Propagator matrix for nonzero chemical potential

Journal club - Lattice field theory

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Why $\mu \neq 0$?

- μ introduces an *inbalance* between q and $\bar{q}$.
- Quark number density:

$$n_q = \frac{T}{V} \frac{\partial \log \mathcal{Z}(\mu, T)}{\partial \mu}.$$

- Study of (dense) baryonic matter. Examples:
  - Quark-Gluon Plasma (RHIC).
  - Neutron stars.
  - QCD phase diagram.

QCD phase diagram

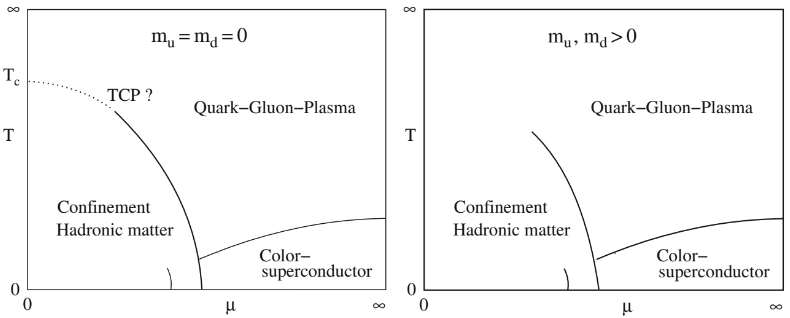

Figure: Conjectured QCD phase diagram [1].

Introduction of μ on the lattice

- Goal: introduce on the lattice a term

$$\mu\bar{\psi}(x)\gamma_4\psi(x).$$

- Linear way:

$$D_{\text{free}}(n|m,\mu) = \sum_{j=1}^{4}\gamma_j \frac{\delta_{n+\hat{j},m} - \delta_{n-\hat{j},m}}{2a} + m\delta_{nm} + \mu\gamma_4\delta_{nm}.$$

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- Divergence in the energy density appears:

$$\epsilon(\mu) - \epsilon(0) \propto \left(\frac{\mu}{a}\right)^2 \xrightarrow{a\to 0} \infty.$$

- Alternative approach: (a la Hasenfratz-Karsch [2])
  Like the QED interaction term :

  - QED: $ig\bar{\psi}(x)\psi A_\nu \gamma^\nu(x)$.
  - μ term: $ig\bar{\psi}(x)\psi A_\nu^{\text{ext.}} \gamma^\nu(x)\delta_{\nu,4}$.

  with $A_4^{\text{ext.}} = -ia\mu/g$.

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  with $A_4^{\text{ext.}} = -ia\mu/g$.
- Appears in the temporal term:

$$D_{\text{free}}(n|m,\mu) = \sum_{j=1}^{3} \gamma_j \frac{\delta_{n+\hat{j},m} - \delta_{n-\hat{j},m}}{2a} + \gamma_4 \frac{e^{\mu}\delta_{n+\hat{j},m} - e^{-\mu}\delta_{n-\hat{j},m}}{2a} + m\delta_{nm}.$$

From now on $\mu \equiv \mu a$.

Sign problem

- Another problem appears:

$$\gamma_5 D(\mu)\gamma_5 = D^\dagger(-\mu).$$

  $D(\mu)$ is not γ_5-hermitian $\Rightarrow \det(D(\mu)) \in \mathbb{C}$.
- Several consequences:
  - $e^{-S_G}\det(D)$ is not a probability distribution.
  - Oscillations.

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- Several consequences:
  - $e^{-S_G}\det(D)$ is not a probability distribution.
  - Oscillations.
- Possible approaches:
  - Imaginary μ: $\mu_{\text{im}} = i\mu$ with $\mu \in \mathbb{R}$.
  - Reweighting

Dirac operator determinant

- Measurement:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}U e^{-S_G} \det(M(\mu))\mathcal{O}}{\int \mathcal{D}U e^{-S_G} \det(M(\mu))}$$

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- We can evaluate it stochastically, e.g. pseudofermions:

$$\det[DD^\dagger] \propto \int \mathcal{D}[\phi_R]\mathcal{D}[\phi_I] e^{-\phi^\dagger (DD^\dagger)^{-1} \phi}$$

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- Exact calculation $\Rightarrow$ different strategies, e.g., *propagator matrix*.

Staggered formulation

- Staggered formulation:

$$D_{\text{stag.}}(n|m,\mu) = \underbrace{\sum_{j=1}^{3} \eta_j \frac{U_j(n)\delta_{n+\hat{j},m} - U_{-j}(n)\delta_{n-\hat{j},m}}{2a}}_{\equiv D^{(3)}_{\text{stag.}}}$$

$$+\eta_4 \frac{e^{\mu a}U_4(n)\delta_{n+\hat{j},m} - e^{-\mu a}U_{-4}(n)\delta_{n-\hat{j},m}}{2a}$$

$$M = D_{\text{stag.}} + m$$

- Definitions:

$$B_i = \eta_4(D^{(3)} + m)|_{t=i}$$

$$U_{N_t-1} = U_4\big|_{t=N_t-1}$$

Propagator matrix derivation

- The derivation is in [3] with a missing factor of 3 in final exponential!
- Axial (temporal) gauge: $U_t(n) = 1 \quad \forall t \neq N_t - 1$.
- Take μ to the last time slice.
- We get $\det(M) =$

$$\det \begin{pmatrix} B_0 & 1 & 0 & \dots & U^\dagger_{N_t-1} e^{-\mu N_t} \\ -1 & B_1 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & B_{N_t-2} & 1 \\ -U_{N_t-1} e^{\mu N_t} & 0 & \dots & -1 & B_{N_t-1} \end{pmatrix}$$

Propagator matrix derivation

- Multiply by $U_{N_t-1}e^{N_t\mu}$ last column: $\det(M) = e^{-3V_sN_t\mu}\times$

$$\times\det\begin{pmatrix} B_0 & 1 & 0 & \dots & U^{\dagger}_{N_t-1}e^{-i\mu N_t} \\ -1 & B_1 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & B_{N_t-2} & 1 \\ -U_{N_t-1}e^{i\mu N_t} & 0 & \dots & -1 & B_{N_t-1} \end{pmatrix}$$

with $V_s = N_xN_yN_z$ spatial (dimensionless) volume.

Propagator matrix derivation

- Multiply from the left by:

$$\begin{pmatrix} 1 & B_0 & 0 & \dots & & & \\ 0 & 1 & 0 & \ddots & & & \\ 0 & 0 & 1 & B_2 & 0 & \dots & \\ 0 & 0 & 0 & 1 & 0 & \dots & \\ & & \dots & & & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 1 & B_{N_t-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Propagator matrix derivation

- Permuting so the first column is the last: $\det(M) =$

$$e^{-3V_s N_t \mu} \det \begin{pmatrix} \Omega_{01} & 0 & 0 & & & \\ -1 & \Omega_{23} & 0 & & 1 \\ 0 & -1 & \Omega_{45} & & \\ \vdots & \ddots & -1 & \ddots & \\ 0 & 0 & 0 & -1 & \Omega_{N_t-2,N_t-1} U_{N_t-1} e^{N_t} \end{pmatrix}$$

with

$$\Omega_{i,j+1} = \begin{pmatrix} B_j B_{j+1} + 1 & B_j \\ B_{j+1} & 1 \end{pmatrix} = \begin{pmatrix} B_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_{j+1} & 1 \\ 1 & 0 \end{pmatrix}$$

Propagator matrix derivation

- This determinant is easily calculable:

$$\det(M) = e^{3N_t V_s \mu} \det(P - e^{-N_t \mu})$$

with P the **propagator/reduced matrix**:

$$P = -\left(\prod_{j=0,2,4}^{N_t-2} \Omega_{i,j+1} \right) U_{N_t-1}$$
$$= -\left(\prod_{j=0}^{N_t-1} \begin{pmatrix} B_j & 1 \\ 1 & 0 \end{pmatrix} \right) U_{N_t-1}.$$

- Then:

$$\det(M) = e^{3N_t V_s \mu} \prod_{i=1}^{6V_s} (\xi_i - e^{-N_t \mu}).$$

Properties of P

- Advantages:
  - Reduced matrix dimension $\to 6N_s^3$.
  - Independent of μ.
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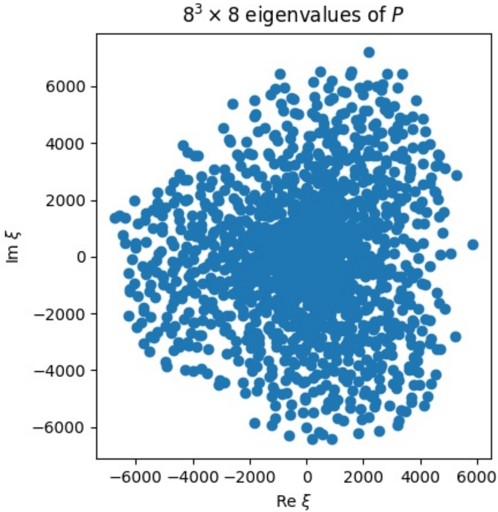
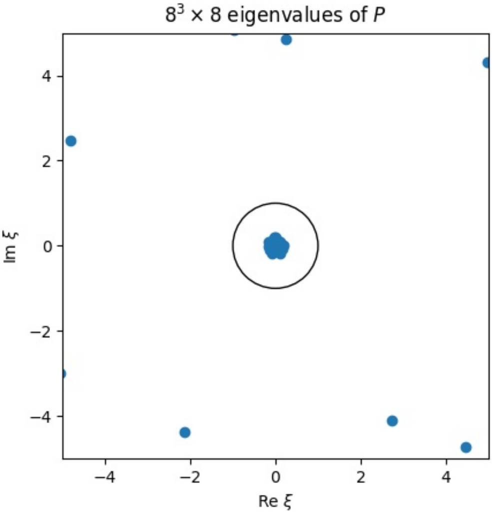
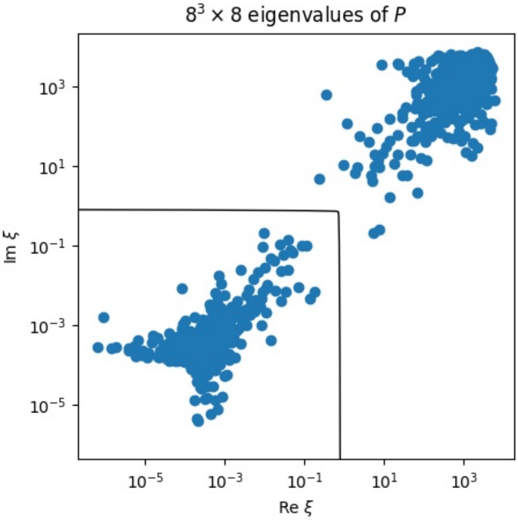
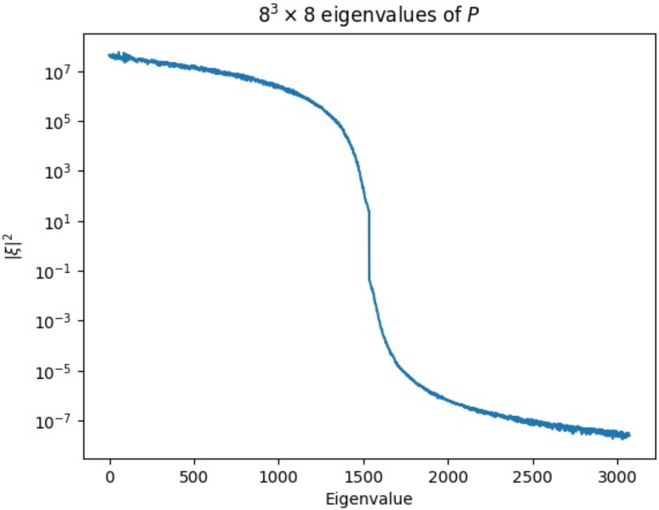
- $\prod_i \xi_i$ with $|\xi_i| < 1$ is real positive.
- Eigenvalues of P are not in the unit circle.


$8^3 \times 8$ eigenvalues of P
6000
4000
2000
0
−2000
−4000
−6000
Im ξ
−6000 −4000 −2000 0 2000 4000 6000
Re ξ


8^3 × 8 eigenvalues of P
4
2
0
−2
−4
Im ξ
−4
−2
0
2
4
Re ξ


$8^3 \times 8$ eigenvalues of P
10^3
10^1
10^{-1}
10^{-3}
10^{-5}
Im ξ
10^{-5}
10^{-3}
10^{-1}
10^1
10^3
Re ξ


$8^3 \times 8$ eigenvalues of P
$|\xi|^2$
10^7
10^5
10^3
10^1
10^{-1}
10^{-3}
10^{-5}
10^{-7}
0
500
1000
1500
2000
2500
3000
Eigenvalue

Reweighting

- Allows to determine $\mathcal{Z}(T', \mu')$ from results evaluated at another T, μ:

$$\mathcal{Z}(T', \mu') = \mathcal{Z}(T, \mu) \left\langle \frac{e^{-S_G[T', \mu']} \det[M(\mu', U)]}{e^{-S_G[T, \mu']} \det[M(\mu, U)]} \right\rangle_{T,\mu}$$

- Require to evaluate $\det(M(\mu))$ exactly $\Rightarrow$ Propagator matrix.
- Fodor, Katz [5]: $\mu = 0$, $T = T_c$ $\rightarrow$ explore QCD critical point.

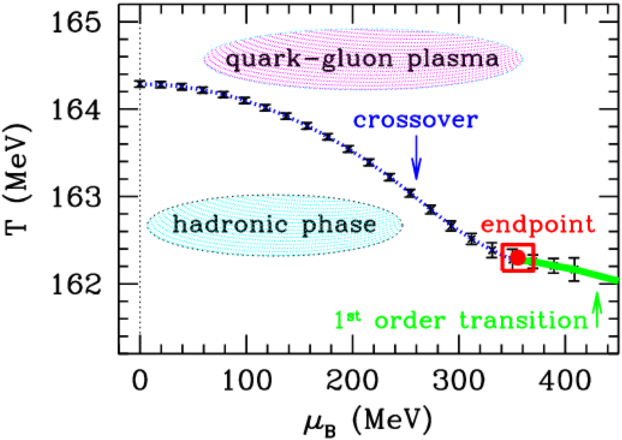


Figure: Study of the QCD critical point [5].

Lee-Yang zeros

- *Lee-Yang zeros* : $\mathcal{Z} = 0 \Rightarrow \ln \mathcal{Z} \to -\infty$: phase transitions.
- These can be related with the propagator matrix.

$$0 = \frac{\mathcal{Z}(\mu)}{\mathcal{Z}(0)} = \left\langle \frac{\det[M(\mu)]}{\det[M(0)]} \right\rangle_{\mu=0} = e^{3N_t V_s \mu} \left\langle \prod_{i=1}^{6V_s} \frac{\xi_i - e^{-N_t \mu}}{\xi_i - 1} \right\rangle_{\mu=0}.$$

- For μ_{im}, look for the zeros and check the behaviour with $1/V$.
- If in $V \to \infty$ there is a zero in $\mu \in \mathbb{R} \Rightarrow$ endpoint.
- Rooting is a technical problem here.

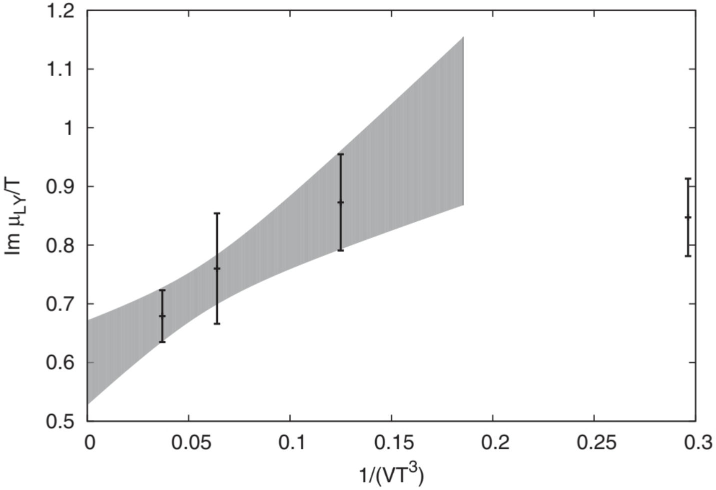


Figure: Behaviour of Im μ of the YL zeros with $1/V$ [4].

Canonical partition functions

- For μ_{im}, fugacity expansion:

$$\mathcal{Z}(T,\mu) = \sum_{n=-l_{\text{max}}}^{l_{\text{max}}} e^{i\mu n/T} \mathcal{Z}_n(T)$$

where n quark number, $z = e^{i\mu/T}$ *fugacity*. Coefficients:

$$\mathcal{Z}_n(T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}(\mu/T) e^{-i\mu n/T} \mathcal{Z}(T, i\mu)$$

Hadron masses

- The $\mathcal{Z}_n$ can be associated with the ξ's:

$$\mathcal{Z}_n(T) = \mathcal{Z}(0) \cdot \left\langle \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}(\mu/T) e^{-i\mu n/T} \left(\frac{\det M(i\mu)}{\det M(0)} \right)^{1/4} \right\rangle .$$

- They can be related with the masses of particles

$$\begin{aligned} F_{n_1,\ldots,n_f} &= -T \ln \mathcal{Z}_{n_1,\ldots,n_f} \\ m_0^{n_1,\ldots,n_f} &= \lim_{T\to 0} F_{n_1,\ldots,n_f} - F_{0,\ldots,0} \end{aligned}$$

so they can be used for studying meson and baryon masses [6]:

$$am_\pi = \lim_{L_t\to\infty} -\frac{1}{L_t} \ln \left\langle \frac{N_t^2}{64} \left| \sum_{k=1}^{3V} \xi_k \right|^2 \right\rangle .$$

Conclusions

- Introducing μ on the lattice creates some technical problems, e.g., the sign problem.
- The propagator matrix is an exact and μ-independent way of dealing with the fermion determinant.
- Several applications to finite density QCD, e.g.:
  - Reweighting.
  - Lee-Yang zeros.
  - Meson/Baryon masses.

References I

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