Electric Fields in Lattice QCD
Lattice Journal Club

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Based mainly on an article by A.Yamamoto, arXiv:1210.8250 [1]

12.11.2021

Outline

Classical Electrodynamics

In classical electrodynamics we work with the electromagnetic field tensor $F = dA$.
Choosing a frame we get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

And in the cartesian coordinates we can get a simple expression for the electric field $E_i \propto F_{0i}$. In particular, in the axial gauge ($A_i = 0$):

$$E_i = -\partial_i A_0. \tag{1}$$

Lattice QCD

In the QFT formalism we work with the following Lagrangian density of the QCD:

$$\mathcal{L}_{QCD} = +\mathcal{L}_{GF} + \mathcal{L}_{FP} \tag{2}$$

In the lattice QCD we introduce gauge fields as link variables[2]:

$$D_\mu \psi \to \frac{1}{2a}\Big(U_\mu(n)\psi(n+\hat{\mu}) - U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu})\Big)\,, \tag{3}$$

where

$$U_\mu = exp(iagA_\mu(n)). \tag{4}$$

Introducing electric fields

We can introduce an electric field:

$$D_\mu = \partial_\mu + igA_\mu + iqA_\mu^{EM} \quad (5)$$

In the lattice approach this introduces extra factors of the $U(1)$ link variable, $exp(\pm iqA_\mu^{em})$ respectively.

Chemical potential

Due to equation 1, electric field enters the lattice action as $exp(aqA_0(x))$. Notice that this is similar to the way one would introduce a chemical potential μ in the exponential form. In fact, external electric field can be treated as a coordinate-dependent chemical potential, so these settings share a lot of similarities and common problems.

At finite temperatures one has to consider both the electric field and the chemical potential.

Sign problem

Non-zero real chemical potential introduces a highly oscillatory complex phase, which makes it very hard to sample the theory. This is the so-called sign problem[2]. A simple way to see the problem is to note that the γ_5-Hermiticity of the Wilson-Dirac operator is broken for the non-zero real chemical potential. Plugging extra factors of $exp(\pm a\mu)$ into the temporal hopping factor of the Dirac-Wilson operator, we get[2]:

$$\gamma_5 D(\mu)\gamma_5 = D^\dagger(-\mu) \neq D^\dagger(\mu). \qquad (6)$$

Sign problem

Lorenzo Dini mentioned some general reasons of the occurrence of this problem, which indicate that it might not be possible to circumvent the sign problem completely. He also discussed some commmon prescriptions of working around it in some settings.

Workarounds

The article features two methods of circumventing the sign problem in the presence of an electric field, both of which were motivated by the similar approaches to the non-zero chemical potetial, which preserve the γ_5-Hermiticity of the Wilson-Dirac operator[3].
In particular,
imaginary chemical potential $\sim$ Euclidian electric field
isospin chemical potential in two-flavour QCD $\sim$ isospin electric charge:

$$q_3 \equiv e\frac{\sigma_3}{2} = \begin{pmatrix} \frac{e}{2} & 0 \\ 0 & -\frac{e}{2} \end{pmatrix} \tag{7}$$

Workarounds

Note that the full Minkowskian treatment can only be done in the quenched lattice QCD[4]. Moreover, a constant Minkowskian electric field is impossible, as the vector potential must be a real number.

The author wanted to study some non-perturbative properties of the quark-antiquark pair creation so he didn't use an Euclidean electric field as it cannot describe particle generation.

Periodic boundary conditions

The author used the Minkowskian electric field in a finite box with periodic boundary conditions. The vector potential in the axial gauge is set to be:

$$A_0(z) = \begin{cases} +E_0(z - \frac{L}{4}) \text{ for } (\frac{L}{2} > z \geq 0) \\ -E_0(z - \frac{3L}{4}) \text{ for } (\frac{L}{2} > z \geq 0) \end{cases}$$

So that the corresponding electric field is given by:

$$E(z) = -\partial_z A_0 = \begin{cases} -E_0 \text{ for } (\frac{L}{2} > z \geq 0) \\ +E_0 \text{ for } (\frac{L}{2} > z \geq 0) \end{cases} \tag{8}$$

and the voltage difference is $V = E_0 L/2$

Periodic boundary conditions

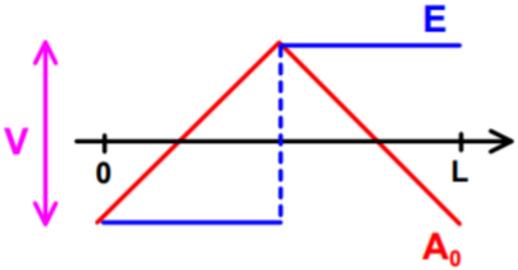


FIG. 1: The configuration of the vector potential $A_0(z)$ and the electric field $E(z)$. The boundary condition is periodic.

Effects of the electric field

When a strong external electric field is applied, quark-antiquark pairs are created by the Schwinger mechanism[5]. This process, however, cannot be observed directly in lattice QCD[1].
In the confinement phase there are two ways to separate the charged particles:
meson condensation $\rightarrow$ this process has a voltage threshold of $m_\pi^+ + m_\pi^-$
deconfinement $\rightarrow$ threshold of m_π^0

Effects of the electric field

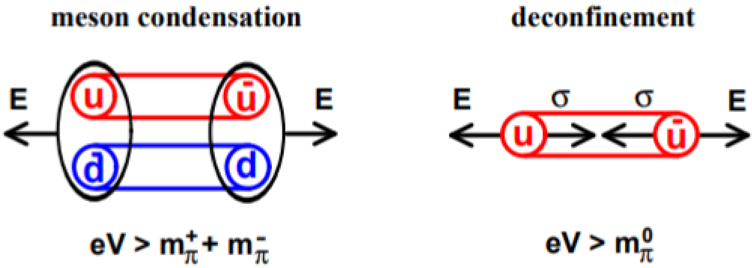


FIG. 2: The meson condensation and the deconfinement.

Effects of the electric field

Note that the latter effect is sensitive to the system volume. When the quark mass is small, the spatial size of the quark-antiquark pair gradually increases. Thus, the charge density gradually appears in a finite volume even if the electric field is smaller than the confining force, while this effect is suppressed in a larger volume.

Setting

The author performed the two-flavour, full QCD simulation with the plaquette gauge action and the Wilson fermion action [1]. For a rectangular $R \times T$ Wilson loop (in the $z - t$ plane) and the vector potential given in we get:

$$\langle W_C(R,T)\rangle = \langle W_{SU(3)}(R,T)\rangle exp(\frac{e}{2}E_0 RT) \tag{9}$$

Charged heavy-quark potential

From equation 9 we get the charged heavy-quark potential:

$$V_C(R) = V_{SU(3)}(R) - \frac{e}{2} E_0 R$$

In this setting it turns out to be consistent with its quenched QCD analogue *Cornell potential*:

$$V_C(R) = \left(\sigma - \frac{e}{2} E_0\right) R + \frac{A}{R} + const \qquad (10)$$

The electric field suppresses the linear confining potential.
At $aeV = 0.96$, the linear confining potential disappears because the electric field is almost the same as the string tension, $a^2 eE_0/2 = 0.08 \approx a^2\sigma$.

Charged heavy-quark potential

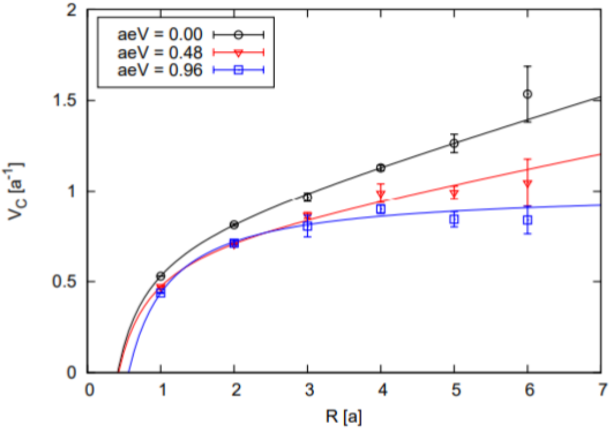


FIG. 3: The charged heavy-quark potential $V_C(R)$.

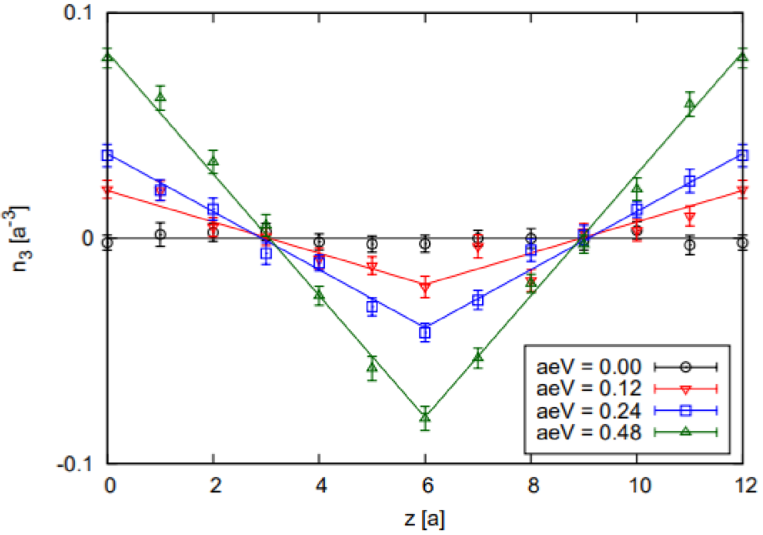
Charge density distribution

Next the author calculated the charge density:

$$n_3(z) \equiv \frac{1}{e} \frac{\partial \ln Z}{\partial A_0(z)}. \quad (11)$$

It is equivalent to an isospin density[3].
Non-zero charge density signifies the appearance of the charged particles in the regions of high voltage, u- and $\bar{d}$- quarks for positive n_3 and $\bar{u}$- and d- quarks for negative.

Charge density distribution



Charge density distribution

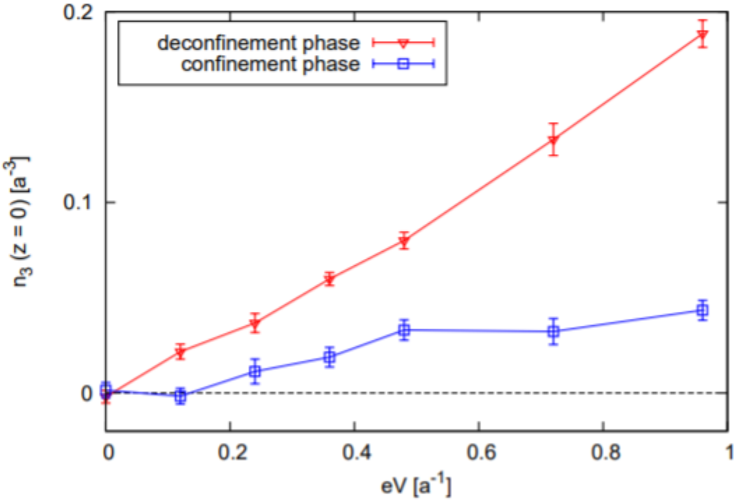
In the final figure the author explores the dependence of $n_3(z=0)$ on voltage V.

In the deconfinement phase, charge density grows monotonically as the created particles flow freely.

The pion mass at $V=0$ is $am_\pi \approx 0.26$. The charge density in the region $eV < 2m_\pi = 0.52$ is generated not by the meson condensation but by the deconfinement in a finite volume.

Asymptotic $L \to \infty$ suggests that the deconfinement happens for $eE/2 \geq \sigma$ in equation 10.

Charge density distribution



Summary

- An external electric field is equivalent to the coordinate-dependent chemical potential. Thus in the general setting it suffers from the sign problem.
- Two common ways to circumvent the sigh problem are to consider an Euclidian electric field or an isospin charge in two-flavour QCD.
- External electric field induces the creation of the quark-antiquark pairs via the Schwinger mechanism, this is a non-equilibrium process
- There are two ways to separate the particles: deconfinement and meson condensation.

References I

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