Wilson Fermions and Chiral Symmetry

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Outline

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$$U'_\mu(n) = \Omega(n) U_\mu(n) \Omega(n+\hat{\mu})^\dagger, \text{ with } U, \Omega \in \mathrm{SU}(3).$$

$$U_{-\mu} \equiv U_\mu(n-\hat{\mu})^\dagger .$$

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$$S_F = a^4 \sum_{n \in \Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu \psi(n+\hat{\mu}) - U_{-\mu}\psi(n-\hat{\mu})}{2a} + m\psi(n) \right).$$

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Notice how in going from the continuum to the lattice:

$$\begin{aligned} A_\mu(x) &\longrightarrow U_\mu(n), \\ \mathfrak{su}(3) &\longrightarrow \mathrm{SU}(3). \end{aligned}$$

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Why Wilson fermions?

Understanding the (naive) Dirac operator I

We can write our naive discretised action as a quadratic form:

$$S_F = a^4 \sum_{n,m \in \Lambda} \sum_{a,b,\alpha,\beta} \overline{\psi}(n)^a_\alpha D(n|m)^{ab}_{\alpha\beta} \psi(m)^b_\beta \ , \ \text{where}$$

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$$D(n|m)^{ab}_{\alpha\beta} = \sum_{\mu=1}^{4} (\gamma_\mu)_{\alpha\beta} \frac{U^{ab}_\mu(n)\delta_{n+\hat{\mu},m} - U^{ab}_{-\mu}(n)\delta_{n-\hat{\mu},m}}{2a} + m\delta_{\alpha\beta}\delta_{ab}\delta_{n,m}.$$

Understanding the (naive) Dirac operator II

Let's Fourier transform $D(n|m)$ for the case of free lattice fermions and obtain the propagator in momentum space:

$$\begin{aligned}\tilde{D}(p|q) &= \frac{1}{|\Lambda|}\sum_{n,m\in\Lambda}\exp\left(-ip\cdot na\right)D(n|m)\exp\left(+iq\cdot ma\right)\\ &= \delta(p-q)\left(m\mathbb{1}+\frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu\sin\left(p_\mu a\right)\right)\\ &= \delta(p-q)\tilde{D}(p)\end{aligned}$$

(Complex conjugate $\longrightarrow$ unitary similarity transformation.)

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Thus, $\tilde{D}(p)^{-1}$ is:

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Indeed $\tilde{D}(p)\tilde{D}(p)^{-1} = \mathbb{1}$!

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We have the continuum pole $p = (0,0,0,0)$!
But the lattice propagator has 15 extra poles (in $d = 4$)..., the so-called doublers.

For all combinations of $p_\mu = 0, \dfrac{\pi}{a}$ we also have poles!

Wilson fermions I

Wilson proposed the following solution: just add a new term to $\tilde{D}(p)$:

$$\tilde{D}(p)_{Wilson} = \tilde{D}(p)_{naive} + \mathbb{1} a^{-1} \sum_{\mu=1}^{4} \left(1 - \cos\left(p_\mu a\right)\right).$$

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3 important properties:

- In the continuum limit, it vanishes.
- It leaves the physical pole untouched.
- Removes all the poles with $p_\mu = \pi/a$!

Wilson fermions II

Now we can undo the Fourier transform on the Wilson term and get our improved operator. It can be written in a compact notation as:

$$D^{(f)}(n|m)_{Wilson} = \left(m^{(f)} + \frac{4}{a}\right)\delta_{\alpha\beta}\delta_{ab}\delta_{n,m} - \frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(\mathbb{1} - \gamma_\mu)_{\alpha\beta}U_\mu^{ab}(n)\delta_{n+\hat{\mu},m},$$

where

$$\gamma_\mu = -\gamma_{-\mu}.$$

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It does not! $\implies$ the $\mathbb{1}$ is the culprit (mass term).

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$U(1)_A$ symmetry of the massless action is *anomalous*. The fermion-integration measure is not invariant.

Chiral symmetry on the lattice

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Nielsen-Ninomiya theorem $\Longrightarrow \nexists$ theory: $\gamma_5 D + D\gamma_5 = 0 \wedge$ free of doublers.

Ginsparg and Wilson came up with a solution:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \neq 0.$$

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Let's assume D fulfills the Ginsparg-Wilson equation. A possible chiral rotation on the lattice can be:

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To do that, we also need to define:

$$\hat{P}_L = \frac{\mathbb{1}-\hat{\gamma}_5}{2},\ \hat{P}_R = \frac{\mathbb{1}+\hat{\gamma}_5}{2},\ \hat{\gamma}_5 = \gamma_5\left(\mathbb{1}-aD\right).$$

Introducing chiral symmetry on the lattice II

The mass term can then be identified in the latttice with:

$$m\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right) = m\overline{\psi}\left(P_L\hat{P}_L + P_R\hat{P}_R\right) = m\overline{\psi}\left(\mathbb{1} - \frac{a}{2}D\right)\psi$$

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And hence, we have a new kind of operator that implements on the lattice the fact that

$$D_m=D+m\left(\mathbb{1}-\frac{a}{2}D\right)=\omega D+m\mathbb{1},\ \text{with}$$

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These are the Ginsparg-Wilson fermions. Notice how in the lattice, the chirality of the fermions depends both on the gauge fields as well as on the neighbouring lattice sites (non-local).

The axial anomaly on the lattice

Let's do an infinitesimal chiral rotation $(M = \mathbb{1}, T_i \in SU(N_f))$:

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The measure transforms as:

$$\mathcal{D}[\psi, \overline{\psi}] = \mathcal{D}[\psi', \overline{\psi'}] \det\left[\mathbb{1} + i\epsilon M\gamma_5 \left(\mathbb{1} - \frac{a}{2}D\right)\right]^2.$$

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For small ϵ, and in particular for $M = \mathbb{1}$:

$$\mathcal{D}[\psi, \overline{\psi}] = \mathcal{D}[\psi', \overline{\psi'}]\left(1 - 2i\epsilon N_f Q_{\mathsf{top}} + \mathcal{O}(\epsilon^2)\right).$$

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- why the naive discretisation fails.
- how Wilson fermions solve the doubling problem.
- that the Wilson fermions explicitly break chiral symmetry (even for the massless case).
- how to introduce the chiral symmetry on the lattice respecting the continuum results.

Thank you for your attention!