Wilson fermions and chiral symmetry

Bastian Brandt

Universität Bielefeld

Based on: Gattringer-Lang, sections 5.2, 7.1 and 7.2

Journal Club – 15.05.2020

Why Wilson fermions?

- Why fermions on the lattice?
  - Fundamental for lattice formulation of full QCD!
- Is the lattice fermion action not unique?
  - **No!**

    As any other observable, actions are only unique in the continuum limit!

    (e.g., irrelevant terms [$\sim a$] can always be added to an observable)
- Why can we not simply use the naive discretisation?
  - Fermion doubling problem!

    We will come to that!

What is chiral symmetry and why is it important?

The QCD action for massless quarks ($N_f = 2$) is invariant under the global

chiral symmetry: $\mathrm{SU}_V(2) \times \mathrm{U}_V(1) \times \mathrm{SU}_A(2) \times \mathrm{U}_A(1)$

symmetry	transformation		status in nature
$\mathrm{SU}_V(2)$	$\psi(x) \to e^{i\theta_V^i \tau_i}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{-i\theta_V^i \tau_i}$	explicitly broken by Δm_q
$\mathrm{U}_V(1)$	$\psi(x) \to e^{i\theta_V^0}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{-i\theta_V^0}$	restored (Baryon number)
$\mathrm{SU}_A(2)$	$\psi(x) \to e^{i\theta_A^i \tau_i \gamma_5}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{i\theta_A^i \tau_i \gamma_5}$	spontaneously broken + explicitly broken by m_q
$\mathrm{U}_A(1)$	$\psi(x) \to e^{i\theta_A^0 \gamma_5}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{i\theta_A^0 \gamma_5}$	anomalously broken + explicitly broken by m_q

τ_i: Pauli matrices

Importance of chiral symmetry:

- **fundamental for hierarchy of hadron masses**
- **effective restoration drives the thermal QCD phase transition**

$\Rightarrow$ **We better try to retain this in our discretisation!**

Contents

1. **The fermion doubling problem**
   - naive discretisation: 15 additional fermions – doublers
2. **Wilson's solution – Wilson fermions**
   - add irrelevant term to action to decouple doublers for $a \to 0$
3. **Properties of the Wilson Dirac operator**
   - additional term breaks chiral symmetry explicitly
4. **Chiral symmetry on the lattice**
   - Nielsen-Ninomiya no-go theorem:
     either doublers or no chiral symmetry
   - workaround:
     Ginsparg-Wilson definition of chiral symmetry on the lattice

1. The fermion doubling problem

Naive discretisation of the Dirac operator

Fermion action for 1 massless flavour in continuum:

$$S_F = \int d^4x d^4y \, \bar{\psi}(x) D(x,y) \psi(y)$$

Dirac operator (Euclidean):

$$D(x,y) = \sum_{\mu=0}^{3} \gamma_\mu D_\mu \delta(x-y) \qquad \text{with} \quad D_\mu = \partial_\mu - igA_\mu(x)$$

Naive discretisation: **symmetrically discretise the derivative**

$$D(x,y) = \frac{1}{2a} \sum_{\mu=0}^{3} \gamma_\mu \left(U_\mu(x) \delta_{x,y-\hat{\mu}} - U_{-\mu}(x) \delta_{x,y+\hat{\mu}} \right)$$

$\hat{\mu}$: unit vector in μ-direction with $|\hat{\mu}| = a$

$U_{-\mu}(x) \equiv U_\mu^\dagger(x - \hat{\mu})$

on the lattice: $x_\mu = a n_\mu$ with $n_\mu = 1, 2, \ldots$

The doubling problem

Particles: **Poles of the free propagator**

Perform a Fourier transformation of the free Dirac operator:

$$D(p,p') = \frac{1}{a^4 N_{\mathrm{lat}}} \sum_{x,y} e^{-ip_\mu x_\mu} D(x,y) e^{ip'_\mu y_\mu} \qquad (N_{\mathrm{lat}} : \# \text{ of latt. points})$$

$$= \frac{1}{a^4 N_{\mathrm{lat}}} \sum_{x} e^{i(p'_\mu - p_\mu)x_\mu} \left[\frac{1}{2a} \sum_{\mu=0}^{3} \gamma_\mu \left(e^{ip'_\mu a} - e^{-ip'_\mu a} \right) \right] = \delta_{p,p'} D(p)$$

Propagator: inverse of the Dirac operator

Continuum

$$D^{-1}(p) = \frac{i \sum_{\mu=0}^{3} \gamma_\mu p_\mu}{p^2}$$

1 pole at $p^2 = 0$ ✓

Lattice

$$D^{-1}(p) = \frac{\frac{i}{a} \sum_{\mu=0}^{3} \gamma_\mu \sin(p_\mu a)}{\frac{1}{a^2} \sum_{\mu=0}^{3} \sin^2(p_\mu a)}$$

✓ physical pole at $p_\mu = 0 \ \forall \mu$

↯ 15 unphysical poles at $p_\mu = 0, \pi/a$

$\Rightarrow$ **doublers**

2. Wilson's solution – Wilson fermions

The Wilson term

To obtain physical results: **need to remove doubler from the theory**

allowed to add terms to the action which vanish for $a \to 0$!

Wilson: add term which modifies doublers and leaves physical pole unchanged.

[Wilson, Confinement of Quarks, Phys. Rev. D10, 2445 (1974)]

$$O_{\mathrm{Wil}}^{\mathrm{free}}(p) = \frac{1}{a}\sum_{\mu}(1-\cos(p_\mu a))$$

Dirac operator in momentum space: (r: Wilson coefficient)

$$D(p) = -\frac{i}{a}\sum_{\mu=0}^{3}\gamma_\mu \sin(p_\mu a) + \frac{r}{a}\sum_{\mu=0}^{3}(1-\cos(p_\mu a))$$

Propagator: $$D^{-1}(p) = \Big[-\frac{i}{a}\sum_{\mu=0}^{3}\gamma_\mu \sin(p_\mu a) + \frac{r}{a}\sum_{\mu=0}^{3}(1-\cos(p_\mu a))\Big]^{-1}$$

$\Rightarrow$ Pole at $(0,0,0,0)$ unchanged ✓

Previous poles with l appearances of $p_\mu = \pi/a$: get mass $m_{\mathrm{doub}} = 2lr/a$

The Wilson term in position space

For the position space action need to know Wilson term $O_{\mathrm{Wil}}(x,y)$.

Fourier back transformation:

$$\begin{aligned} \frac{1}{a}(1-\cos(p_\mu a)) &= \frac{1}{2a}\left(2-e^{ip_\mu a}-e^{-ip_\mu a}\right) \\ \Rightarrow \quad O_{\mathrm{Wil}}^{\mathrm{free}}(x,y) &= a\cdot\frac{1}{2a^2}\sum_{\mu=0}^{3}\left(2\delta_{x,y}-\delta_{x,y-\hat{\mu}}-\delta_{x,y+\hat{\mu}}\right) \end{aligned}$$

$\Rightarrow \quad O_{\mathrm{Wil}}^{\mathrm{free}}(x,y)$: **discretised version of the Laplace operator** $-a/2\partial_\mu\partial_\mu$

Interacting case (gauge invariant):

$$O_{\mathrm{Wil}}\psi(x) = a\frac{1}{2a^2}\sum_{\mu=0}^{3}\left[2\psi(x)-U_\mu(x)\psi(x+\hat{\mu})-U_{-\mu}(x)\psi(x-\hat{\mu})\right]$$

The Wilson Dirac operator

Full Dirac operator:

$$D_{\mathrm{Wil}} = D + rO_{\mathrm{Wil}}$$

Wilson coefficient r:

- free parameter
- multiplies a lattice artefact

  $\Rightarrow$ irrelevant for physical observables (at $a \to 0$)!
- typically set $r = 1$

Putting everything together (massless case):

$$D_{\mathrm{Wil}}(x,y) = -\frac{1}{2a}\sum_{\mu=0}^{3}\left((1-\gamma_\mu)U_\mu(x)\delta_{x,y-\hat{\mu}} + (1+\gamma_\mu)U_{-\mu}(x)\delta_{x,y+\hat{\mu}}\right) + \frac{4}{a}\delta_{x,y}$$

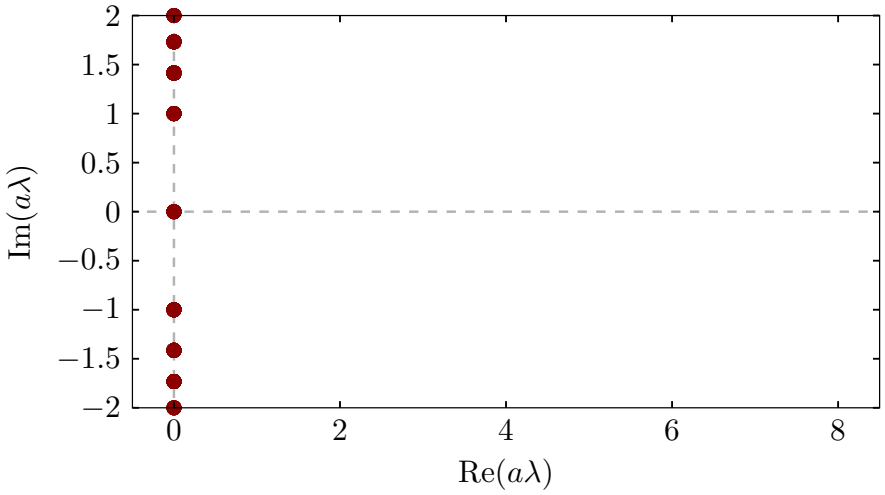
3. Properties of the Wilson Dirac operator

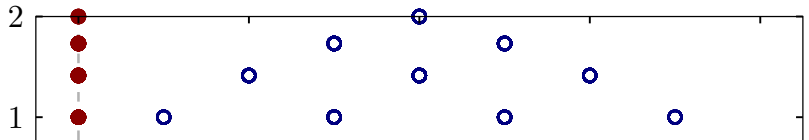
Properties of the Wilson Dirac operator

Property	Naive discretisation	Wilson Dirac operator
Operator	D	$D_{\mathrm{Wil}} = D + O_{\mathrm{Wil}}$
hermiticity	anti-hermitian	γ_5-hermitian
	$D^\dagger = -D$	$\gamma_5 D_{\mathrm{Wil}}^\dagger \gamma_5 = D_{\mathrm{Wil}}$
particles ($a \to 0$)	16	1
CPT	✓	✓

Eigenvalues 4^4 lattice

[eigenvalues: courtesy of Gergő]





Wilson Dirac operator and chiral symmetry

symmetry	transformation		D	D_{Wil}
$\mathrm{SU}_V(2)$	$\psi(x) \to e^{i\theta_V^i \tau_i}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{-i\theta_V^i \tau_i}$	✓	✓
$\mathrm{U}_V(1)$	$\psi(x) \to e^{i\theta_V^0}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{-i\theta_V^0}$	✓	✓
$\mathrm{SU}_A(2)$	$\psi(x) \to e^{i\theta_A^i \tau_i \gamma_5}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{i\theta_A^i \tau_i \gamma_5}$	✓	↯
$\mathrm{U}_A(1)$	$\psi(x) \to e^{i\theta_A^0 \gamma_5}\psi(x)$	$\bar\psi(x) \to \bar\psi e^{i\theta_A^0 \gamma_5}$	✓	↯

Vector transformations:

$$\tau_i \text{ commutes with everything (for } m_q = 0)$$

$\Rightarrow$ $\mathrm{SU}_V(2) \times \mathrm{U}_V(1)$ is a symmetry for D_{Wil}

Axial transformations:

$$D\gamma_5 = -\gamma_5 D \qquad \text{and} \qquad O_{\mathrm{Wil}}\gamma_5 = \gamma_5 O_{\mathrm{Wil}}$$

$\Rightarrow$ $\mathrm{SU}_A(2) \times \mathrm{U}_A(1)$ is explicitly broken by O_{Wil}.

Advantages and disadvantages of D_{Wil}

Disadvantages:

- chiral symmetry broken explicit by O_{Wil}
  - quark mass renormalises additively for Wilson fermions
  - D_{Wil} not protected against negative real eigenvalues
    - makes simulations at small (physical) m_q difficult and expensive
    - conceptually difficult for $N_f = 1 + 1$ simulations
  - complications for studies of QCD phase diagram/thermodynamics

Advantages:

- undistorted particle spectrum

  $\Rightarrow$ usually fermions of choice for spectroscopy and hadron structure
- operator construction/mixings remain simple in different situations

(these things can become very tedious for staggered/twisted-mass fermions)

- cheaper compared to chiral fermion formulations

4. Chiral symmetry on the lattice

Nielsen-Ninomiya no-go theorem and Ginsparg-Wilson equation

To restore $\mathrm{SU}_A(2) \times \mathrm{U}_A(1)$: $\quad D\gamma_5 + \gamma_5 D = 0 \quad (*)$

Nielsen-Ninomiya no-go theorem: [Nielsen, Ninomiya (1981)]

at least either of the three is violated

(i) equation $(*)$ (ii) locality (iii) $N_f = 1$

(Wilson fermions: violate (i))

Way to overcome this: relax equation $(*)$ to

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \qquad \textbf{Ginsparg-Wilson equation}$$

[Ginsparg, Wilson (1982) – rediscovered ∼1997/98]

- term on r.h.s. is a lattice artefact; vanishes for $a \to 0$
- all chiral fermion formulations today obey the GW-equation
  - overlap fermions
  - domain-wall fermions for $N_5 \to \infty$

Take-home messages:

- naive discretisation: 15 additional fermions – doublers
- Wilson fermions:

  add irrelevant term to action to decouple doublers for $a \to 0$
- additional term breaks chiral symmetry explicitly
- Nielsen-Ninomiya no-go theorem:

  either doublers or chiral symmetry breaking
- chiral symmetry at non-zero a:

  Ginsparg-Wilson extension of chiral symmetry to the lattice