathrm{i}A_\mu$$

Linear infinitesimal transformation corresponding to unitary transformations

$$\psi \to \psi' = \Big(\mathbb{1} + \mathrm{i}\varepsilon(x)\lambda\Big)\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\Big(\mathbb{1} + \mathrm{i}\varepsilon(x)\hat{\lambda}\Big).$$

The fermionic action

$$S = \int d^4x\, \bar{\psi}\Big(\gamma_\mu D_\mu + \mathscr{M}\Big)\psi, \quad D_\mu = \partial_\mu + \mathrm{i}A_\mu$$

Linear infinitesimal transformation corresponding to unitary transformations

$$\psi \to \psi' = \Big(\mathbb{1} + \mathrm{i}\varepsilon(x)\lambda\Big)\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\Big(\mathbb{1} + \mathrm{i}\varepsilon(x)\hat{\lambda}\Big).$$

$\lambda = \mathbb{1}, \tau^a, \gamma_5, \gamma_5\tau^a$ and $\hat{\lambda} = -\mathbb{1}, -\tau^a, \gamma_5, \gamma_5\tau^a$ (Dirac,flavor)

$$\delta S = \mathrm{i}\int d^4x \bar{\psi}\Big(\varepsilon\hat{\lambda}\gamma_\mu\partial_\mu + \gamma_\mu\lambda\partial_\mu\varepsilon + \mathrm{i}\varepsilon A_\mu\big(\hat{\lambda}\gamma_\mu + \gamma_\mu\lambda\big) + \varepsilon\big(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\big)\Big)\psi$$

The fermionic action

$$S = \int d^4x\, \bar{\psi}\Big(\gamma_\mu D_\mu + \mathscr{M}\Big)\psi, \quad D_\mu = \partial_\mu + \mathrm{i}A_\mu$$

Linear infinitesimal transformation corresponding to unitary transformations

$$\psi \to \psi' = \Big(\mathbb{1} + \mathrm{i}\varepsilon(x)\lambda\Big)\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\Big(\mathbb{1} + \mathrm{i}\varepsilon(x)\hat{\lambda}\Big).$$

$\lambda = \mathbb{1}, \tau^a, \gamma_5, \gamma_5\tau^a$ and $\hat{\lambda} = -\mathbb{1}, -\tau^a, \gamma_5, \gamma_5\tau^a$ (Dirac,flavor)

$$\delta S = \mathrm{i}\int d^4x \bar{\psi}\Big(\varepsilon\hat{\lambda}\gamma_\mu\partial_\mu + \gamma_\mu\lambda\partial_\mu\varepsilon + \mathrm{i}\varepsilon A_\mu\big(\hat{\lambda}\gamma_\mu + \gamma_\mu\lambda\big) + \varepsilon\big(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\big)\Big)\psi$$

$$\hat{\lambda}\gamma_\mu + \gamma_\mu\lambda = 0,$$

$$\partial_\mu(\varepsilon\psi) = \partial_\mu(\varepsilon)\psi + \varepsilon\partial_\mu(\psi).$$

$$\delta S = \mathrm{i} \int d^4x\, \varepsilon \Big(-\partial_\mu (\bar{\psi}\gamma_\mu \lambda \psi) + \bar{\psi}\big(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\big)\psi \Big)$$

$$\delta S = \mathrm{i} \int d^4x\, \varepsilon \Big(-\partial_\mu (\bar{\psi}\gamma_\mu \lambda \psi) + \bar{\psi}\big(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\big)\psi \Big)$$

$$\boxed{\partial_\mu (\bar{\psi}\gamma_\mu \lambda \psi) = \bar{\psi}\big(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\big)\psi.}$$

$$\delta S = \mathrm{i}\int d^4x\,\varepsilon\Big(-\partial_\mu(\bar\psi\gamma_\mu\lambda\psi)+\bar\psi\big(\hat\lambda\mathscr{M}+\mathscr{M}\lambda\big)\psi\Big)$$

$$\boxed{\partial_\mu(\bar\psi\gamma_\mu\lambda\psi)=\bar\psi\big(\hat\lambda\mathscr{M}+\mathscr{M}\lambda\big)\psi.}$$

$$\lambda=\mathbb{1},\hat\lambda=-\mathbb{1}, \qquad \partial_\mu V_\mu=0, \qquad V_\mu=\frac{1}{2}\bar\psi\gamma_\mu\psi,$$

$$\lambda=\tau^a,\hat\lambda=-\tau^a, \qquad \partial_\mu V^a_\mu=2\bar\psi[\mathscr{M},\tau^a]\psi, \qquad V^a_\mu=\frac{1}{2}\bar\psi\gamma_\mu\tau^a\psi,$$

$$\lambda=\gamma_5,\hat\lambda=\gamma_5, \qquad \partial_\mu A_\mu=4\bar\psi\mathscr{M}\gamma_5\psi\ (+?), \qquad A_\mu=\frac{1}{2}\bar\psi\gamma_\mu\gamma_5\psi,$$

$$\lambda=\gamma_5\tau^a,\hat\lambda=\gamma_5\tau^a, \qquad \partial_\mu A^a_\mu=\underbrace{2\bar\psi\{\mathscr{M},\tau^a\}\gamma_5\psi,}_{\text{nonsinglet AWI}} \qquad A^a_\mu=\frac{1}{2}\bar\psi\gamma_\mu\gamma_5\tau^a\psi.$$

$$\delta S = \mathrm{i}\int d^4x\,\varepsilon\Big(-\partial_\mu(\bar\psi\gamma_\mu\lambda\psi)+\bar\psi\big(\hat\lambda\mathscr{M}+\mathscr{M}\lambda\big)\psi\Big)$$

$$\boxed{\partial_\mu(\bar\psi\gamma_\mu\lambda\psi)=\bar\psi\big(\hat\lambda\mathscr{M}+\mathscr{M}\lambda\big)\psi.}$$

$$\lambda=\mathbb{1},\hat\lambda=-\mathbb{1}, \qquad \partial_\mu V_\mu=0, \qquad V_\mu=\frac{1}{2}\bar\psi\gamma_\mu\psi,$$

$$\lambda=\tau^a,\hat\lambda=-\tau^a, \qquad \partial_\mu V^a_\mu=2\bar\psi[\mathscr{M},\tau^a]\psi, \qquad V^a_\mu=\frac{1}{2}\bar\psi\gamma_\mu\tau^a\psi,$$

$$\lambda=\gamma_5,\hat\lambda=\gamma_5, \qquad \partial_\mu A_\mu=4\bar\psi\mathscr{M}\gamma_5\psi\ (+?), \qquad A_\mu=\frac{1}{2}\bar\psi\gamma_\mu\gamma_5\psi,$$

$$\lambda=\gamma_5\tau^a,\hat\lambda=\gamma_5\tau^a, \qquad \partial_\mu A^a_\mu=\underbrace{2\bar\psi\{\mathscr{M},\tau^a\}\gamma_5\psi,}_{\text{nonsinglet AWI}} \qquad A^a_\mu=\frac{1}{2}\bar\psi\gamma_\mu\gamma_5\tau^a\psi.$$

These identities hold as expectation values.
O can be non-zero inside space-time region and zero outside.
may be considered to be valid locally.
also valid for renormalized quantities.

To compare with experimental results, the quantities need to renormalized.

$$P^{(r)a} = Z_P\, P^a, \qquad V^{(r)a} = Z_V\, V^a, \qquad A^{(r)a} = Z_A\, A^a,$$

$$m^{(r)} = Z_m\, m, \qquad \langle \bar{\psi}\psi \rangle^{(r)} = Z_S\, \langle \bar{\psi}\psi \rangle, \qquad m = m_{bare} + m_{res}.$$

To compare with experimental results, the quantities need to renormalized.

$$P^{(r)a} = Z_P\, P^a, \qquad V^{(r)a} = Z_V\, V^a, \qquad A^{(r)a} = Z_A\, A^a,$$

$$m^{(r)} = Z_m\, m, \qquad \langle\bar\psi\psi\rangle^{(r)} = Z_S\, \langle\bar\psi\psi\rangle\,, \qquad m = m_{bare} + m_{res}.$$

$$P^+ = P^1 - \mathrm{i}P^2 = \frac{1}{2}\bar\psi[\tau^1 - \mathrm{i}\tau^2)\gamma_5\psi = \bar d\gamma_5 u,$$

$$P^- = P^1 + \mathrm{i}P^2 = \frac{1}{2}\bar\psi[\tau^1 + \mathrm{i}\tau^2)\gamma_5\psi = \bar u\gamma_5 d,$$

$$A^+_\mu = A^1_\mu - \mathrm{i}A^2_\mu = \frac{1}{2}\bar\psi\gamma_\mu(\tau^1 - \mathrm{i}\tau^2)\gamma_5\psi,$$

$$A^-_\mu = A^1_\mu + \mathrm{i}A^2_\mu = \frac{1}{2}\bar\psi\gamma_\mu(\tau^1 + \mathrm{i}\tau^2)\gamma_5\psi.$$

To compare with experimental results, the quantities need to renormalized.

$$P^{(r)a} = Z_P\, P^a, \qquad V^{(r)a} = Z_V\, V^a, \qquad A^{(r)a} = Z_A\, A^a,$$

$$m^{(r)} = Z_m\, m, \qquad \langle\bar\psi\psi\rangle^{(r)} = Z_S\, \langle\bar\psi\psi\rangle\,, \qquad m = m_{bare} + m_{res}.$$

$$P^+ = P^1 - \mathrm{i}P^2 = \frac{1}{2}\bar\psi[\tau^1 - \mathrm{i}\tau^2)\gamma_5\psi = \bar d\gamma_5 u,$$

$$P^- = P^1 + \mathrm{i}P^2 = \frac{1}{2}\bar\psi[\tau^1 + \mathrm{i}\tau^2)\gamma_5\psi = \bar u\gamma_5 d,$$

$$A^+_\mu = A^1_\mu - \mathrm{i}A^2_\mu = \frac{1}{2}\bar\psi\gamma_\mu(\tau^1 - \mathrm{i}\tau^2)\gamma_5\psi,$$

$$A^-_\mu = A^1_\mu + \mathrm{i}A^2_\mu = \frac{1}{2}\bar\psi\gamma_\mu(\tau^1 + \mathrm{i}\tau^2)\gamma_5\psi.$$

For degenerate masses the AWI reads

$$\partial_\mu A^{(r)a}_\mu = 2m^{(r)}P^{(r)a}, \qquad \text{where } P^{(r)a} = \frac{1}{2}\bar\psi\tau^a\gamma_5\psi$$

The matrix element with single pion at rest

$$\langle 0|\phi^{(r)a}(x)|\pi^b(\mathbf{p}=0)\rangle = \delta_{ab}\, e^{-M_\pi t},$$

The matrix element with single pion at rest

$$\langle 0|\phi^{(r)a}(x)|\pi^b(\mathbf{p}=0)\rangle = \delta_{ab}\, e^{-M_\pi t},$$

The propagator

$$\langle 0|\phi^{(r)a}(\mathbf{p}=0,t)\phi^{(r)b}(0)|0\rangle = \frac{1}{2M_\pi\sqrt{\omega}}\delta_{ab}\, e^{-M_\pi t},$$

The matrix element with single pion at rest

$$\langle 0|\phi^{(r)a}(x)|\pi^b(\mathbf{p}=0)\rangle = \delta_{ab}\, e^{-M_\pi t},$$

The propagator

$$\langle 0|\phi^{(r)a}(\mathbf{p}=0,t)\phi^{(r)b}(0)|0\rangle = \frac{1}{2M_\pi\sqrt{\omega}}\delta_{ab}\, e^{-M_\pi t},$$

The decay $\pi \to e\nu_e$ is described by a Lagrangian, it has the coupling term

$$\frac{G_F}{\sqrt{2}} cos\theta_c(\bar{u}\gamma_\mu(1-\gamma_5)d)(\bar{e}\gamma_\mu(1-\gamma_5)\nu_e),$$

this is an examlple of operator product expansion. This leads to the axial vector current between π and vacuum. The relation to physical pion defines F_π as

$$\partial_\mu A_\mu^{(r)a} = M_\pi^2 F_\pi \phi^{(r)a}$$

The divergence of the axial vector current acts as interpolating field operator of the pion

$$\partial_\mu \langle 0|A_\mu^{(r)a}(x)\pi^b(\mathbf{p}=0)\rangle = M_\pi^2 F_\pi \delta_{ab}\, e^{-M_\pi t},$$

this is partially conserved axial vector current or PCAC relation. This allows also to compute F_π from the pair correlators between A_4 and $\partial_t A_4$

$$\langle A_4^{(r)+}(\mathbf{p}=0,t)A_4^{(r)-}(0)\rangle \sim \frac{M_\pi F_\pi^2}{\sqrt{\omega}}\, e^{-M_\pi t},$$

The divergence of the axial vector current acts as interpolating field operator of the pion

$$\partial_\mu \langle 0|A_\mu^{(r)a}(x)\pi^b(\mathbf{p}=0)\rangle = M_\pi^2 F_\pi \delta_{ab}\, e^{-M_\pi t},$$

this is partially conserved axial vector current or PCAC relation. This allows also to compute F_π from the pair correlators between A_4 and $\partial_t A_4$

$$\langle A_4^{(r)+}(\mathbf{p}=0,t)A_4^{(r)-}(0)\rangle \sim \frac{M_\pi F_\pi^2}{\sqrt{\omega}}\, e^{-M_\pi t},$$

compare with nonsinglet AWI

$$\boxed{M_\pi^2 F_\pi \langle 0|\phi^{(r)a}|\pi\rangle = \langle 0|\partial_\mu A_\mu^{(r)a}|\pi\rangle = 2m^{(r)}\langle 0|P^{(r)a}|\pi\rangle,}$$

$$\langle 0|\delta O|0\rangle - \langle 0|O\delta S|0\rangle = 0.$$

$$\delta S = \int d^4x \left(-\partial_\mu(\bar{\psi}\gamma_\mu\lambda\psi) + \underbrace{\bar{\psi}\left(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\right)\psi}_{=0} \right).$$

$$\langle 0|\delta O|0\rangle - \langle 0|O\delta S|0\rangle = 0.$$

$$\delta S = \int d^4x \left(-\partial_\mu(\bar{\psi}\gamma_\mu\lambda\psi) + \underbrace{\bar{\psi}\left(\hat{\lambda}\mathscr{M} + \mathscr{M}\lambda\right)\psi}_{=0} \right).$$

$$\lambda = \hat{\lambda} = \gamma_5 \frac{1}{2}(\tau^1 + \mathrm{i}\tau^2),$$

$$O = P^+(0) = \bar{d}(0)\,\gamma_5\,u(0), \quad \langle 0|\delta P^+|0\rangle = \langle \bar{u}(0)u(0) + \bar{d}(0)d(0)\rangle\,,$$

$$\langle 0|\delta O|0\rangle - \langle 0|O\delta S|0\rangle = 0.$$

$$\delta S = \int d^4x \left(-\partial_\mu(\bar\psi\gamma_\mu\lambda\psi) + \underbrace{\bar\psi\big(\hat\lambda\mathscr{M} + \mathscr{M}\lambda\big)\psi}_{=0} \right).$$

$$\lambda = \hat\lambda = \gamma_5 \frac{1}{2}(\tau^1 + \mathrm{i}\tau^2),$$

$$O = P^+(0) = \bar d(0)\,\gamma_5\,u(0), \quad \langle 0|\delta P^+|0\rangle = \langle \bar u(0)u(0) + \bar d(0)d(0)\rangle\,,$$

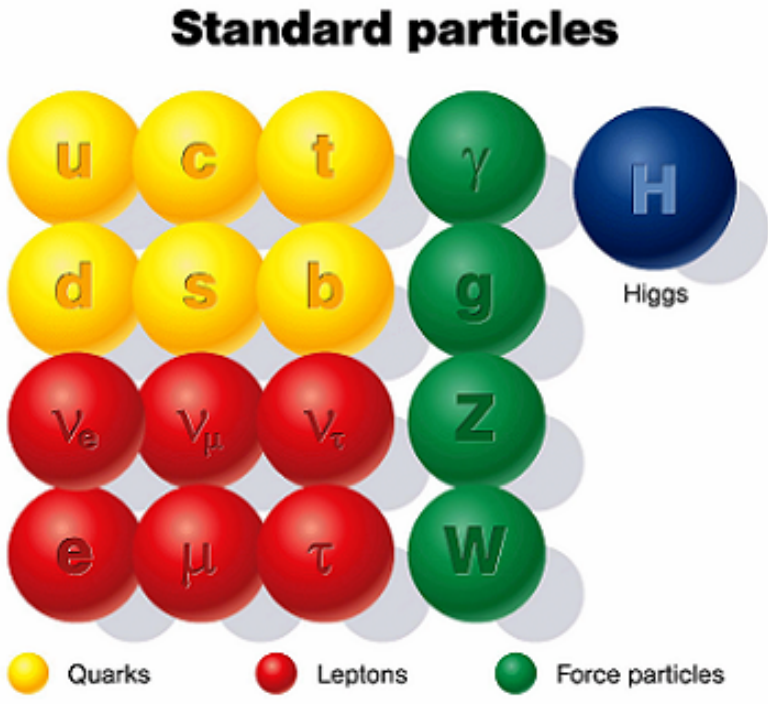
$$-\int d^4x\ \langle \partial_\mu A_\mu^{(r)-}(x) P^+(0)\rangle = \langle \bar u(0)u(0) + \bar d(0)d(0)\rangle\,,$$

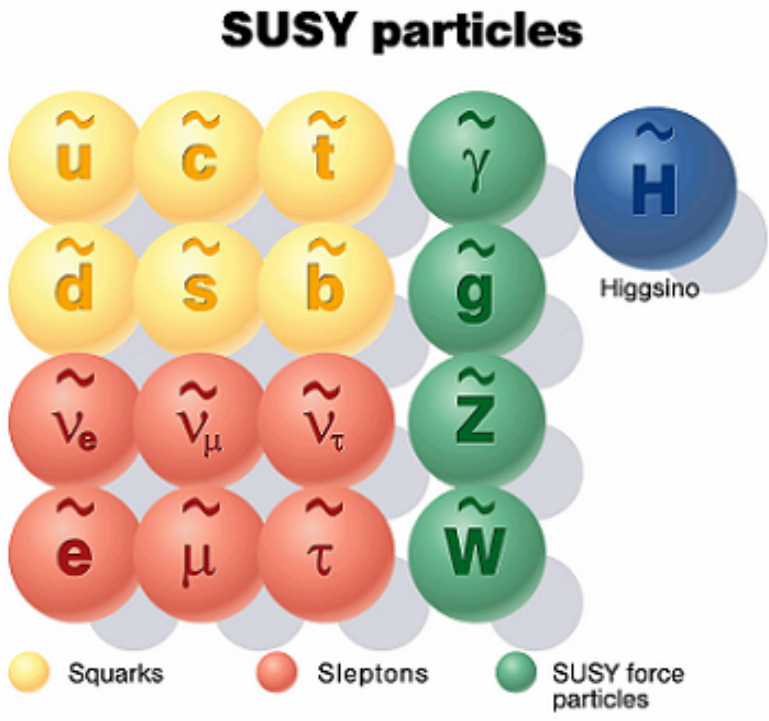
$$-\frac{F_\pi^2 M_\pi^4}{2m^r} \underbrace{\int d^4x\ \langle \phi^-(x)\phi^+(0)\rangle}_{2M_\pi^{-2} \text{ at } \mathbf{p}=0} = N_f \Sigma^{(r)},$$

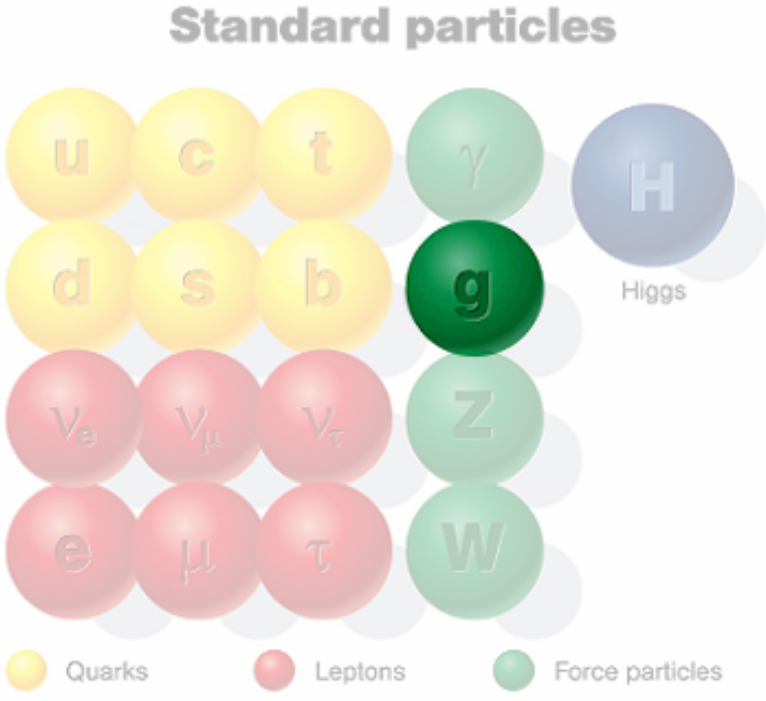
$$F_\pi^2 M_\pi^2 = -m^{(r)} N_f \Sigma^{(r)}.$$

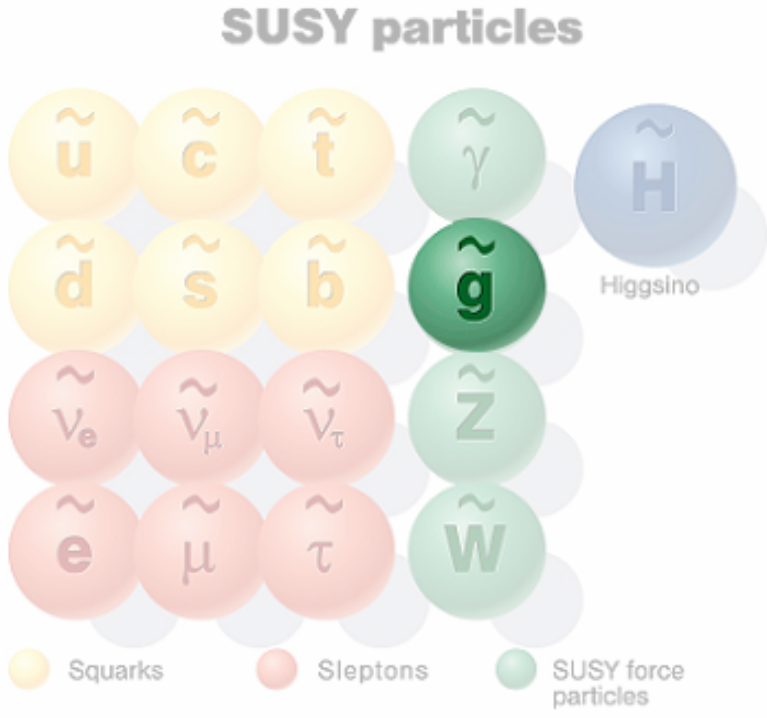
Gell–Mann–Oakes–Renner (GMOR) relation

Numerical analysis of SUSY Ward identities and continuum extrapolation.


Standard particles
u
c
t
γ
H
Higgs
d
s
b
g
ν_e
ν_μ
ν_τ
Z
e
μ
τ
W
Quarks
Leptons
Force particles


SUSY particles
$\tilde{u}$
$\tilde{c}$
$\tilde{t}$
$\tilde{\gamma}$
$\tilde{H}$
Higgsino
$\tilde{d}$
$\tilde{s}$
$\tilde{b}$
$\tilde{g}$
$\tilde{\nu}_e$
$\tilde{\nu}_\mu$
$\tilde{\nu}_\tau$
$\tilde{Z}$
$\tilde{e}$
$\tilde{\mu}$
$\tilde{\tau}$
$\tilde{W}$
Squarks
Sleptons
SUSY force particles





$\mathscr{N}=1$ supersymmetric Yang-Mills theory

Spectrum

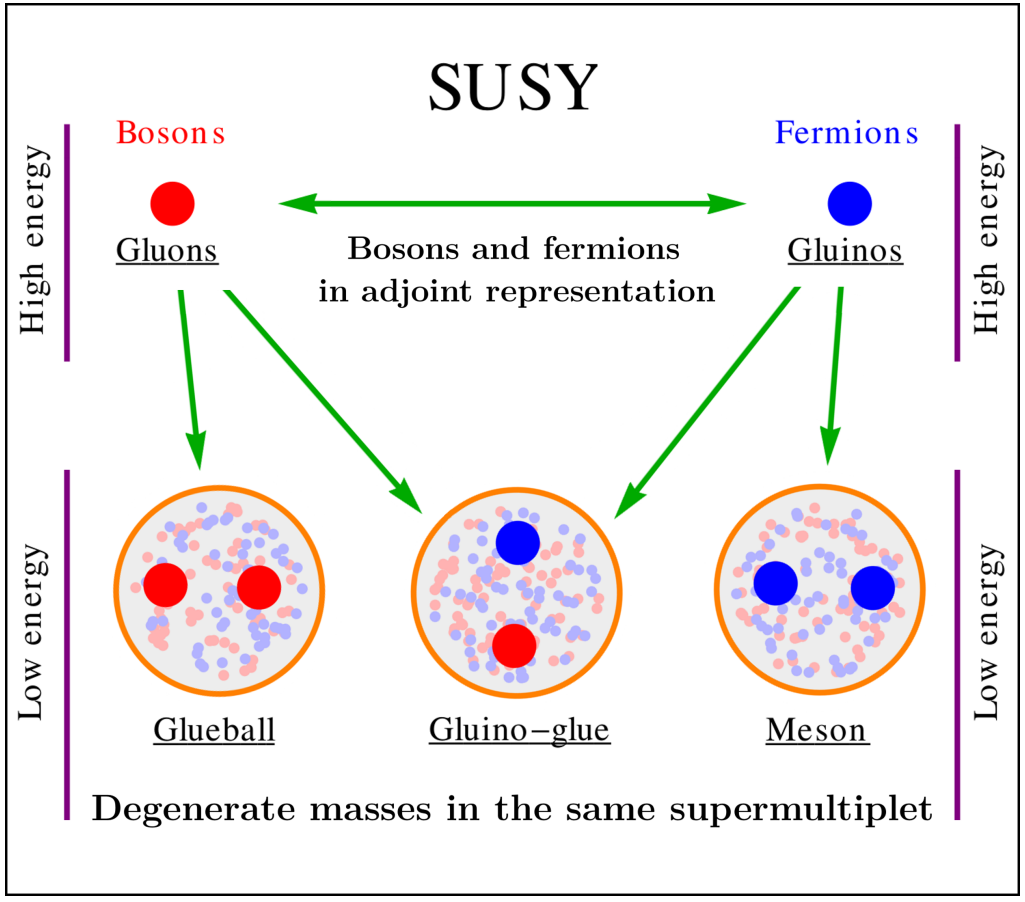


Figure: Qualitative: Formation of bound states at low energy.

Supersymmetric Yang-Mills theory

$$S_{SYM} = \int d^4x \Big\{ -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \frac{\mathrm{i}}{2} \bar{\lambda}_a \gamma_\mu (\mathscr{D}_\mu \lambda)^a - \frac{1}{2} m_{\tilde{g}} \bar{\lambda}_a \lambda^a \Big\}$$

- Field strength tensor
  $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \mathrm{i}\, g\, f^a_{bc} A^b_\mu A^c_\nu$, Gauge field A^a_μ (gluon)

Supersymmetric Yang-Mills theory

$$S_{SYM} = \int d^4x \Big\{ -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \frac{\mathrm{i}}{2} \bar{\lambda}_a \gamma_\mu (\mathscr{D}_\mu \lambda)^a - \frac{1}{2} m_{\tilde{g}} \bar{\lambda}_a \lambda^a \Big\}$$

- Field strength tensor
  $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \mathrm{i}\, g\, f^a_{bc} A^b_\mu A^c_\nu$, Gauge field A^a_μ (gluon)
- Covariant derivative in adjoint representation
  $(\mathscr{D}_\mu \lambda)^a = \partial_\mu \lambda^a + g\, f^a_{bc} A^b_\mu \lambda^c$, $a = 1, \ldots, N_c^2 - 1$
  fermion field λ^a (gluino)

Supersymmetric Yang-Mills theory

$$S_{SYM} = \int d^4x \Big\{ -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} + \frac{i}{2} \bar{\lambda}_a \gamma_\mu (\mathscr{D}_\mu \lambda)^a - \frac{1}{2} m_{\tilde{g}} \bar{\lambda}_a \lambda^a \Big\}$$

- Field strength tensor
  $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - i g f^a_{bc} A^b_\mu A^c_\nu$, Gauge field A^a_μ (gluon)
- Covariant derivative in adjoint representation
  $(\mathscr{D}_\mu \lambda)^a = \partial_\mu \lambda^a + g f^a_{bc} A^b_\mu \lambda^c$, $a = 1, \ldots, N_c^2 - 1$
  fermion field λ^a (gluino)
- Gluino mass term $\boxed{\frac{1}{2} m_{\tilde{g}} \bar{\lambda}_a \lambda^a}$ breaks SUSY softly
- In contrast to QCD:
  1) adjoint representation of SU(N_c)
  2) Majorana condition, $\boxed{\bar{\lambda}_a = \lambda_a^T C}$
  3) λ^a is Majorana spinor field, "$N_f = \frac{1}{2}$"

SUSY Ward identities

Noether's Theorem in classical theory $\rightarrow$ WIs in quantum theory

SUSY Ward identities

Noether's Theorem in classical theory $\rightarrow$ WIs in quantum theory

$$\boxed{\left\langle \partial^{\mu} S_{\mu}(x) Q(y) \right\rangle = -\left\langle \frac{\delta Q(y)}{\delta \bar{\varepsilon}(x)} \right\rangle .}$$

SUSY Ward identities

Noether's Theorem in classical theory $\rightarrow$ WIs in quantum theory

$$\boxed{\left\langle \partial^{\mu} S_{\mu}(x) Q(y) \right\rangle = -\left\langle \frac{\delta Q(y)}{\delta \bar{\varepsilon}(x)} \right\rangle .}$$

- $S_{\mu}(x)$ is super current
- $Q(y)$ is local insertion operator
- $\bar{\varepsilon}(x)$ is fermionic parameter
- RHS of equation is contact term, which is zero if Q is localised at space-time points different from x

SUSY transformations on the lattice with P, T, Majorana nature, and gauge invariance:

$$\delta U_\mu(x) = -\frac{\mathrm{i}ga}{2}\left(\bar{\varepsilon}(x)\gamma_\mu U_\mu(x)\lambda(x) + \bar{\varepsilon}(x+\hat{\mu})\gamma_\mu\lambda(x+\hat{\mu})U_\mu(x)\right),$$

$$\delta\lambda(x) = +\frac{1}{2}P^{(cl)}_{\mu\nu}(x)\sigma_{\mu\nu}\varepsilon(x).$$

SUSY transformations on the lattice with P, T, Majorana nature, and gauge invariance:

$$\delta U_\mu(x) = -\frac{iga}{2}\left(\bar{\varepsilon}(x)\gamma_\mu U_\mu(x)\lambda(x) + \bar{\varepsilon}(x+\hat{\mu})\gamma_\mu\lambda(x+\hat{\mu})U_\mu(x)\right),$$

$$\delta\lambda(x) = +\frac{1}{2}P^{(cl)}_{\mu\nu}(x)\sigma_{\mu\nu}\varepsilon(x).$$

Above transformations result in following Ward identities:

$$\boxed{\left\langle \nabla_\mu S_\mu(x) Q(y)\right\rangle = m_{\tilde{g}}\left\langle \chi(x)Q(y)\right\rangle + \left\langle X_S(x)Q(y)\right\rangle - \left\langle \frac{\delta Q(y)}{\delta\bar{\varepsilon}(x)}\right\rangle,}$$

$$S_\mu(x) = -\frac{2i}{g}\mathrm{tr}\left[P^{(cl)}_{\rho\nu}(x)\sigma_{\rho\nu}\gamma_\mu\lambda(x)\right], \chi(x) = \frac{2i}{g}\mathrm{tr}\left[P^{(cl)}_{\mu\nu}(x)\sigma_{\mu\nu}\lambda(x)\right].$$

SUSY transformations on the lattice with P, T, Majorana nature, and gauge invariance:

$$\delta U_\mu(x) = -\frac{iga}{2}\left(\bar{\varepsilon}(x)\gamma_\mu U_\mu(x)\lambda(x) + \bar{\varepsilon}(x+\hat{\mu})\gamma_\mu\lambda(x+\hat{\mu})U_\mu(x)\right),$$

$$\delta\lambda(x) = +\frac{1}{2}P^{(cl)}_{\mu\nu}(x)\sigma_{\mu\nu}\varepsilon(x).$$

Above transformations result in following Ward identities:

$$\boxed{\left\langle \nabla_\mu S_\mu(x) Q(y)\right\rangle = m_{\tilde{g}}\left\langle \chi(x)Q(y)\right\rangle + \left\langle X_S(x)Q(y)\right\rangle - \left\langle \frac{\delta Q(y)}{\delta\bar{\varepsilon}(x)}\right\rangle,}$$

$$S_\mu(x) = -\frac{2i}{g}\mathrm{tr}\left[P^{(cl)}_{\rho\nu}(x)\sigma_{\rho\nu}\gamma_\mu\lambda(x)\right], \chi(x) = \frac{2i}{g}\mathrm{tr}\left[P^{(cl)}_{\mu\nu}(x)\sigma_{\mu\nu}\lambda(x)\right].$$

$\chi(x)$: due to bare gluino mass ($m_{\tilde{g}}$)
$X_S(x)$: due to lattice regularization (a)
} *Break SUSY*

Renormalization

$$\boxed{Z_S\left\langle \nabla_\mu S_\mu(x) Q(y)\right\rangle + Z_T\left\langle \nabla_\mu T_\mu(x) Q(y)\right\rangle = m_S\left\langle \chi(x) Q(y)\right\rangle + O(a).}$$

Z_S and Z_T are renormalization coefficients, $m_S = m_{\tilde{g}} - \bar{m}$.

$$T_\mu(x) = \frac{2\mathrm{i}}{g}\mathrm{tr}\left[P^{(cl)}_{\mu\nu}(x)\gamma_\nu\lambda(x)\right], Q(y) = \frac{2\mathrm{i}}{g}\sum_{i<j}\mathrm{tr}\left[P^{(cl)}_{ij}(y)\sigma_{ij}\lambda(y)\right].$$

Renormalization

$$Z_S\left\langle \nabla_\mu S_\mu(x) Q(y)\right\rangle + Z_T\left\langle \nabla_\mu T_\mu(x) Q(y)\right\rangle = m_S\left\langle \chi(x) Q(y)\right\rangle + O(a).$$

Z_S and Z_T are renormalization coefficients, $m_S = m_{\tilde{g}} - \bar{m}$.

$$T_\mu(x) = \frac{2\mathrm{i}}{g}\mathrm{tr}\left[P^{(cl)}_{\mu\nu}(x)\gamma_\nu\lambda(x)\right], Q(y) = \frac{2\mathrm{i}}{g}\sum_{i<j}\mathrm{tr}\left[P^{(cl)}_{ij}(y)\sigma_{ij}\lambda(y)\right].$$

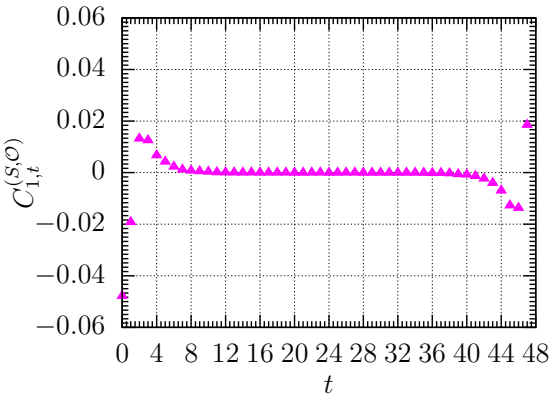
Zero spatial momentum WIs and expansion in a basis of 16 Dirac matrices; the surviving contributions form a set of two non-trivial independent equations:

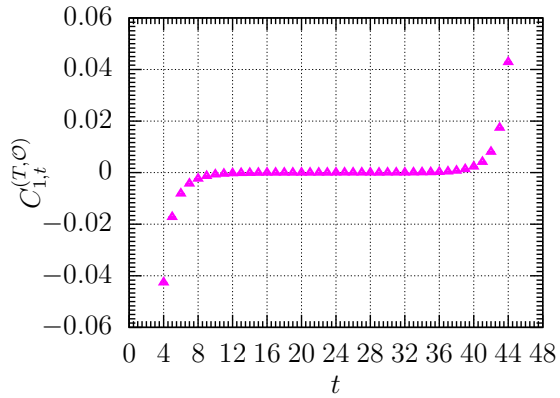
$$1C^{(S,\mathscr{O})}_{1,t} + AC^{(T,\mathscr{O})}_{1,t} = BC^{(\chi,\mathscr{O})}_{1,t},$$
$$1C^{(S,\mathscr{O})}_{\gamma_4,t} + AC^{(T,\mathscr{O})}_{\gamma_4,t} = BC^{(\chi,\mathscr{O})}_{\gamma_4,t}.$$

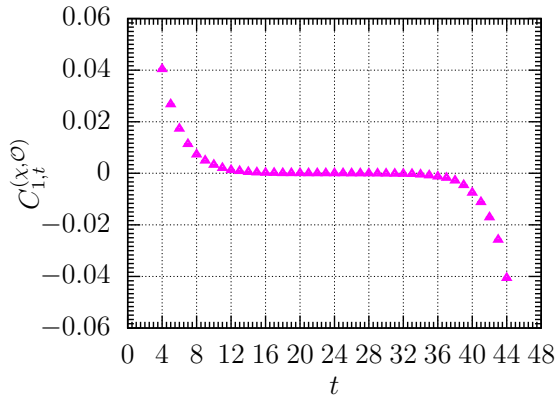
$$A = Z_T Z_S^{-1},\ B = am_S Z_S^{-1},$$

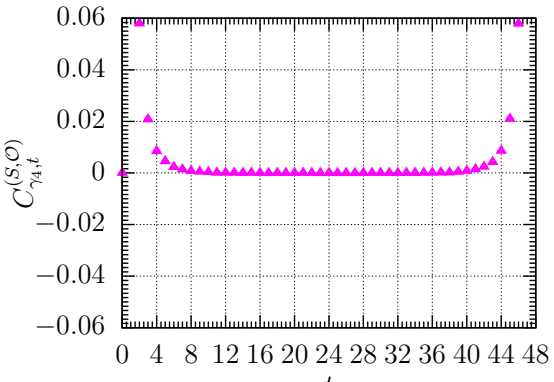
$$1x_{b,t} + Ay_{b,t} - Bz_{b,t} = 0, \qquad b = 1,2$$

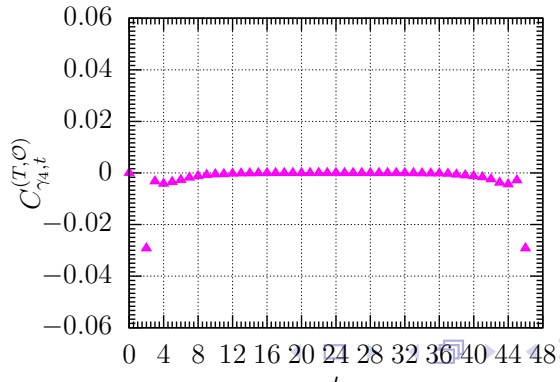
Numerical results of correlation functions











The Global method

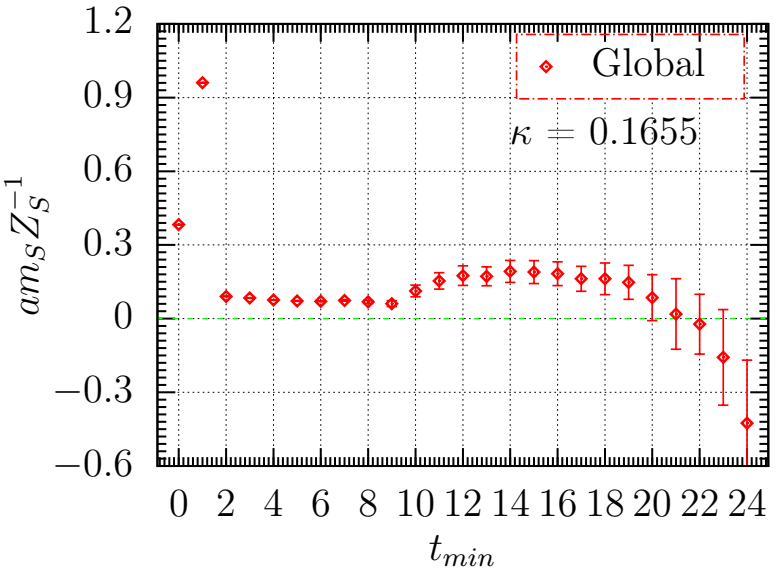
Calculate A and B by minimizing the quantity:

$$\sum_{i=1}^{2} \sum_{t=t_{min}}^{t_{max}} \frac{(x_{i,t} + A y_{i,t} - B z_{i,t})^2}{\sigma_t^2}$$

The Global method

Calculate A and B by minimizing the quantity:

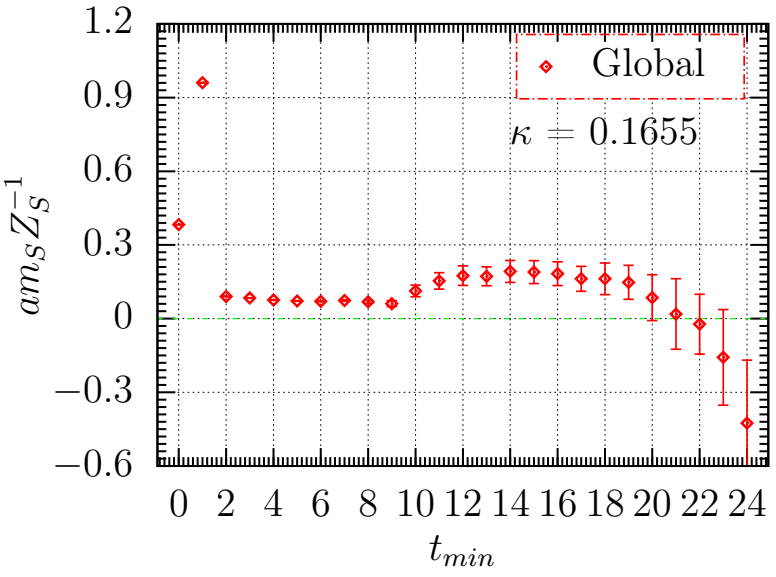
$$\sum_{i=1}^{2} \sum_{t=t_{min}}^{t_{max}} \frac{(x_{i,t} + A y_{i,t} - B z_{i,t})^2}{\sigma_t^2}$$



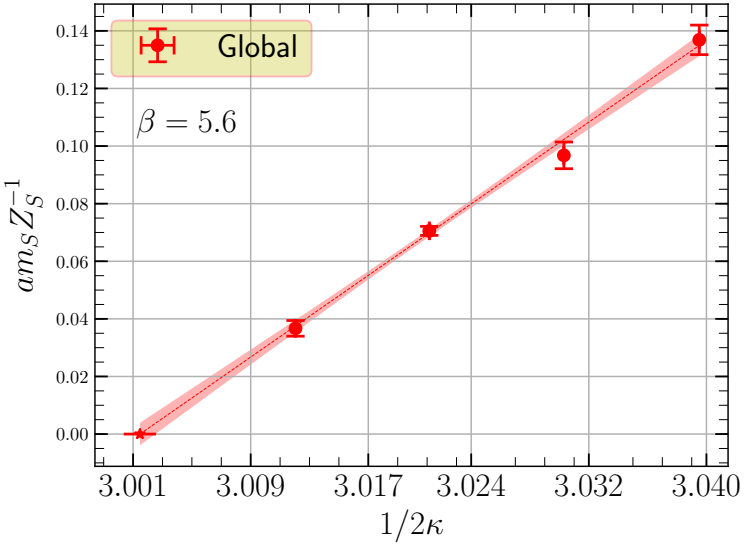
The Global method

Calculate A and B by minimizing the quantity:

$$\sum_{i=1}^{2}\sum_{t=t_{min}}^{t_{max}} \frac{(x_{i,t} + A y_{i,t} - B z_{i,t})^2}{\sigma_t^2}$$



Extrapolation to the Chiral limit using the subtracted gluino mass ($am_S Z_S^{-1}$)



The generalised least squared (GLS) method

$$1x_{b,t} + Ay_{b,t} - Bz_{b,t} = 0, \quad b = 1, 2$$

The generalised least squared (GLS) method

$$1x_{b,t} + Ay_{b,t} - Bz_{b,t} = 0, \quad b = 1,2$$

$$\boxed{\sum_\alpha A_\alpha \hat{x}_{i\alpha} = 0,} \quad i = (b,t), \alpha = 1,2,3$$

The generalised least squared (GLS) method

$1x_{b,t} + Ay_{b,t} - Bz_{b,t} = 0, \quad b = 1,2$

$$\boxed{\sum_\alpha A_\alpha \hat{x}_{i\alpha} = 0,} \quad i = (b,t), \alpha = 1,2,3$$

We employ the method of maximum likelihood

$$L = \frac{1}{2} \sum_{i,\alpha,j,\beta} (A_\alpha \hat{x}_{i\alpha})(D^{-1})_{ij}(A_\beta \hat{x}_{j\beta}), \quad D_{ij} = \sum_{\alpha,\beta} A_\alpha A_\beta (\langle x_{i\alpha} x_{j\beta} \rangle - \hat{x}_{i\alpha}\hat{x}_{j\beta})$$

find A_α such that L is minimum

The generalised least squared (GLS) method

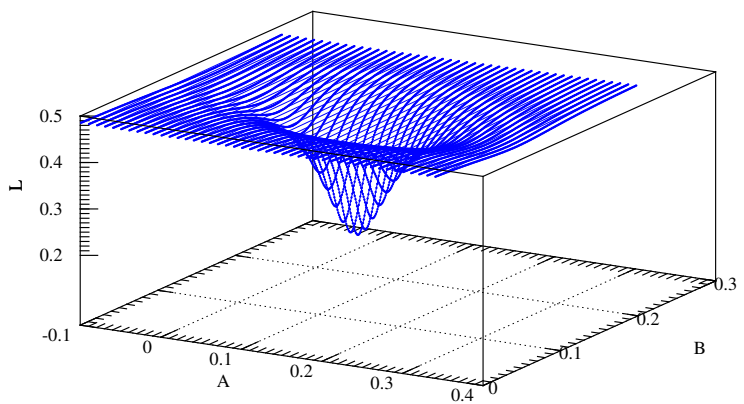
$1x_{b,t} + Ay_{b,t} - Bz_{b,t} = 0, \quad b = 1,2$

$$\boxed{\sum_\alpha A_\alpha \hat{x}_{i\alpha} = 0,} \quad i = (b,t), \alpha = 1,2,3$$

We employ the method of maximum likelihood

$$L = \frac{1}{2} \sum_{i,\alpha,j,\beta} (A_\alpha \hat{x}_{i\alpha})(D^{-1})_{ij}(A_\beta \hat{x}_{j\beta}), \quad D_{ij} = \sum_{\alpha,\beta} A_\alpha A_\beta (\langle x_{i\alpha} x_{j\beta} \rangle - \hat{x}_{i\alpha}\hat{x}_{j\beta})$$

find A_α such that L is minimum



(g) Determination of A and B.

The generalised least squared (GLS) method

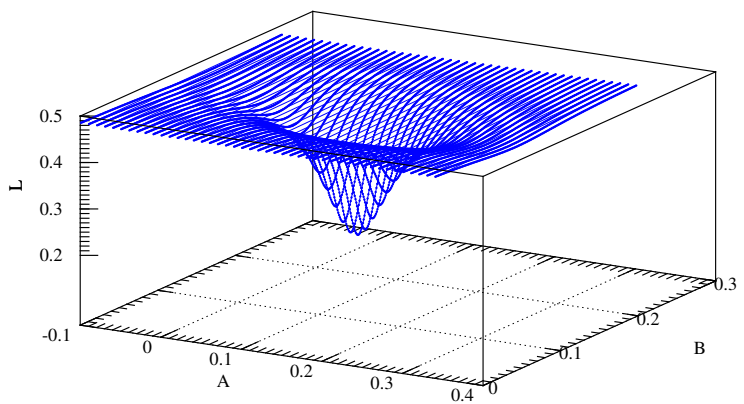
$1x_{b,t} + Ay_{b,t} - Bz_{b,t} = 0, \quad b = 1,2$

$$\boxed{\sum_\alpha A_\alpha \hat{x}_{i\alpha} = 0,} \quad i = (b,t), \alpha = 1,2,3$$

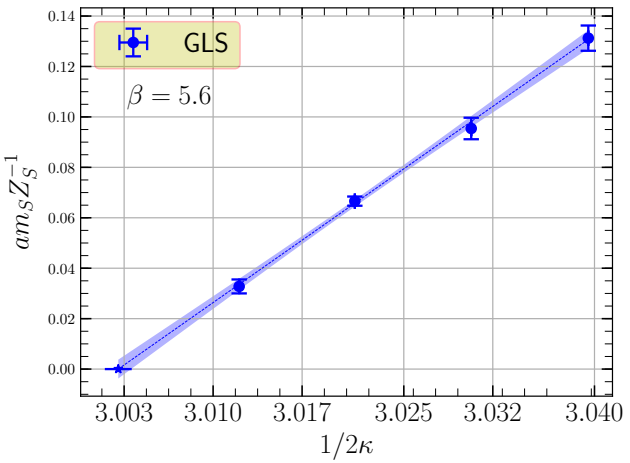
We employ the method of maximum likelihood

$$L = \frac{1}{2} \sum_{i,\alpha,j,\beta} (A_\alpha \hat{x}_{i\alpha})(D^{-1})_{ij}(A_\beta \hat{x}_{j\beta}), \quad D_{ij} = \sum_{\alpha,\beta} A_\alpha A_\beta (\langle x_{i\alpha} x_{j\beta} \rangle - \hat{x}_{i\alpha}\hat{x}_{j\beta})$$

find A_α such that L is minimum



(i) Determination of A and B.



(j) Chiral extrapolation.

The adjoint Pion

- The $m_{a-\pi}$ is obtained numerically in simulations of $\mathcal{N}=1$ SYM theory
- It is being used for the tuning of chiral limit

The adjoint Pion

- The $m_{a-\pi}$ is obtained numerically in simulations of $\mathcal{N}=1$ SYM theory
- It is being used for the tuning of chiral limit

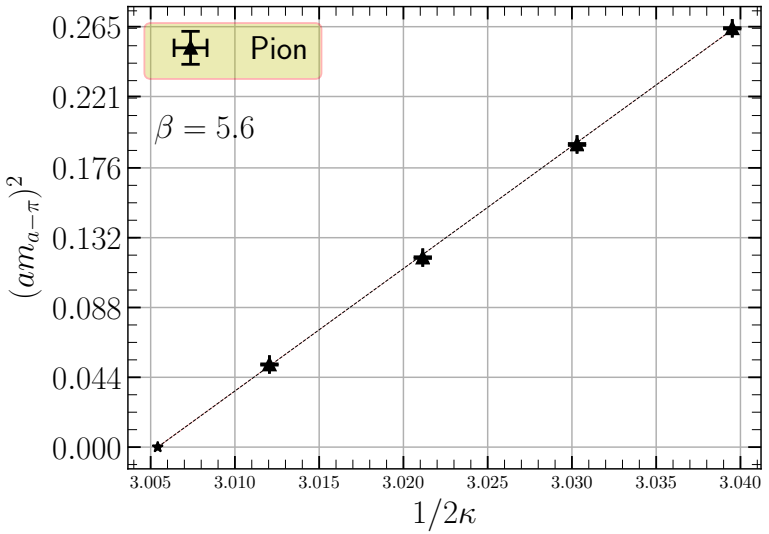


Figure: Extrapolation to the chiral limit using $m^2_{a-\pi}$.

The adjoint Pion

- The $m_{a-\pi}$ is obtained numerically in simulations of $\mathcal{N}=1$ SYM theory
- It is being used for the tuning of chiral limit

$$(am_{a-\pi})^2 \propto am_S Z_S^{-1}$$

(analogous to GOR relation)

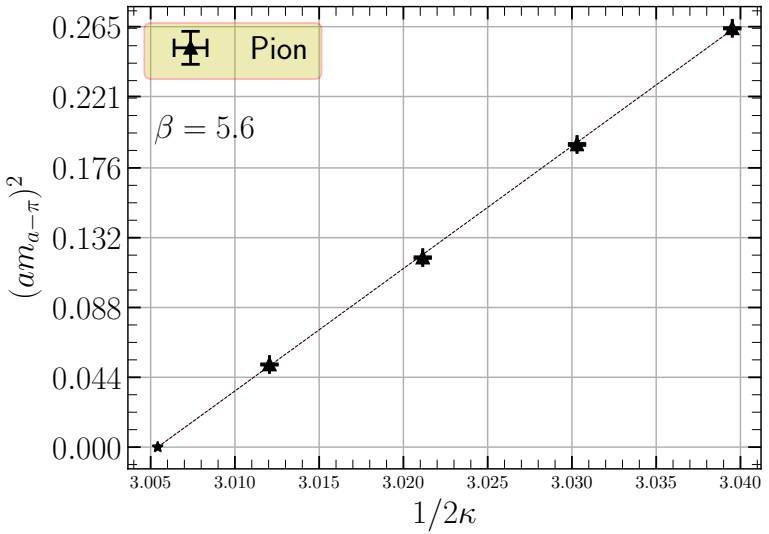
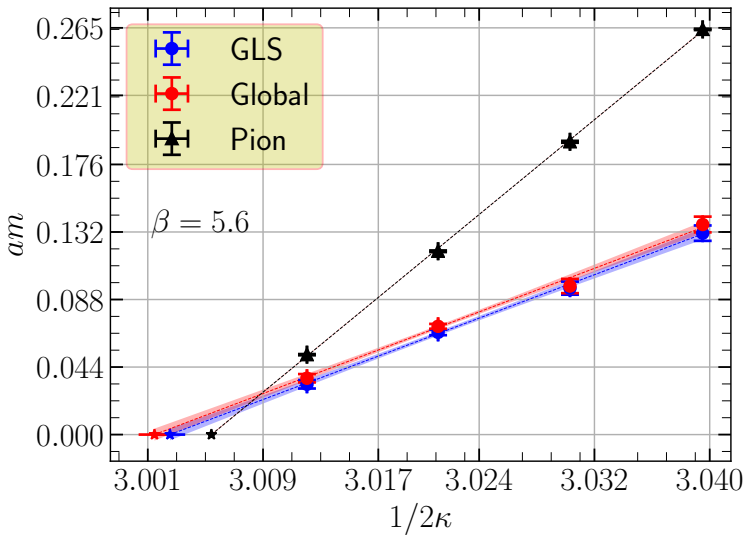
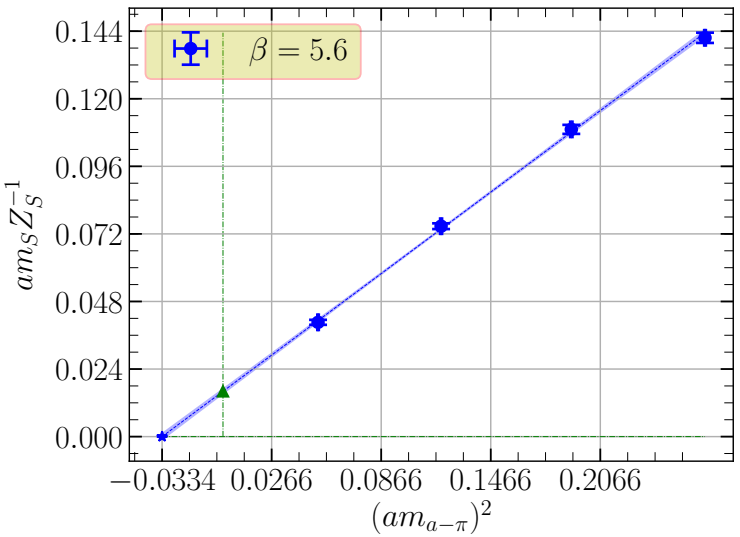


Figure: Extrapolation to the chiral limit using $m_{a-\pi}^2$.

Chiral limit



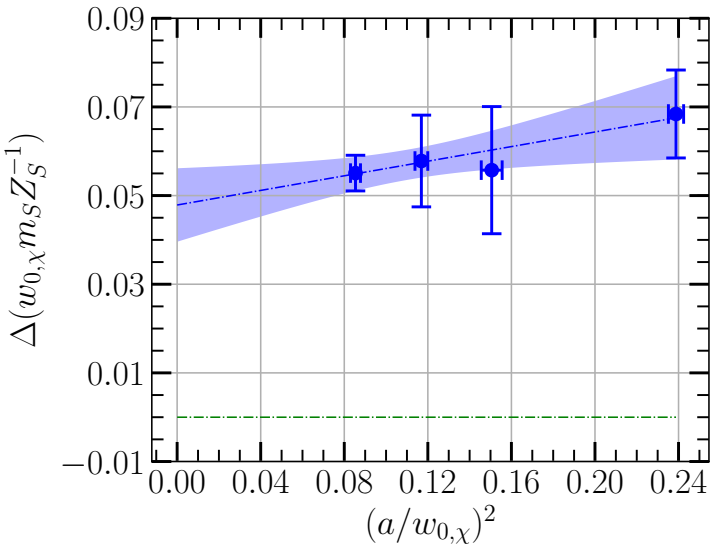
(a) Comparison.



(b) $\Delta(am_S Z_S^{-1})$.

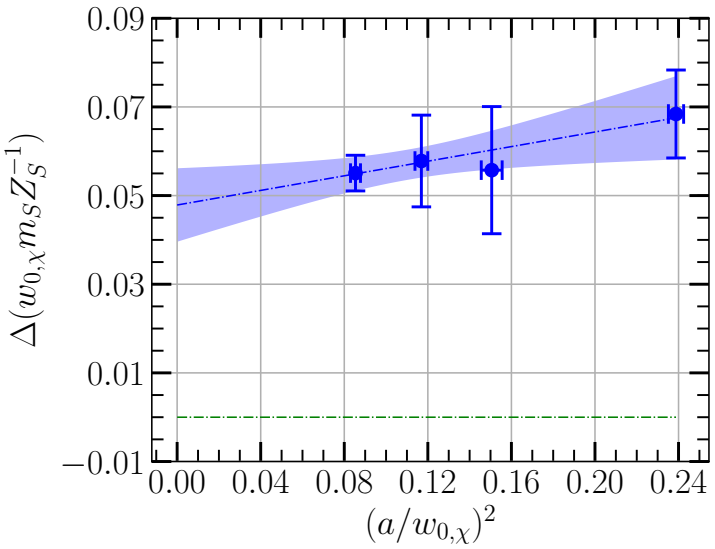
Figure: 4(a): Comparison of $am_S Z_S^{-1}$ from two methods with $m_{a-\pi}^2$. 4(b): $\Delta(am_S Z_S^{-1})$ using $m_{a-\pi}^2$ and $am_S Z_S^{-1}$.

Continuum limit



Continuum limit in physical scale using t_{min}=3

Continuum limit



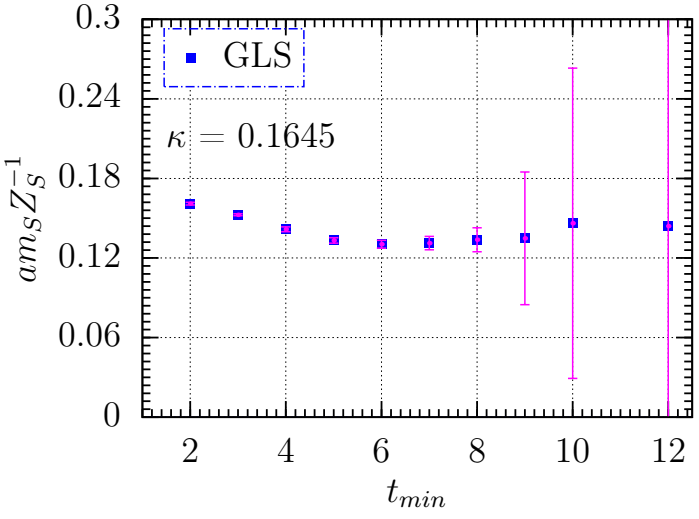
Continuum limit in physical scale using t_{min}=3

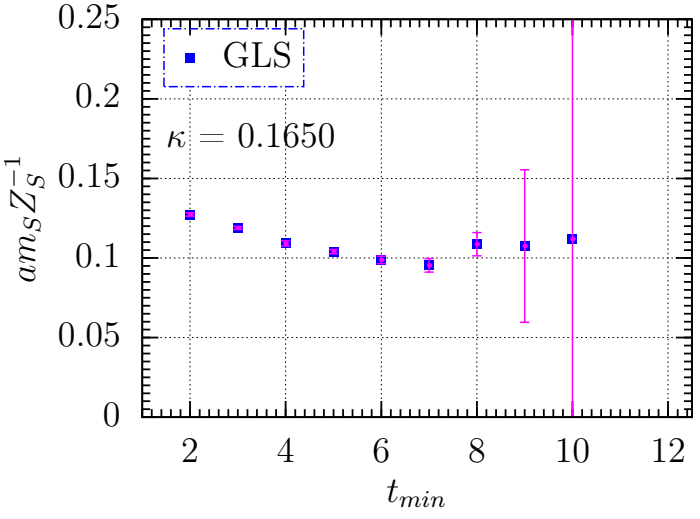
Handling of discretisation effects

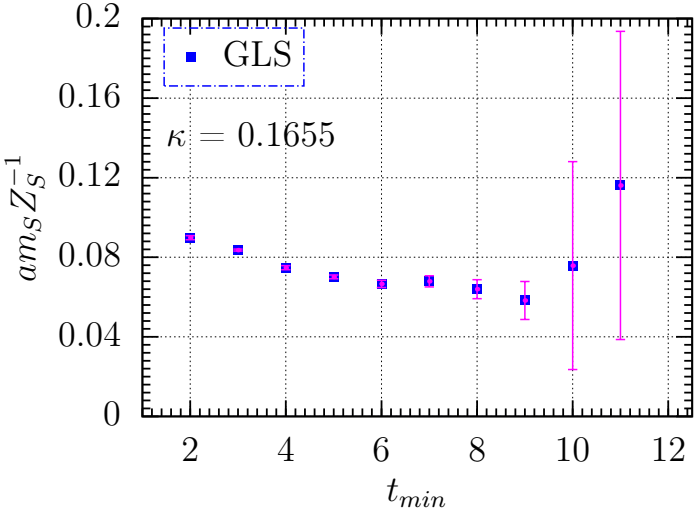
What about $O(a)$? in

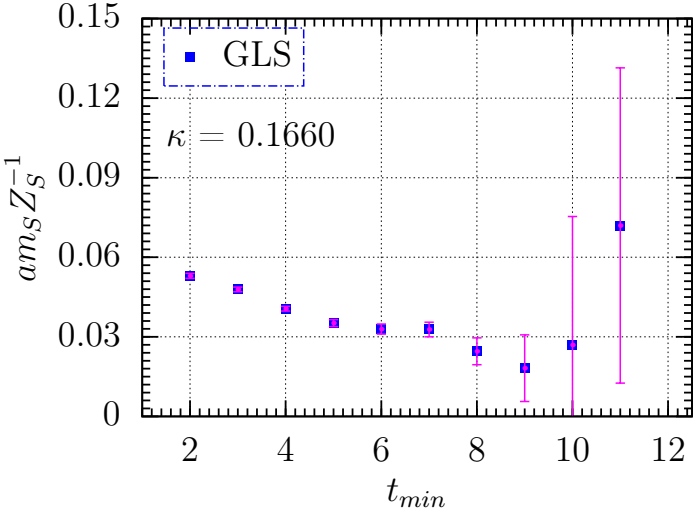
$$Z_S\langle\nabla_\mu S_\mu(x)Q(y)\rangle + Z_T\langle\nabla_\mu T_\mu(x)Q(y)\rangle = m_S\langle\chi(x)Q(y)\rangle + O(a).$$

Sufficiently large t_{min}, $\beta = 5.6$









Fixed physical time slice distance

$$t_{min,\beta_i} = t_{min,\beta_0} \frac{m_{g\tilde{g},\beta_0}}{m_{g\tilde{g},\beta_i}}, \quad t_{min,\beta_i} = t_{min,\beta_0} \frac{w_{0,\beta_i}}{w_{0,\beta_0}},$$

where $i = 1, 2, 3$, $\beta_0 = 5.4$ and $\beta_i = 5.45, 5.5, 5.6$.

Fixed physical time slice distance

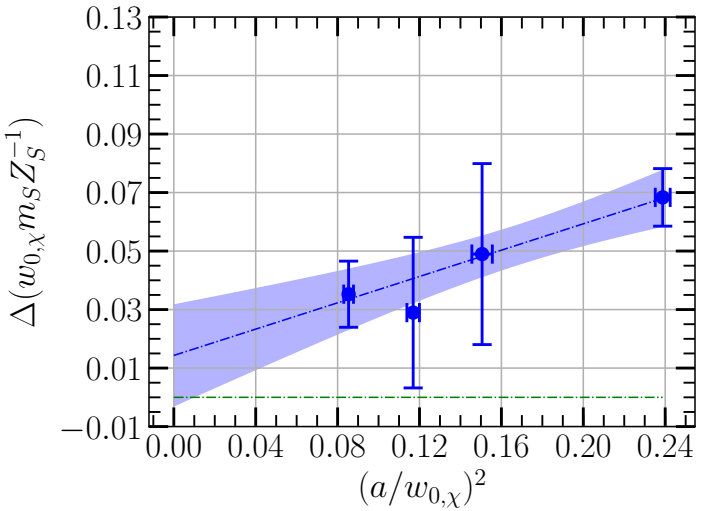
$$t_{min,\beta_i} = t_{min,\beta_0} \frac{m_{g\tilde{g},\beta_0}}{m_{g\tilde{g},\beta_i}}, \quad t_{min,\beta_i} = t_{min,\beta_0} \frac{w_{0,\beta_i}}{w_{0,\beta_0}},$$

where $i = 1,2,3$, $\beta_0 = 5.4$ and $\beta_i = 5.45, 5.5, 5.6$.

β	t_{min} from $m_{g\tilde{g}}$	t_{min} from w_0
5.4	4	4
5.45	5	5
5.5	5	6
5.6	7	7

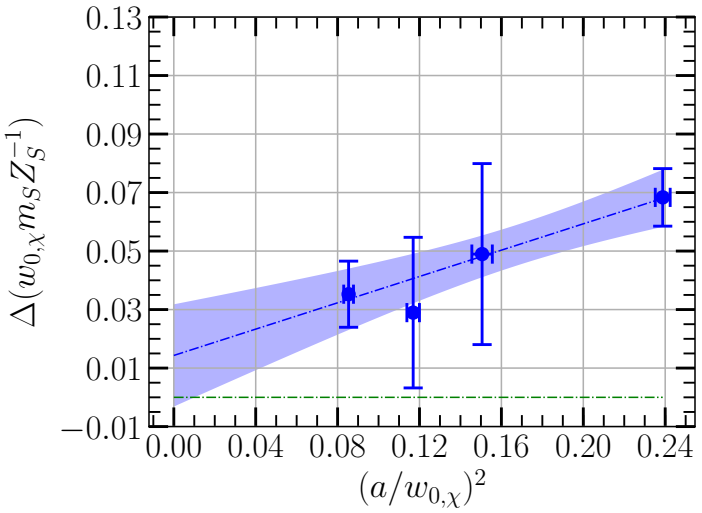
Table: The choice of t_{min} at a fixed physical time slice distance from the gluino-glue mass ($m_{g\tilde{g}}$) and from the physical scale w_0.

Continuum limit

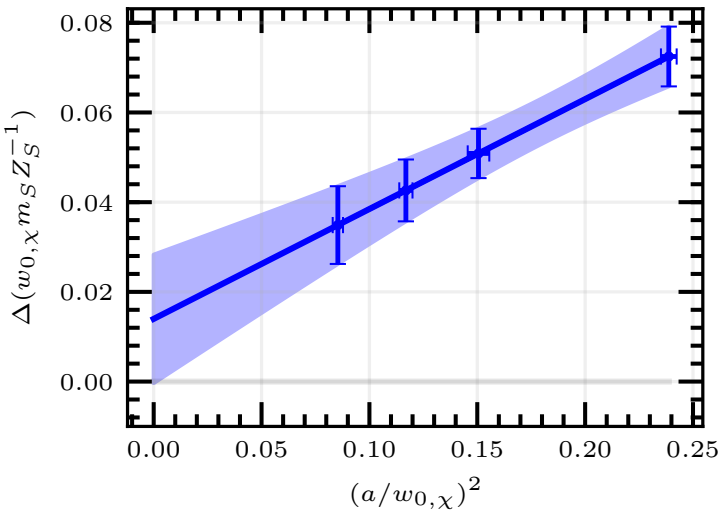


(a) Fit by two step procedure.

Continuum limit



(c) Fit by two step procedure.



(d) Fit by one step procedure.

Prediction from effective actions

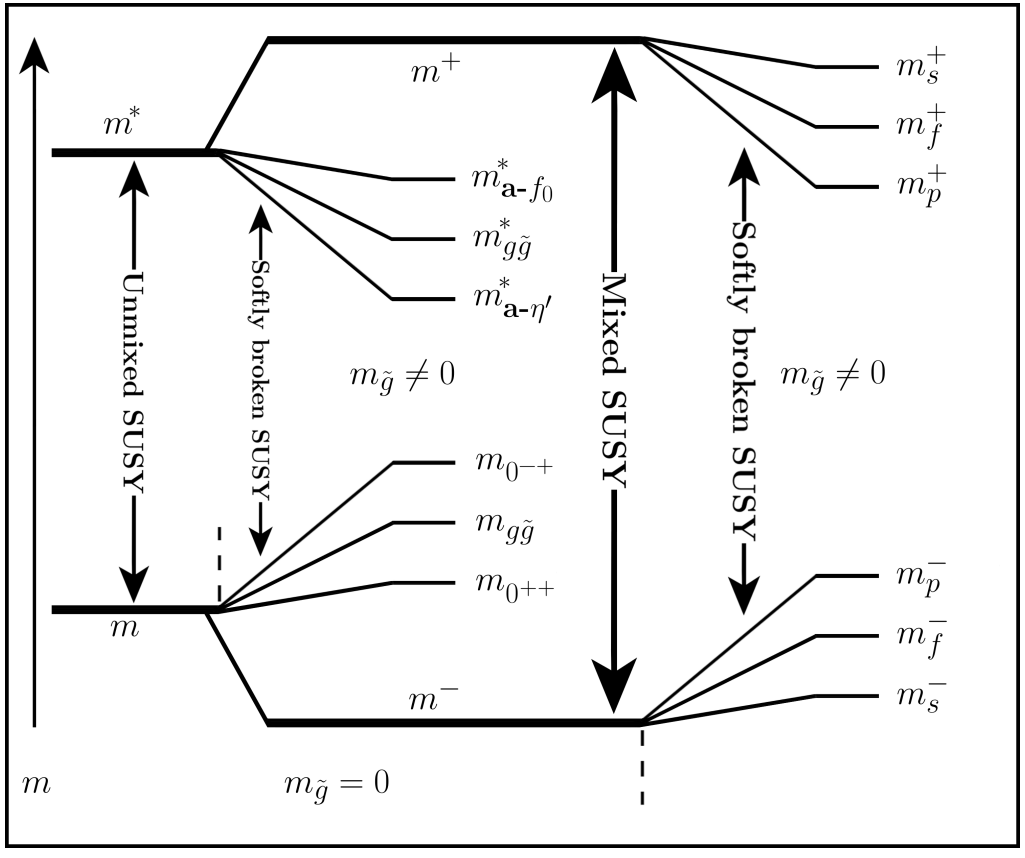


Figure: Spectrum predicted by the VY and its generalisation form two distinct supermultiplets. The degenracy of masses is shifted when $m_{\tilde{g}}$ is switched on.

Chiral Ward identity

$$SU(N_f)_L \times SU(N_f)_R \times U_A(1) \times U_V(1)$$

Chiral Ward identity

$$SU(N_f)_L \times SU(N_f)_R \times U_A(1) \times U_V(1)$$

Non-singlet case:

Vector trans. $\psi' = e^{iw^a T^a}\psi \to \langle \partial^\mu V^a_\mu(x) O(y)\rangle = 0$

Axial trans. $\psi' = e^{iw^a T^a \gamma_5}\psi \to \langle \partial^\mu A^a_\mu(x) O(y)\rangle = 2m\langle P^a(x) O(y)\rangle$

Chiral Ward identity

$$SU(N_f)_L \times SU(N_f)_R \times U_A(1) \times U_V(1)$$

Non-singlet case:

Vector trans. $\psi' = e^{iw^a T^a}\psi \rightarrow \langle \partial^\mu V^a_\mu(x) O(y)\rangle = 0$

Axial trans. $\psi' = e^{iw^a T^a \gamma_5}\psi \rightarrow \langle \partial^\mu A^a_\mu(x) O(y)\rangle = 2m\langle P^a(x) O(y)\rangle$

constants c_A, c_V in O(a) improved currents, $A^a_{I,\mu} = A^a_\mu + a c_A \tilde{\partial}_\mu P^a$
$\rightarrow$ decay constants etc.

Chiral Ward identity

$$SU(N_f)_L \times SU(N_f)_R \times U_A(1) \times U_V(1)$$

Non-singlet case:

Vector trans. $\psi' = e^{iw^a T^a}\psi \rightarrow \langle \partial^\mu V^a_\mu(x) O(y)\rangle = 0$

Axial trans. $\psi' = e^{iw^a T^a \gamma_5}\psi \rightarrow \langle \partial^\mu A^a_\mu(x) O(y)\rangle = 2m\langle P^a(x) O(y)\rangle$

constants c_A, c_V in O(a) improved currents, $A^a_{I,\mu} = A^a_\mu + a c_A \tilde{\partial}_\mu P^a$
$\rightarrow$ decay constants etc.

Singlet case:

The matrix element of divergence of axial flavour singlet current between two on-shell states

$$Z_A \langle \alpha | \partial^\mu J^5_\mu | \beta \rangle = 2\langle \alpha | \bar{\psi}\gamma_5 \bar{m}\psi | \beta \rangle + Z_F \langle \alpha | \mathrm{tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right] | \beta \rangle$$

Chiral Ward identity

$$SU(N_f)_L \times SU(N_f)_R \times U_A(1) \times U_V(1)$$

Non-singlet case:

Vector trans. $\psi' = e^{iw^a T^a}\psi \rightarrow \langle \partial^\mu V^a_\mu(x) O(y)\rangle = 0$

Axial trans. $\psi' = e^{iw^a T^a \gamma_5}\psi \rightarrow \langle \partial^\mu A^a_\mu(x) O(y)\rangle = 2m\langle P^a(x) O(y)\rangle$

constants c_A, c_V in O(a) improved currents, $A^a_{I,\mu} = A^a_\mu + a c_A \tilde{\partial}_\mu P^a$
$\rightarrow$ decay constants etc.

Singlet case:

The matrix element of divergence of axial flavour singlet current between two on-shell states

$$Z_A \langle \alpha | \partial^\mu J^5_\mu | \beta \rangle = 2\langle \alpha | \bar{\psi}\gamma_5 \bar{m}\psi | \beta \rangle + Z_F \langle \alpha | \mathrm{tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right] | \beta \rangle$$

1). mass term 2). Adler-Bell-Jackiw anomaly $\rightarrow$ Topologial charge