Wilson Flow and Applications

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Outline

Motivation

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Maps from supersymmetry that take interacting theories to free theories (Nicolai maps H. Nicolai 1980).

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Many applications: renormalisation, separation of topological sectors...

Flow equation

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- A continuous map $\mathcal{F}$: $V \to U$ can be thought of as a diff. eq. evolving U through an extra dimension t.
- One can obtain this differential equation by considering infinitesimal transformations:

$$U \to U + \epsilon Z(U)U + O(\epsilon^2),$$

whose continuous composition is given by

$$\dot{U}_t = Z_t(U_t)U_t,$$

and Z lives in the algebra.

Flow equation II

The choice $Z_t = -g_0^2\, \partial^a_{x,\mu} S_{\mathrm{w}}$, where S_{w} is the Wilson action, is commonly used.
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- in order to have a trivialising map, we can ask for:

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- Since $\ln \det \mathcal{F}_{t,*}(V)$ is equal to the LHS of the above eq.:

$$S(\mathcal{F}_t(V)) - \ln \det \mathcal{F}_{t,*}(V) = (1-t)S(\mathcal{F}_t(V)) - C_t,$$

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- and we see how for $t = 1$ we obtain a trivialising map.

Solutions for the flow generator I

A better equation to find Z_t is:

$$\sum_{x,\mu}\left[\partial^a_{x,\mu} Z^a(U_s) - t\partial^a_{x,\mu} S(U) Z^a(U_s)\right] = S(U) + \dot{C}_t.$$

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Why the choice of the Wilson action? $\Longrightarrow$ simplicity

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Thus, to first order the adequate flow generator is just the gradient of the Wilson action!

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- Locality guaranteed where series converges. But the latter is not clear.
- At least to leading order, the use of the Wilson flow decreases the action for increasing t:

$$\frac{\mathrm{d}}{\mathrm{dt}} S_{\mathrm{w}}(U_t) = -\frac{3}{16} \sum_{x,\mu} \partial^a_{x,\mu} S_{\mathrm{w}}(U_t) \partial^a_{x,\mu} S_{\mathrm{w}}(U_t) \leq 0.$$

Wilson Flow and Renormalisation

Wilson flow and perturbation theory

The continuum flow equations are given by

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$$B_\mu = \sum_{k=1}^{\infty} g_0^k B_{\mu,k},$$

and inserting that in the above eq. we obtain a series of equations. To leading order, we have that:

$$B_{\mu,1}(t,x) = \int d^4y \frac{e^{-(x-y)^2/4t}}{(4\pi t)^2} A_\mu(y).$$

Explicitly, we see that to this order the gauge potential is averaged over a volume whose mean-square radius is $\sqrt{8t}$.

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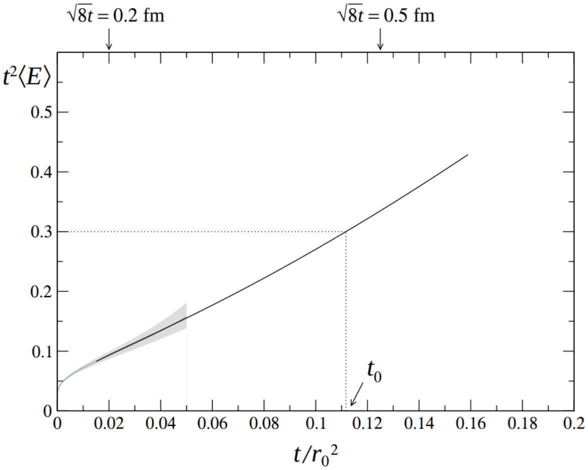
It does from the next order and onward. Using the relation between the bare and renormalised couplings in the $\overline{\text{MS}}$ scheme and going to an energy scale $q = (8t)^{-1/2}$:

$$\langle E \rangle = \frac{3}{4\pi t^2} \alpha(q)[1 + k_1 \alpha(q) + O(\alpha^2)].$$

Flow dependence of $t^2\langle E\rangle$



M. Lusher 2010

What about fermions?

We can also flow the fermionic fields M. Lusher 2013. The simplest extension is to keep the evolution of the gauge fields untouched and:

$$\partial_t \chi_t = D_\mu D_\mu \chi_t, \quad \chi_t|_{t=0} = \psi.$$

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Also, similarly to the gauge field case, now the energy scale of the renormalised operators can be understood as driven by the change in t.

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But we can approximate any (classical) gauge field. Topological sectors included but not separated.

Recall that for $t > 0$, the Wilson action tends to decrease!

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We can define a way to measure the smoothness of a certain gauge field as:

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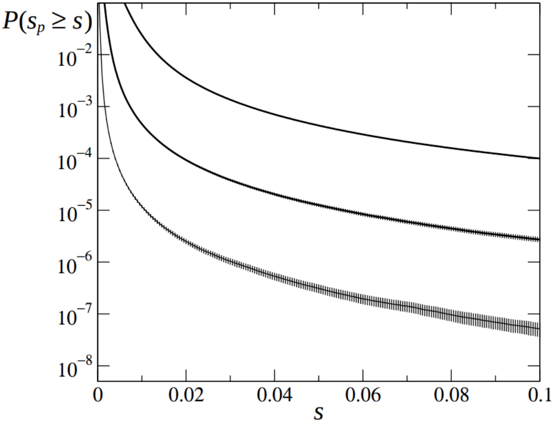
Large values for h not favoured $\Longrightarrow$ large values for the plaquettes are suppressed.

Suppression of the plaquettes



Top to bottom $a = 0.1, 0.07, 0.05$ fm M. Lusher 2010

Wilson Flow and Topology III

The gauge fields on a lattice divide into disconnected topological sectors under particular smoothness conditions M. Lusher 1982 A. Phillips et al 1986.

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Long story short, fields satisfying:

$$h < 0.067$$

are included into a particular subspace and fall into a particular topological sector.

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Long story short, fields satisfying:

$$h < 0.067$$

are included into a particular subspace and fall into a particular topological sector.

Hence, Wilson flow dynamically drives the gauge fields into smooth and and well-defined topological configurations.

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We have talked about:

- How the flow equation can be obtained.
- That Wilson flow drives fields to configurations decreasing the Wilson action.
- That the gauge and fermionic fields renormalise at positive flow time.
- An intuition for how disconnected topological sectors appear at $t > 0$.

Thank you for your attention!