# Spectrum of Dirac operator and Banks-casher relation

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# Introduction

$\mathscr{L}_{QCD}$ **is symmetric under** $SU(2)_L \times SU(2)_R \times U(1)_A \times U(1)_V$ **for**

$$m_u = m_d = 0$$

**Realistic World :** $u$ **and** $d$ **quarks have small masses** $\longrightarrow$ **chiral symmetry is approximate.**

$SU(2)_L \times SU(2)_R$ **is broken spontaneously and** $U(1)_A$ **is broken due to quantum corrections at** $T = 0$**.**

**Consequences : Massless (almost) pions, Mass difference between** $p$ **and** $N^*(1535)$**, Mass difference between** $\eta$ **and** $\eta'$

**Order Parameter,** $\langle \bar{\psi}\psi \rangle \neq 0$

**Banks-Casher relation,** $\langle \bar{\psi}\psi \rangle$ **relate to the eigenvalues of the Dirac operator**

$$\langle \bar{\psi}\psi \rangle = \langle \ TrD^{-1} \rangle = \langle \sum_i Re \frac{1}{\lambda_i} \rangle \quad \boxed{??}$$

$$\langle\bar{\psi}\psi\rangle = \frac{\partial lnZ}{\partial m}$$

$$\langle\bar{\psi}\psi\rangle = \lim_{V\to\infty}\int\frac{\rho(\lambda)}{\lambda^2+m^2}d\lambda$$

$$\langle\bar{\psi}\psi\rangle = \lim_{m\to 0, V\to\infty}\pi\rho(0)$$

**Banks-Casher relation**

- The LQCD Path integral can be written as,

$$Z = \int DU\, D\psi\, D\bar{\psi}\; e^{-S_G + S_F}$$

$S_G$ is gauge part of the action and $S_F$ is the fermion part of the action,

$$S_F = \bar{\psi} M \psi$$

The eigen value density one defines,

$$\rho = \frac{1}{ZV}\int DU\,(\det M)\, e^{-S_G}\; \underline{\delta(\lambda - \lambda_k[U])}$$

Now, the chiral condensate,

$$\Sigma = \frac{1}{V}\sum_x \langle \bar{\psi}_x \psi_x \rangle$$

$$= \frac{1}{V}\langle \mathrm{Tr}\; M^{-1} \rangle$$

$$= \frac{1}{V}\left\langle \sum_r \frac{2m}{\lambda_r^2+m^2} \right\rangle$$

# Eigen-values of the Free Dirac Fermions

**The naive Dirac operator can be written as,**

$$M = \gamma_\mu D_\mu + m$$

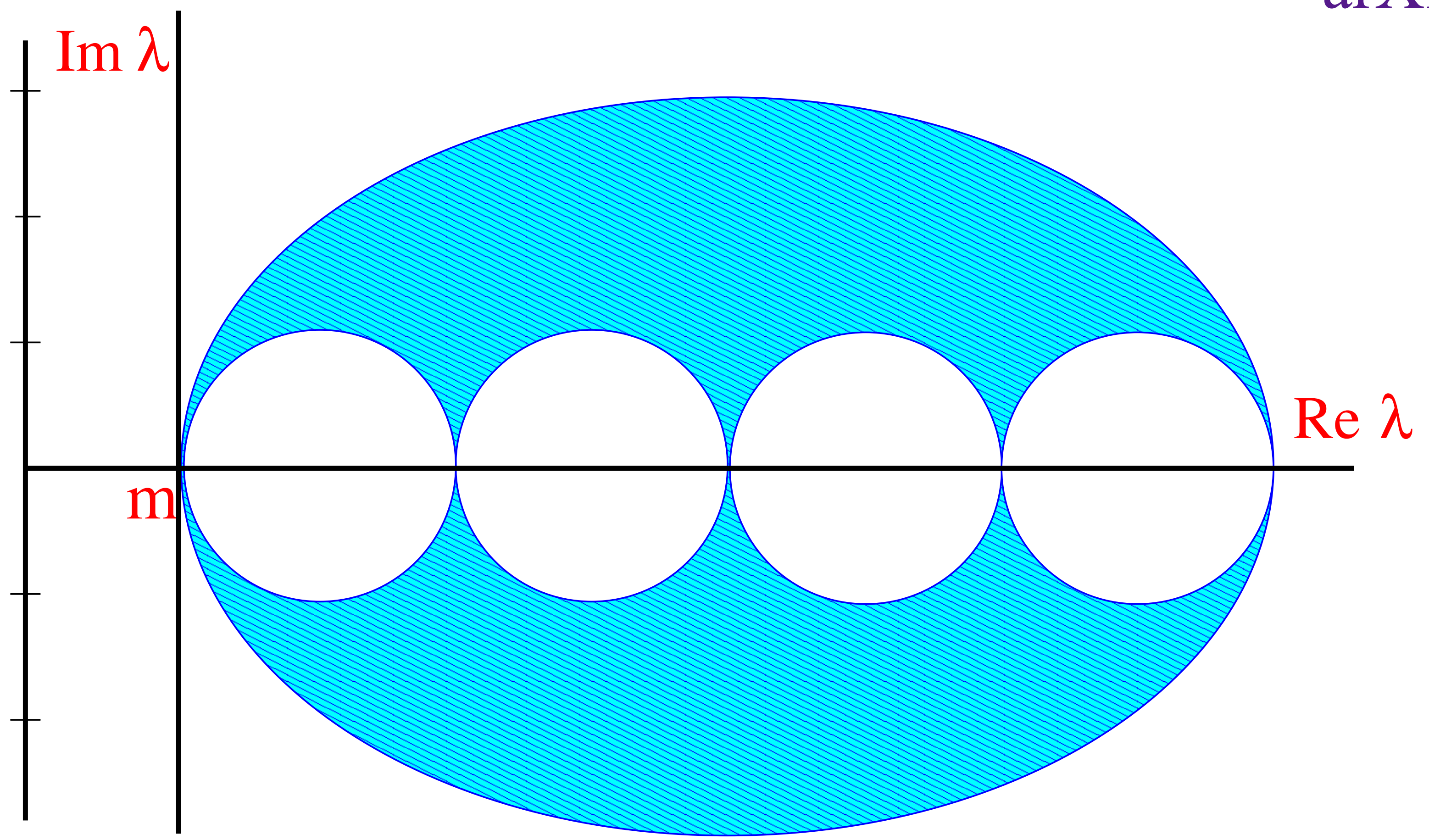
**For , $U = 1$, this can be diagonalise easily in the momentum space, $m \pm i\sqrt{\sum_{\mu=1}^{4} \sin(p_\mu)^2}$ , complex eigenvalues.**

**The eigenvalues Wilson Dirac operator ($D_W$ ) can be written as,**

$$m + r\sum_{\mu=1}^{4}(1 - \cos(p_\mu)) \pm i\sqrt{\sum_{\mu=1}^{4} \sin(p_\mu)^2}$$

**Eigenvalue spectrum of $\gamma_5 D_W$??**

arXiv:hep-lat/0511052




**Eigenvalue spectrum of free $D_W$ operator.**

- Interesting thing will be to study the eigenvalues of the Dirac operator in background gauge field configurations.
- The density of small Dirac eigenvalues related to spontaneous chiral symmetry breaking: Banks and Casher relation.
- lattice discretizations of the Dirac operator obey the Ginsparg-Wilson relation have the corresponding eigenvalues on circles in the complex plane.

GW fermions, $D\gamma_5 + \gamma_5 D = a D \gamma_5 D$

or, $\gamma_5 D \gamma_5 + D = a\, \gamma_5 D \gamma_5 D$

$\therefore$ $\boxed{D^\dagger + D = a D^\dagger D}$ , lets say $D$ has eigenvalue $\lambda$ and $D^\dagger$ has $\lambda^*$.

$\therefore$ $\lambda^* + \lambda = a \lambda^* \lambda$

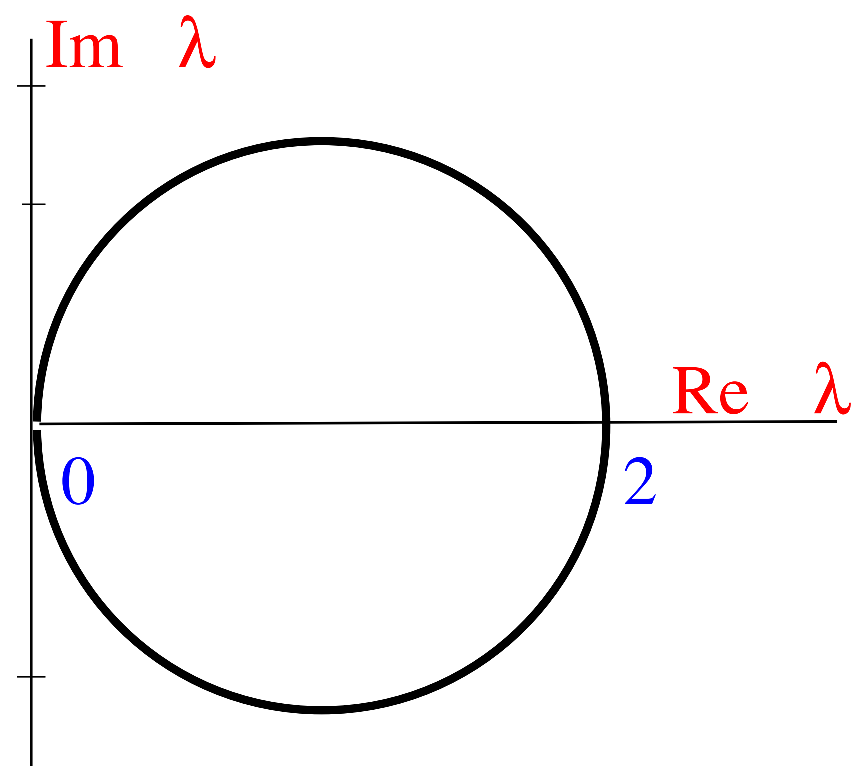
writing, $\lambda = \lambda_r + i \lambda_{Im}$

$$\left(\lambda_r - \frac{1}{a}\right)^2 + \lambda_{Im}^2 = \frac{1}{a^2}$$

This is equation of Circle, center at $(1/a, 0)$ and also radius of $a$.

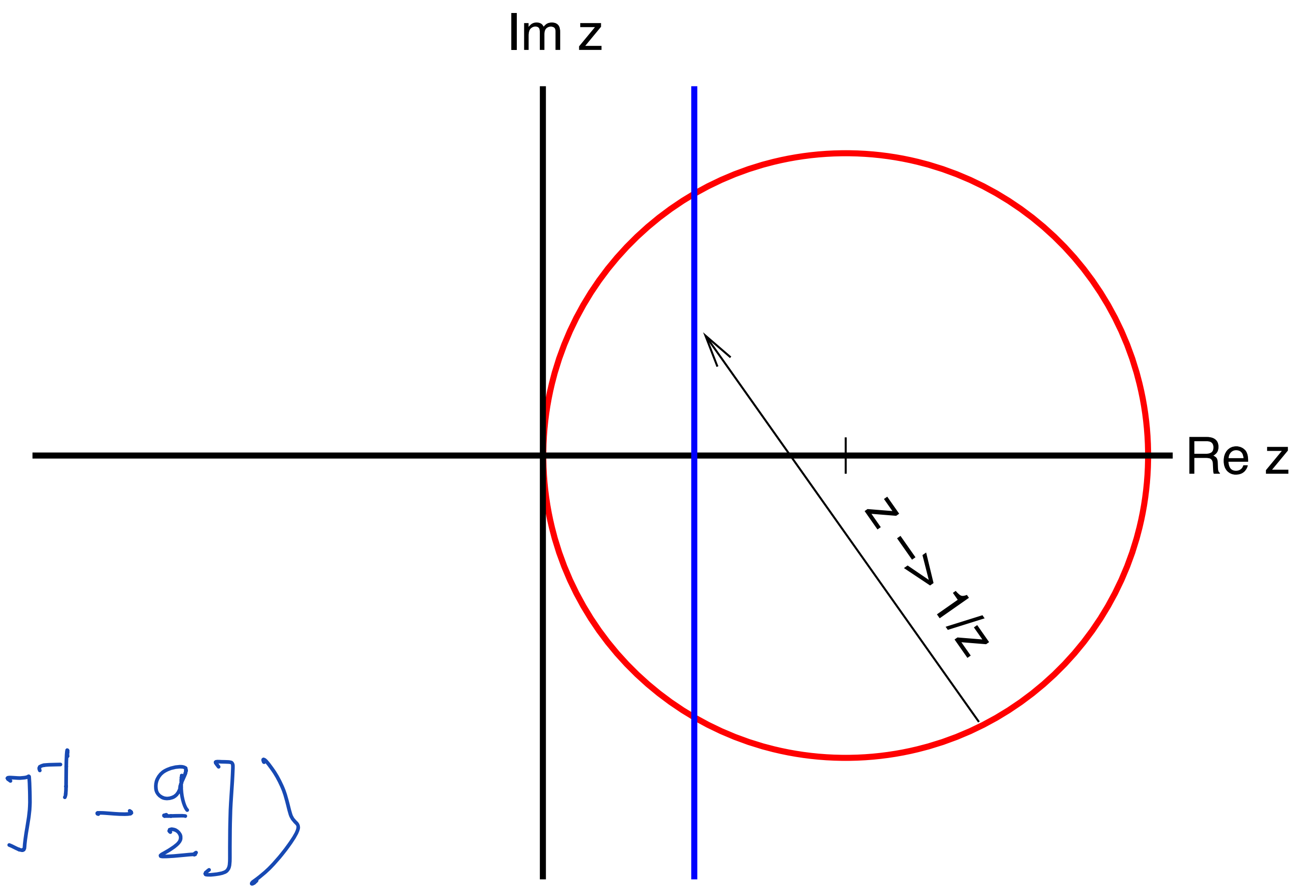
So, the parametric equation for the circle,

$$\lambda = \frac{1}{a}\left(1 - e^{i\theta}\right), \quad \theta \in [-\pi, \pi]$$



$$Z = \frac{1}{a}(1 - e^{i\theta})$$

$$1/Z = \frac{a}{2} + i\frac{a}{2}\frac{sin(\theta)}{1 - cos(\theta)}$$




$$\langle \bar{\Psi} \psi \rangle (a, m, V) = \frac{1}{V} \langle tr\, D_{GW}^{-1} \rangle$$

$$= \frac{1}{\omega V} \langle tr \left[ [\omega D + m]^{-1} - \frac{a}{2} \right] \rangle$$

where, $\underline{\omega = 1 - \frac{am}{2}}$

$$= (n_+ + n_-)\left(\frac{1}{m} - \frac{a}{2}\right) + \sum_{\lambda_i \neq 2/a, 0} \left( \frac{1}{\omega \lambda_i + m} - \frac{a}{2} \right)$$

$$= \lim_{V \to \infty} \frac{1}{V} \frac{(n_+ + n_-)\, \omega}{m} + \cdots\cdots$$

$$\simeq \lim_{m \to 0, V \to \infty} \pi \rho(0)$$

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## Remarks on the chiral phase transition in chromodynamics

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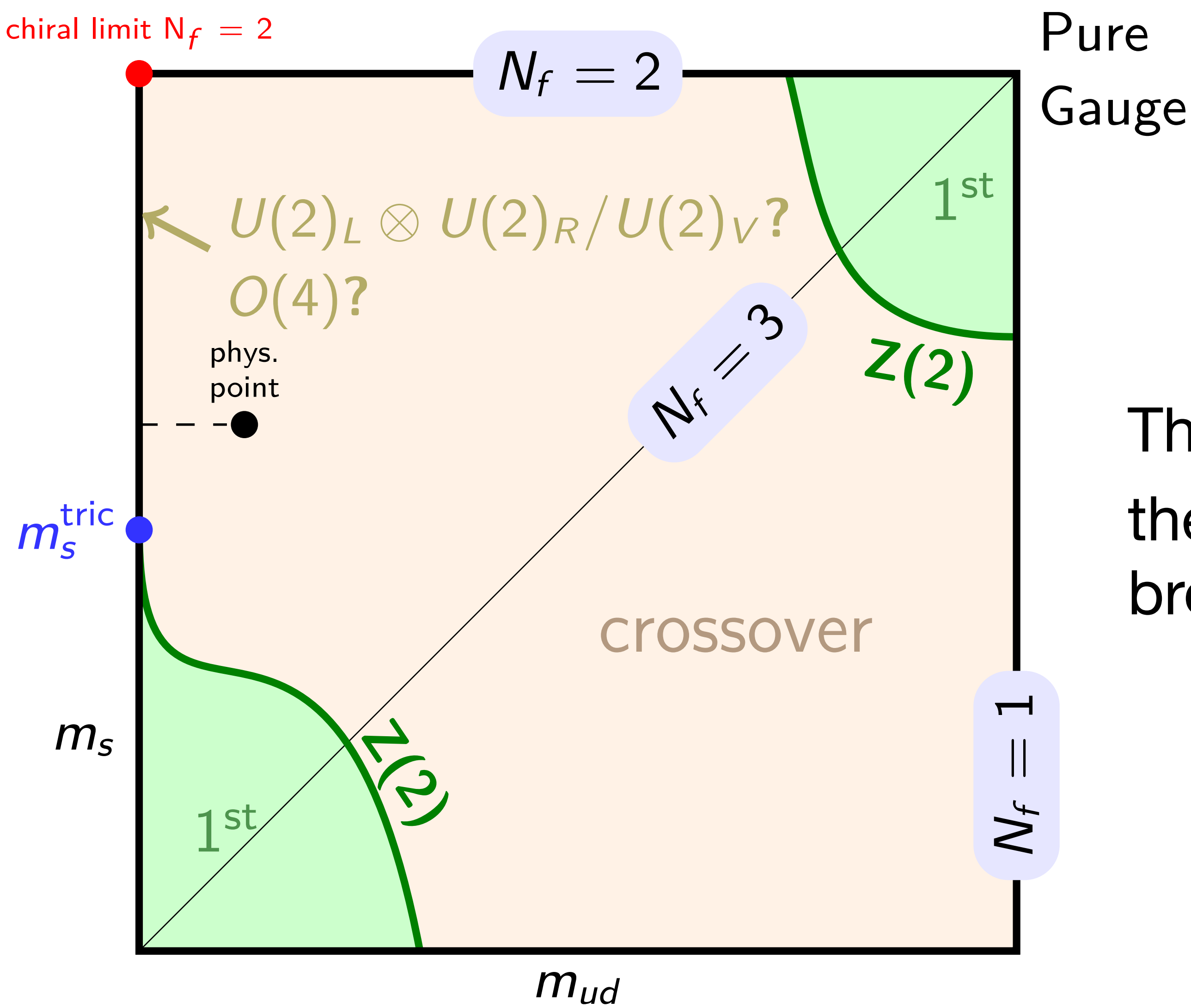
The phase transition restoring chiral symmetry at finite temperatures is considered in a linear $\sigma$ model. For three or more massless flavors, the perturbative $\epsilon$ expansion predicts the phase transition is of first order. At high temperatures, the $U_A(1)$ symmetry will also be effectively restored.




The chiral transition ($N_f$=2+1, $\mu$=0) belongs to the O(4) universality class if $U(1)_A$ is still broken at $T_c$.

$$\langle \bar{\psi}\psi \rangle = \int_0^{\alpha} d\lambda \, \frac{2m_l \, \rho(\lambda)}{m^2 + \lambda^2} \quad \text{— (I)}$$

$$\chi = \int_0^{\alpha} d\lambda \, \frac{4m_l^2 \, \rho(\lambda)}{(m_l^2 + \lambda^2)^2} \quad \text{— (II)}$$

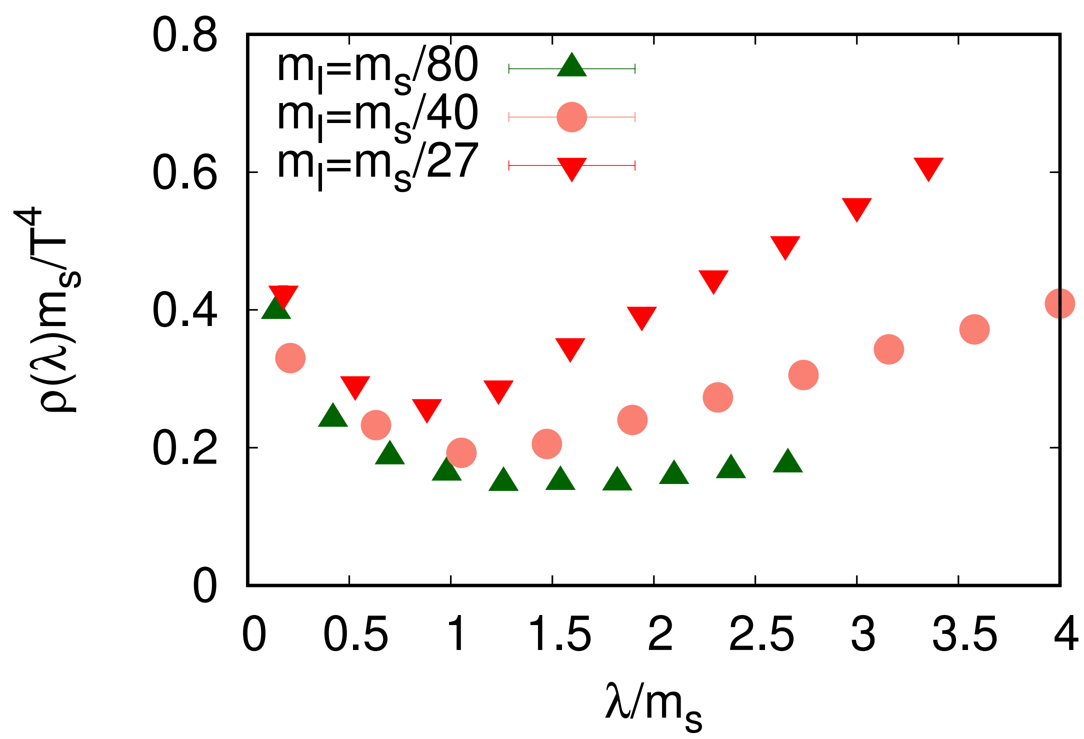
At, $T \neq 0$, $SU(2)_L \times SU(2)_R$ get restored,
$m \to 0$

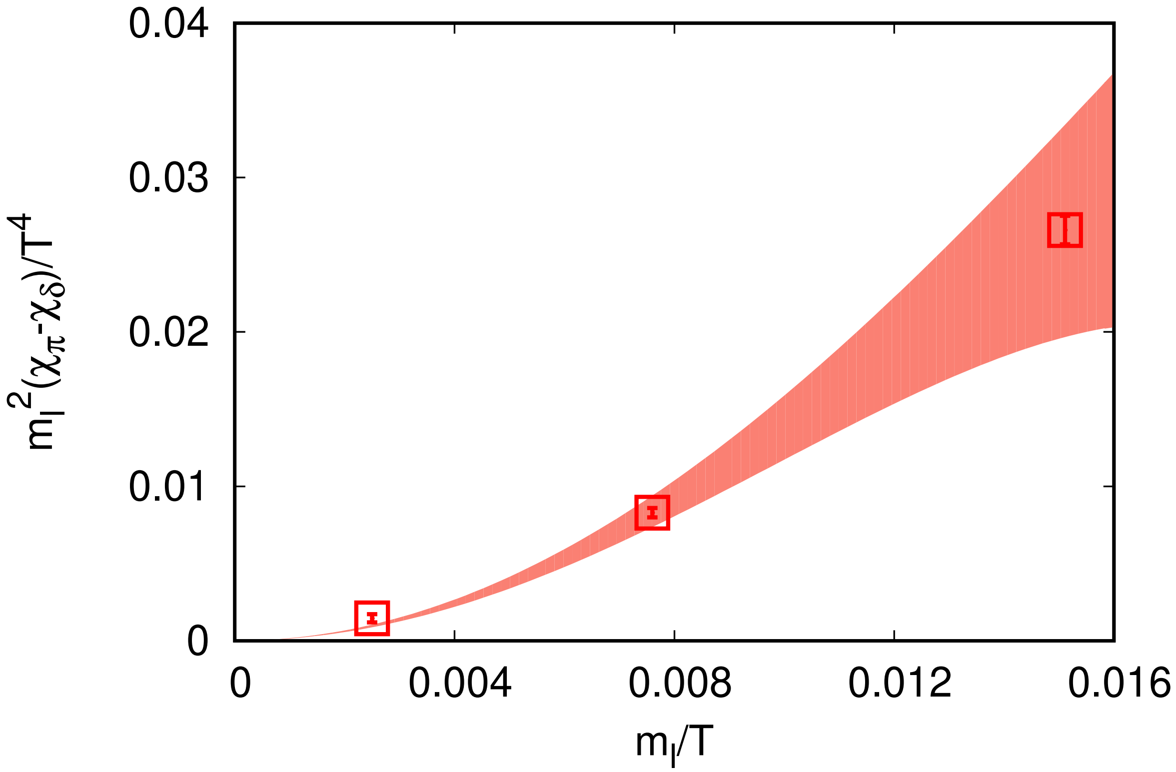
$$\langle \bar{\psi}\psi \rangle = \int_0^{\alpha} d\lambda \, \frac{2m_l \, \rho(\lambda)}{m^2 + \lambda^2} \propto (T - T_c)^{\beta}$$

$\boxed{\text{if it's } O(4)}$

**Behaviour of $\rho(\lambda)$ close to $T_c$ will be relevant for $U_A(1)$ breaking.**

$\chi$ **Quantifies the $U(1)_A$ breaking at finite temperature.**




Kaczmarek et. al, arXiv:2102.06136

$$T \sim 1.05T_c$$

$U_A(1)$ **still broken!!**

**Please also look at, arXiv:2103.05954, arXiv:1205.3535**