Fluctuation determinants in the presence of instantons

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These are a complimentary set of notes for the *Lattice Journal Club* presentation, held on *Friday November 18th, 2022*. The notes largely cover the same material presented during the journal club meeting, but also discuss further details on certain topics. Additional information on derivations and related topics can be found in the references. Enjoy!

Contents

1 Topological susceptibility in QCD

In the context of QCD with a non-zero θ-term, we known that a given field configuration will contribute to the partition function differently, according to its *topological charge* Q:

$$Q = \frac{1}{16\pi^2} \int d^4x \ \mathrm{tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu} \in \mathbb{Z} \tag{1}$$

The quantity $\tilde{F}_{\mu\nu}$ is known as the *dual field strength*, and is given in terms of $F_{\mu\nu}$ in the following way:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} F_{\sigma\rho} \tag{2}$$

For reasons which will be explained momentarily, it is of particular interest to quantify the fluctuations of the topological charge Q. To this end, we introduce a quantity known as the *topological susceptibility* in the following way:

$$\chi(T) = \left. \frac{\partial^2 F(\theta, T)}{\partial \theta^2} \right|_{\theta=0} \tag{3}$$

It turns out that the topological susceptibility $\chi(T)$ is related to the *QCD axion mass* in the following simple way:

$$m_a^2 = \chi(T)/f_a^2 \tag{4}$$

With f_a being the *axion scale*. This relation holds true because the effective axion potential $V(a, T)$, is fundamentally related to the free energy density $F(\theta, T)$ in the following way:

$$V(a, T) = F(a/f_a, T) - F(0, T) \tag{5}$$

Similarly to equation (4), higher order moments of the topological charge can used to extract further information about the effective axion potential [3], which is important in understanding the dynamics of inflation in the presence of axions.

In what follows, we will discuss how to compute the topological susceptibility $\chi(T)$, at high temperatures T, such that QCD might be described reliably by perturbative methods. We start by extracting the leading order θ-dependent contribution to the partition function $Z(\theta, T)$, which is of course related to the free energy density by: $F(\theta, T) = -T \ln Z(\theta, T)/V$, with V being the volume of space.

1.1 The QCD partition function

In Euclidean signature, the standard $SU(N)$ Yang Mills action, minimally coupled to N_f fermion species, takes the following form:

$$S_{\text{QCD}} = \frac{1}{g^2}\int d^4x \left[\frac{1}{2}\operatorname{tr} F_{\mu\nu}F_{\mu\nu} + \sum_{s=1}^{N_f}\bar{\psi}_s(\not{D} + m_s)\psi_s\right] \tag{6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Additionally, we need to include the so called *θ-term* in perturbation theory, to account for the topological structure of the QCD vacuum:

$$S_\theta = \int d^4x \left[-\frac{i\theta}{16\pi^2}\operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu}\right] \tag{7}$$

As per usual when dealing with non-Abelian gauge theories, we also obtain two additional contributions to the total action from the *Faddeev-Popov* gauge-fixing procedure. These are the *gauge fixing* contribution S_{gf} and the *ghost fields* contribution S_{gh}. The precise forms of these contributions depend on our choice of *gauge fixing functional* $G(A_\mu)$:

$$S_{\text{gf}} = \frac{1}{g^2}\int d^4x \operatorname{tr}\left[-\frac{1}{2}G^2(A_\mu)\right] \tag{8}$$

$$S_{\text{gh}} = \frac{1}{g^2}\int d^4x \operatorname{tr}\left[\bar{c}\,\frac{\delta G(A_\mu + D_\mu\omega)}{\delta\omega}\,c\right] \tag{9}$$

Where A_μ is the gauge field, c and $\bar{c}$ is the *ghost field* and *anti ghost field* respectively, $\omega \in \mathfrak{su}(N)$, and $D_\mu = \partial_\mu + [A_\mu, \cdot\,]$ is the covariant derivative for adjoint fields.

As stated above, we want to extract the leading θ-dependent part of the total partition function, in order to approximate the topological susceptibility $\chi(T)$. In the perturbative regime, this leading order contribution will come from field configurations which minimize the total action and have non-zero topological charge Q. Thus, we need to compute the partition function around these minimizing configurations, and subsequently add up the individual contributions to obtain the leading order θ-dependence.

It turns out that the minimizing solutions in question exactly corresponds to *classical* field configurations A^{cl} with non-zero topological charge. Such solutions are known as *topological instantons*, the properties of which we shall now discuss in further detail.

1.2 Instanton solutions

We now wish to identify the classical solutions for the gauge field A_μ, with non-zero topological charge, under the assumption that all other fields vanish. In this case, the only part of the action which contribute to the classical equations of motion is S_{QCD}. It yields the following result:

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \tag{10}$$

The above equations are horribly complicated *second order non-linear* equations in A_μ, and attempting to find solutions would most certainly be a rather painful process. Fortunately, there exists for our case a better approach. The trick is to rewrite S_{QCD} as follow:

$$S_{\text{QCD}} = \frac{1}{4g^2}\int d^4\ \operatorname{tr}\Big[F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\Big]^2 \pm \frac{1}{2g^2}\int d^4\ \operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} \tag{11}$$

Given that we are looking for solutions with a some fixed topological charge Q, we see that the second integral is unchanging, implying that S_{QCD} is minimal exactly when:

$$\tilde{F}_{\mu\nu} = \pm F_{\mu\nu} \tag{12}$$

The above equations are *first order non-linear* in A_μ; a considerable improvement from the second order non-linear equation one gets from the standard Euler-Lagrange approach. Solutions for which $\tilde{F}_{\mu\nu} = F_{\mu\nu}$ are said to be *selfdual*, and solutions which satisfy $\tilde{F}_{\mu\nu} = -F_{\mu\nu}$ said to be *anti-selfdual*.

The most general solution to (12) can be found via the *Atiyah Drinfeld Hitchin Manin* (*ADHM*) construction [6]. We will not go into the details on this construction here. For our purposes, it will be sufficient to consider the following subset of solutions to (12):

$$\textit{Selfdual}: \quad A^{\text{cl}}_\mu = -\frac{1}{2}\bar{\eta}^a_{\mu\nu}\sigma^a\partial_\nu \ln \Pi \tag{13}$$

$$\textit{Anti-selfdual}: \quad A^{\text{cl}}_\mu = -\frac{1}{2}\eta^a_{\mu\nu}\sigma^a\partial_\nu \ln \Pi \tag{14}$$

where σ^a are the standard *Pauli matrices*, and the quantity Π is given by:

$$\Pi = 1 + \sum_{n=1}^{|Q|}\frac{\rho_n^2}{(x-x_n)^2} \tag{15}$$

The objects $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are known as *'t Hooft symbols*. They are respectively selfdual and anti-selfdual. One can think of these symbols as 4-dimensional representations of the $\mathfrak{su}(2)$ generators. The free parameters x_n and ρ_n are known as *collective coordinates* of the classical solutions (*the more general solutions found via the ADHM construction turns out to depend on exactly the same collective coordinates as the* (86,87) *solutions*). We will later see the appearance of additional collective coordinates, related to the embedding of the 2×2 matrix valued solutions (86,87) into $N \times N$ matrices.

1.3 The dilute instanton gas approximation (DIGA)

Now that we identified (*at least a subset of*) all classical solutions of the total action, we are now in a position to say something quantitative about the θ-dependence of the total partition function $Z(\theta, T)$. In order to do this, we employ an approximation method known as the *dilute instanton gas approximation (DIGA)*, which we shall now discuss in more detail.

It turns out that, given certain assumptions, we can treat all classical solutions with $|Q| > 1$ (*also the more general ones found via the ADHM construction)*, as super-positions of solutions with $|Q| = 1$. This method of approximation works well when all individual instanton sizes ρ_n, are much smaller than the distances $|x_n - x_m|$ between the individual instantons. This is known as the *small constituent instanton (SCI)* limit [6].

You might object to the idea of only including instanton configurations in the path integral, which obey the SCI limit. After all, there are no a priori reasons why this would make for a reasonable approximation, since we are excluding a large part of the collective coordinate space. It turns out however, that at finite temperature T, single instantons with size greater than $(\pi T)^{-1}$ are exponentially suppressed in the path integral [8]. Thus, at high temperatures, only a small region of collective coordinate space will have constituent instantons close enough together to see significant deviations between single-instanton super positions and genuine multi-instanton configurations.

Now that we discussed the justifications for employing the DIGA, we can use it to approximate the total partition function as a product of single instanton and anti-instanton partition functions:

$$Z(\theta, T) \approx \sum_{n=0}^{\infty} \sum_{\bar{n}=0}^{\infty} \frac{1}{n!\bar{n}!} Z_I^n(T) Z_{\bar{I}}^{\bar{n}}(T) e^{i(n-\bar{n})\theta} \tag{16}$$

The objects $Z_I(T)$ and $Z_{\bar{I}}(T)$ are the partitions function computed around the background of a single instanton and anti-instanton respectively. The contributions from S_θ are accounted for separately via the exponential factors. The factorial factors are there to account for the fact that the (anti)-instantons are indistinguishable from one another.

Using that $Z_I(T) = Z_{\bar{I}}(T)$ (*this follows from the fact that the action is time-reversal invariant, and that anti-instantons are time-reversed instantons*), the expression (16) can be evaluated explicitly, yielding the following result:

$$Z(\theta, T) \approx \exp\{Z_I(T) e^{i\theta}\} \exp\{Z_I(T) e^{-i\theta}\} = \exp\{2 Z_I(T) \cos\theta\} \tag{17}$$

We can now readily find an expression for the topological susceptibility $\chi(T)$ in the DIGA, by employing the formula (3):

$$\chi(T) = \left. \frac{\partial^2 F(\theta, T)}{\partial \theta^2} \right|_{\theta=0} \approx \frac{2T}{V} Z_I(T) \tag{18}$$

Thus, we see that (*given we work in the DIGA*), the topological susceptibility $\chi(T)$ is proportional to the single-instanton partition function $Z_I(T)$. With this result in mind, we now move on to discuss how to approximate this object in a consistent manner.

2 Fluctuation determinants and zero modes

Our goal is now to approximate the single-instanton partition function $Z_I(T)$, at finite temperature T. In these notes, we will only discuss how to do this up to quadratic order in field fluctuations around the instanton background. This might initially sound like a trivial task, since any quadratic action yields a Gaussian path integral, which when evaluated yields the determinant of the quadratic form M describing the action:

$$\int \mathcal{D}\Phi \ \exp\left\{-S_{\text{cl}} - \frac{1}{2} \Phi \cdot M \cdot \Phi \right\} = e^{-S_{\text{cl}}} \left[\det M\right]^{-1/2} \tag{19}$$

Where Φ is the field contents of the theory in question, and $S_{\text{cl}} \equiv S[\Phi^{\text{cl}}]$ (*the above result assumes the fields Φ to be bosonic. A similar result of course holds for fermionic fields*).

On the surface of things, equation (19) seems to be the end of the story, concerning quadratic approximations to path integrals. There is however an important caveat to the result (19); it only holds when M is *invertible*, e.i. when M has no eigenvectors with zero eigenvalues. There are many physically interesting examples in which this assumptions does not hold. Critically, the single-instanton partition function is one of those examples! In fact, every time we expand the action of a theory around some classical solutions Φ^{cl} which depend on some set of free parameters γ_i, the quadratic form will have *zero modes* (*eigenvectors with zero eigenvalues*) of the form:

$$Z_i = \frac{\partial \Phi^{\text{cl}}}{\partial \gamma_i} \tag{20}$$

The above result is straight forwardly proven, using the following two facts:

$$\frac{\delta S[\Phi^{\text{cl}}]}{\delta \Phi(x)} = 0 \tag{21}$$

$$\frac{\delta^2 S[\Phi^{\text{cl}}]}{\delta \Phi(x)\delta \Phi(y)} = M(x, y) \tag{22}$$

Combining the two results above, we readily obtain the desired result:

$$0 = \frac{\partial}{\partial \gamma_i}\frac{\delta S[\Phi^{\text{cl}}]}{\delta \Phi(x)} = \int d^d y \; \frac{\delta^2 S[\Phi^{\text{cl}}]}{\delta \Phi(x)\delta \Phi(y)} \cdot \frac{\partial \Phi^{\text{cl}}(y)}{\partial \gamma_i} = \int d^d y \; M(x, y) \cdot Z_i(y) \tag{23}$$

Now that we know to be wary of any potential zero modes, we need to discuss how to systematically handle them for our present case of computing $Z_I(T)$. Before we do so however, it will be useful to look at a slightly simpler example, without the complicating factor of gauge symmetry.

2.1 Zero mode fixing and collective coordinates

Consider a theory with a single scalar field $\tilde{\phi}$. Let this theory posses a set of classical solutions ϕ^{cl}, parametrized by a single collective coordinate γ. For convenience , we make the following field redefinition:

$$\tilde{\phi}(x) = \phi^{\text{cl}}(x, \gamma) + \phi(x) \tag{24}$$

The action up to quadratic order, is given in terms of the quadratic form $M(\phi^{\text{cl}})$:

$$S = S_{\text{cl}} + \frac{1}{2g^2}\phi \cdot M(\phi^{\text{cl}}) \cdot \phi \tag{25}$$

Where g is some coupling constant. As already discussed, we can obtain the zero modes of the theory by taking the derivative of ϕ^{cl} with respect to the collective coordinate γ:

$$Z \equiv F_0 = g\frac{\partial \phi^{\text{cl}}}{\partial \gamma} \quad , \quad \lambda_0 = 0 \tag{26}$$

The quadratic form will have a number of other eigenvectors with non-zero eigenvalues. We collectively denote all eigenvectors F_α and their corresponding eigenvalues λ_α:

$$M \cdot F_\alpha = \lambda_\alpha F_\alpha \tag{27}$$

Since the quadratic form M can be taken to be *symmetric* (*Hermitian*), we can expand the field ϕ in terms of the eigenvectors of M:

$$\phi = \sum_\alpha \xi_\alpha F_\alpha \tag{28}$$

We can also define an inner product $\langle \cdot | \cdot \rangle$ between the eigenvectors F_α via the usual L^2 inner product. With respect to this inner product, we take the eigenvectors to be orthogonal:

$$\langle F_\alpha | F_\beta \rangle = \frac{1}{g^2}\int d^d x \; F_\alpha(x) F_\beta(x) = u_\alpha \delta_{\alpha\beta} \tag{29}$$

Given the orthogonality condition with respect to the above inner product, we can rewrite the quadratic order action in terms of ξ_α and u_α:

$$S = S_{\text{cl}} + \frac{1}{2}\sum_\alpha \lambda_\alpha u_\alpha \xi_\alpha \xi_\alpha \tag{30}$$

With this form of the action in mind, we can *define* the path integral measure $\mathcal{D}\phi$, to have the following form:

$$\mathcal{D}\phi \equiv \prod_\alpha d\xi_\alpha \sqrt{\frac{u_\alpha}{2\pi}} \tag{31}$$

Using the above measure, it is straight forward to integrate the quadratic order action S. We obtain the following, deceptively familiar looking result:

$$\int \mathcal{D}\phi \; e^{-S[\phi]} = \int d\xi_0 \sqrt{\frac{u_0}{2\pi}} \; e^{-S_{\text{cl}}} \left[\det{}' M\right]^{-1/2} \tag{32}$$

The notation $\det' M$ reminds us to leave out $\lambda_0 = 0$ from the product of eigenvalues. It is not a priori clear how to carry out the integration over ξ_0, and so we would like to exchange it for an integration over the collective coordinate γ. We achieve this by using a trick similar to the well known *Faddeev-Popov* procedure. First, we introduce a constraint function $f(\gamma)$:

$$f(\gamma) = -\langle \phi | Z \rangle = -\left\langle \tilde{\phi} - \phi^{\text{cl}}(\gamma) \middle| Z \right\rangle \tag{33}$$

Next, we make use of the following delta-function identity:

$$\begin{aligned} 1 = \int d\gamma\, \delta(f(\gamma))\, \frac{\partial f(\gamma)}{\partial \gamma} &= \int d\gamma\, \delta(\langle \phi | Z \rangle) \left[\frac{1}{g} \langle Z | Z \rangle - \left\langle \phi \middle| \frac{\partial Z}{\partial \gamma} \right\rangle \right] \\ &= \int d\gamma\, \delta(\xi_0 u_0) \left[\frac{u_0}{g} - \left\langle \phi \middle| \frac{\partial Z}{\partial \gamma} \right\rangle \right] \end{aligned} \tag{34}$$

Since the term proportional to $\left\langle \phi \middle| \frac{\partial Z}{\partial \gamma} \right\rangle$ is sub-leading compared to the term proportional to u_0, we will ignore it in what follows. Inserting (34) into the path integral (32), we find that it now takes the following form:

$$\int \mathcal{D}\phi\, e^{-S[\phi]} = \int \frac{d\gamma}{\sqrt{2\pi g^2}}\, [u_0]^{1/2}\, e^{-S_{\text{cl}}}\, [\det' M]^{-1/2} \tag{35}$$

For the case of multiple zero modes Z^i, a very similar result holds [4]:

$$\int \mathcal{D}\phi\, e^{-S[\phi]} = \int \left[\prod_i \frac{d\gamma_i}{\sqrt{2\pi g^2}} \right] [\det U]^{1/2}\, e^{-S_{\text{cl}}}\, [\det' M]^{-1/2} \tag{36}$$

Where the matrix U^{ij} is simply given by the inner products of the individual zero modes: $U^{ij} = \langle Z^i | Z^j \rangle$. The above result is the one we will need in our continued discussion of the single-instanton partition function $Z_I(T)$.

2.2 The single-instanton partition function

We now proceed to the task of expanding the action $S = S_{\text{QCD}} + S_{\text{gf}} + S_{\text{gh}}$ to quadratic order. Similarly to the example from the previous subsection, we start by making a convenient field redefinition; this time from the complete field $\tilde{A}_\mu$ to the fluctuation A_μ:

$$\tilde{A}_\mu = A_\mu^{\text{cl}} + A_\mu \tag{37}$$

We now need to choose an appropriate gauge-fixing functional $G(A_\mu)$. It will be convenient to choose a gauge fixing functional which preserves the gauge symmetry of the classical field A_μ^{cl} [9]. One such choice looks as follow:

$$G(A_\mu) = D_\mu^{\text{cl}} A_\mu \tag{38}$$

Where $D_\mu^{\text{cl}} = \partial_\mu + [A_\mu^{\text{cl}}, \cdot]$, when acting on adjoint fields like A_μ. From this point onwards, we will only be working with D_μ^{cl}, and so we will drop the superscript and just write D_μ. With the above choice of gauge fixing functional, the gauge fixing S_{gf}, and ghost field actions S_{gh}, take the following forms respectively:

$$S_{\text{gf}} = \frac{1}{g^2} \int d^4x \left[-\frac{1}{2} \left(D_\mu A_\mu\right)^2 \right] \tag{39}$$

$$S_{\text{gh}} = \frac{1}{g^2} \int d^4x \left[\bar{c}\, D^2\, c + A_\mu \left[\bar{c}, D_\mu\, c\right] \right] \tag{40}$$

The total action $S = S_{\text{QCD}} + S_{\text{gf}} + S_{\text{gh}}$, can now be expanded to quadratic order, yielding the following result:

$$S = S_{\text{cl}} + \frac{1}{g^2} \left[\frac{1}{2} A \cdot M_A \cdot A + \bar{c} \cdot M_{\text{gh}} \cdot c + \sum_s \bar{\psi}_s \cdot M_{\psi_s} \cdot \psi_s \right] \tag{41}$$

Where, for the single instanton and anti-instanton solutions:

$$S_{\text{cl}} = \frac{8\pi^2}{g^2} \tag{42}$$

The particular form of each of the quadratic forms M_A, M_{gh} and M_{ψ_s}, are all list below for future reference [8]:

$$M_A = \left(-D^2 \delta_{\mu\nu} - 2F_{\mu\nu}\right)_{\text{adj}} \tag{43}$$

$$M_{\text{gh}} = \left(-D^2\right)_{\text{adj}} \tag{44}$$

$$M_{\psi_s} = \left(\not{D} + m_s\right)_{\text{fund}} \tag{45}$$

Where in the above, the subscripts $_{\text{adj}}$ and $_{\text{fund}}$ denotes whether D_μ is taken to act in the *adjoint* or *fundamental* representation of $SU(N)$ respectively.

2.3 The instanton zero modes

In order to compute the zero mode matrix $U^{ij} = \langle Z^i | Z^j \rangle$, we first have to identify the exact zero modes of the single-instanton background. This mostly comes down to a straight forward computation of the derivatives of the single instanton solution A^I_μ, with respect to the collective coordinates. There is however one complication to this simple task, which arise from the presence of gauge symmetry. In the previous subsection, we chose a gauge-fixing functional $G(A_\mu)$, which in turn force us to work in the gauge given by:

$$G(A_\mu) = D_\mu A_\mu = 0 \tag{46}$$

It is not a priori guaranteed that the collective coordinate derivatives will respect this particular gauge choice, but we can make them conform by acting with appropriate gauge transformations ω^i:

$$Z^i_\mu = \frac{\partial A^I_\mu}{\partial \gamma_i} + D_\mu \omega^i \tag{47}$$

For convenience, we will write out the single-instaton solution A^I_μ explicitly:

$$A^I_\mu(x) = -\frac{1}{2}\bar{\eta}^a_{\mu\nu}\sigma^a \partial_\nu \ln\left[1 + \frac{\rho^2}{(x-X)^2}\right] \tag{48}$$

Let us first tackle the zero modes associated with the collective coordinates X_ν. In this case, taking $\omega^{(\nu)} = A^I_\nu$ will shift us back to the right gauge:

$$Z^{(\nu)}_\mu = \frac{\partial A^I_\mu}{\partial X_\nu} + D_\mu A^I_\nu = -\frac{\partial A^I_\mu}{\partial x_\nu} + D_\mu A^I_\nu = F^I_{\mu\nu} \tag{49}$$

Which, on the basis of the equations of motion $D_\mu F^I_{\mu\nu} = 0$, obeys the correct gauge. Next, we turn to the zero mode associated with the collective coordinate ρ. For this case, no gauge transformation is needed to shift the zero mode back to the correct gauge, e.i. $\omega^{(\rho)} = 0$. By explicit computation, one finds [4]:

$$Z^{(\rho)}_\mu = -2\frac{\rho\,\bar{\eta}^i_{\mu\nu}\sigma^i\,x_\nu}{(x^2+\rho^2)^2} \tag{50}$$

At this point, we have to address a comment made back in subsection (86), about the existence of additional collective coordinates beyond, for the present case, the five given by

X_ν, ρ. Since the solution (48) is only $\mathfrak{su}(2)$-valued, we have to embed this solution into an $\mathfrak{su}(N)$ element. One particular way of doing this looks as follow:

$$(A^I_\mu)_{\mathfrak{su}(N)} = \begin{bmatrix} (A^I_\mu)_{\mathfrak{su}(2)} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{bmatrix} \tag{51}$$

The above embedding is clearly not unique. One can change between all possible embeddings by acting with global $SU(N)$ transformations on the embedding chosen above. Some $SU(N)$ transformations however, leaves the above embedding unchanged. These are the $SU(N-2)$ transformation acting on the lower right corner, and the $U(1)$ transformations generated by the last remaining Cartan generator of $SU(N)$. Thus, the subgroup of $SU(N)$ which transform between distinct embeddings is given by.

$$\frac{SU(N)}{SU(N-2)\times U(1)} \tag{52}$$

The coordinates on the above group manifold, of which there are $4N-5$, acts as additional collective coordinates. Any element Ω of the algebra, associated to the above subgroup, can of course be expressed in terms of its generators:

$$\Omega = \theta_a T^a + \theta_k T^k \quad , \quad a = 1,2,3 \quad , \quad k = 4,5,\dots,4N-5 \tag{53}$$

Here, the 3 generators T^a generate the $SU(2)$ of the upper left corner, whereas the $4(N-2)$ generators T^k generate what remains. The zero modes associated with the coordinates θ_a and θ_k take the following forms [4]:

$$Z^{(a)}_\mu = \frac{\partial A^I_\mu}{\partial \theta_a} + D_\mu \omega^{(a)} = [A^I_\mu, T^a] + D_\mu \omega^{(a)} = D_\mu \left[\frac{x^2}{x^2+\rho^2} T^a\right] \tag{54}$$

where

$$\omega^{(a)} = -\frac{\rho^2}{x^2+\rho^2} T^a \tag{55}$$

$$Z^{(k)}_\mu = \frac{\partial A^I_\mu}{\partial \theta_k} + D_\mu \omega^{(k)} = [A^I_\mu, T^k] + D_\mu \omega^{(k)} = D_\mu \left[\sqrt{\frac{x^2}{x^2+\rho^2}} T^k\right] \tag{56}$$

where

$$\omega^{(k)} = \left[\sqrt{\frac{\rho^2}{x^2+\rho^2}} - 1\right] T^k \tag{57}$$

With all the instanton zero modes now computed, we move on to the computation of $U^{ij} = \langle Z^i | Z^j \rangle$. For the setup in question, the exact form of the inner product is given by:

$$\left\langle Z^i | Z^j \right\rangle = \frac{2}{g^2} \int d^4x \ \mathrm{tr}\, Z^i_\mu Z^j_\mu \tag{58}$$

Given this inner product, we can now evaluate all elements of U^{ij}. Remarkable, it turns out that most elements of U^{ij} vanish, leaving only non-zero inner products between zero modes of the same "type". All these non-zero elements are listed below [4]:

$$U^{(\mu\nu)} = \frac{8\pi^2}{g^2}\delta^{\mu\nu} \tag{59}$$

$$U^{(\rho\rho)} = \frac{16\pi^2}{g^2} \tag{60}$$

$$U^{(ab)} = \frac{4\pi^2\rho^2}{g^2} \tag{61}$$

$$U^{(kl)} = \frac{2\pi^2\rho^2}{g^2} \tag{62}$$

With the above matrix elements at hand, we finally obtain an expression for the square root of the determinant of U^{ij}:

$$[\det U]^{1/2} = \frac{2^{2N+7}}{\rho^5}\left(\frac{\pi\rho}{g}\right)^{4N} \tag{63}$$

This concludes our discussion of the zero modes of the single-instanton, and how to handle them in the path integral. We now move on to the final section of these notes, where we derive a final expression for the single instanton partition function $Z_I(T)$ to quadratic order.

3 Evaluating the fluctuation determinants

Referring back to equation (36) and our discussions in subsection (2.2), we see that $Z_I(T)$ to quadratic order in field fluctuations, can be expressed in the following way:

$$Z_I(T) = \int \left[\prod_i d\gamma_i\right] n(\gamma) \tag{64}$$

Where the object $n(\gamma)$, refereed to as the *instanton density*, is given by:

$$n(\gamma) = \left[\prod_i \frac{1}{\sqrt{2\pi g^2}}\right] [\det U]^{1/2} \, [\det' M_A]^{-1/2} \, \det M_{\mathrm{gh}} \left[\prod_s \det M_{\psi_s}\right] e^{-\frac{8\pi^2}{g^2}} \tag{65}$$

Here, $i = 1, 2, \dots, 4N$ and $\gamma = (X_\nu, \rho, \theta_a, \theta_k)$. Through appropriate field redefinitions, and by making use of the symmetries of the action $S = S_{\text{QCD}} + S_{\text{gf}} + S_{\text{gh}}$, it can be shown that all determinants contained in $n(\gamma)$ depend only on the collective coordinate ρ. In particular, this means that we can replace the integration over the compact group generated by (θ_a, θ_k), with the volume ν of the correspondence group manifold. Thus, we can now write:

$$Z_I(T) = \int d^4X\, d\rho\, n(\rho) \tag{66}$$

Where the instanton density $n(\rho)$ is now given as:

$$n(\rho) = \left[\frac{1}{2\pi g^2}\right]^{2N} \nu\, [\det U]^{1/2}\, [\det{}' M_A]^{-1/2}\, \det M_{\text{gh}} \left[\prod_s \det M_{\psi_s}\right] e^{-\frac{8\pi^2}{g^2}} \tag{67}$$

With the group volume ν given by the following:

$$\nu \equiv \text{Vol}\left[\frac{SU(N)}{SU(N-2)\times U(1)}\right] = \frac{2^{4N-5}\pi^{2N-2}}{(N-1)!(N-2)!} \tag{68}$$

Inserting the explicit forms of ν and $[\det U]^{1/2}$ into the instanton density, we find that it takes the following form:

$$n(\rho) = \frac{4}{\pi^2}\frac{[\det{}' M_A]^{-1/2}\, \det M_{\text{gh}} \prod_s \det M_{\psi_s}}{(N-1)!(N-2)!}\frac{1}{\rho^5}\left(\frac{4\pi^2\rho^2}{g^2}\right)^{2N} e^{-\frac{8\pi^2}{g^2}} \tag{69}$$

All which is now left to be done, are the evaluation of the quadratic form determinants $\det{}' M_A$, $\det M_{\text{gh}}$ and $\det M_{\psi_s}$. This job is significantly simplified by the existence of certain revelations between these determinants, which we shall now discuss in further detail.

3.1 Determinant relations

Let us first set our focus on the operator $M_{\text{gh}} = (-D^2)_{\text{adj}}$. For convenience in what follows, we denote the eigenvectors of this operator F_α, and the corresponding eigenvalues λ_α^2, which we can take to be positive since $(-D^2)_{\text{adj}}$ is positive definite:

$$(-D^2)_{\text{adj}} F_\alpha = \lambda_\alpha^2 F_\alpha \tag{70}$$

We now set our attention on the operator $M_A = (-D^2\delta_{\mu\nu} - F_{\mu\nu})_{\text{adj}}$. Rather remarkably, it turns out that for every eigenvector of F_α of M_{gh} with eigenvalue $\lambda_\alpha^2 \neq 0$, we can explicitly construct 4 eigenvectors of M_A, all with the same eigenvalue λ_α^2 [9]:

$$(F_\alpha)_\mu^a = \bar{\eta}_{\mu\nu}^a D_\nu F_\alpha \quad , \quad a = 1, 2, 3 \tag{71}$$

$$(F_\alpha)_\mu^0 = D_\mu F_\alpha \tag{72}$$

Thus, we conclude that:

$$\det{}' M_A = [\det\left(-D^2\right)_{\text{adj}}]^4 \tag{73}$$

A similar relation exists between the operators $M_{\psi_s} = (\not{D} + m_s)_{\text{fund}}$ and $(-D^2)_{\text{fund}}$, but only in the limit where m_s is so small as to only affect the zero mode of $(\not{D})_{\text{fund}}$ (*in the single-instanton background, $(\not{D})_{fund}$ has a single fermionic zero mode. We will not need to discuss this any further in these notes*):

$$\det\left(\not{D} + m_s\right)_{\text{fund}} \approx m_s \det{}'(\not{D})_{\text{fund}} \tag{74}$$

In this limit, for every eigenvector F_α of $(-D^2)_{\text{fund}}$ with eigenvalue λ_α^2, we can construct 2 *left chiral* and 2 *right chiral* eigenvectors of $(\not{D})_{\text{fund}}$, all with eigenvalue λ_α [9]:

$$(F_\alpha)_L = F_\alpha P_L v \tag{75}$$

$$(F_\alpha)_R = \not{D} F_\alpha P_R v \tag{76}$$

With v a constant 4-component Dirac spinor. Thus, we now conclude that:

$$\det M_{\psi_s} \approx m_s [\det\left(-D^2\right)_{\text{fund}}]^2 \tag{77}$$

Finally, there also exists a relations between $\det(-D^2)_{\text{adj}}$, $(-D^2)_{\text{fund}}$, and the determinants $\det(-D^2)_1$, $\det(-D^2)_{1/2}$, where the subscripts $_1$ and $_{1/2}$ indicates whether the operators are acting on spin-1 and spin-$\frac{1}{2}$ representations of $SU(2)$. The relations in question reads as follow:

$$\det\left(-D^2\right)_{\text{adj}} = \det\left(-D^2\right)_1 [\det\left(-D^2\right)_{1/2}]^{2(N-2)} \tag{78}$$

$$\det\left(-D^2\right)_{\text{fund}} = \det\left(-D^2\right)_{1/2} \tag{79}$$

The first relations ultimately arise from the fact that the adjoint representation of $SU(N)$, decompose in the following way when restricted to $SU(2)$:

$$\mathbf{N^2 - 1} \rightarrow \mathbf{3} \oplus \mathbf{2}^{\oplus 2(N-2)} \oplus \mathbf{1}^{\oplus (N-2)^2} \tag{80}$$

Similarly, the second relation stems from how the fundamental representation of $SU(N)$, decompose when restricted to $SU(2)$:

$$\mathbf{N} \to \mathbf{2} \oplus \mathbf{1}^{\oplus N-2} \tag{81}$$

Combining all these determinant relations, we find that the product of $\det' M_A$, $\det M_{\text{gh}}$ and all the $\det M_{\psi_s}$ can be written as:

$$\det' M_A \, \det M_{\text{gh}} \prod_s \det M_{\psi_s} = \frac{[\det(-D^2)_{1/2}]^{2N_f} \prod_s m_s}{\det(-D^2)_1 \, [\det(-D^2)_{1/2}]^{2(N-2)}} \tag{82}$$

Where N_f is the number of fermion species: $s = 1, 2, \ldots, N_f$. We have refrained from combining the spin-$\frac{1}{2}$ contributions from bosonic and fermionic determinants in the above expression, since these needs slightly different treatments at non-zero temperature.

3.2 Zero temperature result

We are now left with the task of evaluating the determinants of $-D^2$, acting in the spin-1 and spin-$\frac{1}{2}$ representations of $SU(2)$. We start by focusing on the zero temperature case. Even in this limit, evaluating the determinants is no trivial task, and we shall not go into detail on those computations here. Instead, we will simply quote *'t Hooft's* results from [7]:

$$\ln \det\big(-D^2/-\partial^2\big)_1 = \alpha(1) + \frac{2}{3}\ln(\mu\rho) \tag{83}$$

$$\ln \det\big(-D^2/-\partial^2\big)_{1/2} = \alpha(\frac{1}{2}) + \frac{1}{6}\ln(\mu\rho) \tag{84}$$

Where $\alpha(1) \approx 0.443\ 307$ and $\alpha(\frac{1}{2}) \approx 0.145\ 873$. The new parameter μ is a regulator mass, needed to make the determinants converge. One also finds that this regularization introduces one factor of μ for every zero mode (*of which there are* $4N$), and one factor of μ for each fermion species (*of which there are* N_f). Putting together all determinant results, and inserting these into (69), we find the following final expression for the instanton density at zero temperature:

$$n(\rho, T=0) = \frac{4}{\pi^2} \frac{e^{-\alpha(1) - 2(N-2-N_f)\alpha(\frac{1}{2})}}{(N-1)!(N-2)!} \frac{1}{\rho^5} \left(\frac{4\pi^2}{g^2}\right)^{2N} \left[\prod_s \rho\, m_s\right] e^{-\frac{8\pi^2}{g^2} + \frac{1}{3}(11N - 2N_f)\ln(\rho\mu)} \tag{85}$$

Notice that the term $\frac{1}{3}(11N - 2N_f)\ln(\rho\mu)$ in the last exponential, exactly coincides with the one-loop *beta function* for $SU(N)$ Yang Mills coupled to N_f fermions, and thus serves to renormalize $1/g^2$, as is required by the *Renormalization Group*.

3.3 The thermal corrections

At non-zero temperature T, the single instanton and single anti-instanton solutions are no longer proper solutions of the equations of motion, since these solutions are not periodic, with period $\beta = T^{-1}$. Fortunately, we can easily solve this issue, by making use of the zero temperature multi-instanton solutions:

$$\textit{Selfdual}: \quad A_\mu^{\text{cl}} = -\frac{1}{2}\bar{\eta}^a_{\mu\nu}\sigma^a\partial_\nu \ln \Pi \tag{86}$$

$$\textit{Anti-selfdual}: \quad A_\mu^{\text{cl}} = -\frac{1}{2}\eta^a_{\mu\nu}\sigma^a\partial_\nu \ln \Pi \tag{87}$$

In order to have β-periodic solutions, we need to make the following choice of collective coordinates: $x_n = n\beta\hat{t}$ and $\rho_n = \rho$. The quantity Π then takes takes the following form:

$$\Pi = 1 + \sum_{n=-\infty}^{\infty} \frac{\rho^2}{\left(x - n\beta\hat{t}\right)^2} \tag{88}$$

Where $\hat{t} = (1, 0, 0, 0)$. We will refer to these solutions as the periodic (anti)-instanton. Expanding around a periodic instanton will lead to different forms for the zero modes, compared to those at zero temperature. Remarkably, it turns out that the inner product matrix $U^{ij} = \langle Z^i | Z^j \rangle$ at temperature T, is completely independent of T and identical to the zero temperature result. Thus, we only need to re-evaluate $\det(-D^2/-\partial^2)$ for spin-1 and spin-$\frac{1}{2}$.

For the evaluation of $\det(-D^2/-\partial^2)$ at finite T, we start by splitting the determinant into its temperature dependent and temperature independent parts:

$$\ln\det\left(-D^2/-\partial^2\right) = \ln\det\left(-D^2/-\partial^2\right)\big|_{T=0} + \delta \tag{89}$$

where

$$\delta = \int_0^T dT' \, \frac{\partial}{\partial T'} \, \mathrm{Tr}\ln\left(-D^2/-\partial^2\right)\big|_{T'} = \int_0^T dT' \, \mathrm{Tr}\ln\Delta \, \frac{\partial}{\partial T'} \left(-D^2\right)\big|_{T'} \tag{90}$$

Here, Δ is the scalar propagator $(-D^2)^{-1}$, in the presence of a periodic instanton, at finite temperature. This propagator can be constructed from the known zero temperature periodic instanton propagator $\tilde{\Delta}$ [10], by way of the following argument: The propagator $\tilde{\Delta}$ obeys the equation:

$$-D^2\tilde{\Delta}(x, y) = \delta(x - y) \tag{91}$$

We want the propagator Δ to obey the equation:

$$-D^2\Delta^{\pm}(x,y) = \sum_{n=-\infty}^{\infty} (\pm 1)^n \delta(x - y - n\beta\hat{t}) \tag{92}$$

The $\pm$ is chosen based on whether the boundary conditions on the propagator are periodic $(+)$ or anti-periodic $(-)$. From equation (91), we can easily construct an object which obeys equation (92):

$$-D^2 \sum_{n=-\infty}^{\infty} (\pm 1)^n \tilde{\Delta}(x, y + n\beta\hat{t}) = \sum_{n=-\infty}^{\infty} (\pm 1)^n \delta(x - y - n\beta\hat{t}) \tag{93}$$

Thus, we conclude that $\Delta^{\pm}$ is given by:

$$\Delta^{\pm}(x,y) = \sum_{n=-\infty}^{\infty} (\pm 1)^n \Delta(x, y + n\beta\hat{t}) \tag{94}$$

Using the expression above, along side the known expression for $\tilde{\Delta}$ in the background of a periodic instanton, it is now possible to evaluate δ for spin-1 and spin-$\frac{1}{2}$ representations. This has been done by *Gross, Pisarski and Yaffe* [8], and their results are presented below:

$$\delta_{1/2} = \frac{1}{3}\eta\lambda^2 + A(\lambda) \tag{95}$$

$$\delta_1 = \frac{4}{3}\lambda^2 + 16A(\lambda) \tag{96}$$

Where $\lambda = \pi\rho T$. Furthermore, $\eta = 1$ for the case of periodic boundary conditions, and $\eta = -\frac{1}{2}$ for anti-periodic boundary conditions. The quantity $A(\lambda)$ is partially numerically evaluated, and takes the following form:

$$A(\lambda) = -\frac{1}{12}\ln\big(1 + \lambda^2/3\big) + c_1(1 + c_2\lambda^{-3/2})^{-8} \tag{97}$$

Where $c_1 = 0.012\,897\,64$ and $c_2 = 0.158\,58$. Once again, putting together all determinant results, and inserting these into (69), we find the following final expression for the instanton density at finite temperature T:

$$n(\rho,T) = n(\rho,T=0)\exp\left\{-\frac{1}{3}\lambda^2(2N + N_f) - 12A(\lambda)\left[1 + \frac{1}{6}(N - N_f)\right]\right\} \tag{98}$$

Observe that the above expression is highly exponentially suppressed for values of ρ larger than $(\pi T)^{-1}$, thus confirming the validity of the dilute instanton gas approximation (*DIGA*), at high temperatures. With this remark, we conclude these notes.

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