How to measure the magnetic susceptibility of QCD matter on the lattice?

Gergely Markó*

* University of Bielefeld, Department of Theoretical Physics

Based on:

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- Continuum definition of the magnetic susceptibility
- Magnetic fields on the lattice
- Lattice methods for χ
- Magnetic susceptibility of QCD matter

Continuum definition of the magnetic susceptibility

- In classical electrodynamics Ampere's law in matter is written in terms of the *magnetizing field* $\mathbf{H}$.
- The *magnetic field* in a given point ($\mathbf{B}$) adds up from the magnetizing field and the *magnetization*, $\mathbf{M}$: $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.
- The magnetization is a reaction of the matter to the magnetizing field.
- To leading order (classically almost always enough) this response is described by the magnetic susceptibility:

$$\mathbf{M} = \chi_H \mathbf{H} \,.$$

- Based on the behaviour (sign and magnitude of χ_H) matter is either paramagnetic ($\chi_H \gtrapprox 0$) or diamagnetic ($\chi_H \lessapprox 0$).
- (Sometimes the permeability, μ is used instead, which connects $\mathbf{H}$ to $\mathbf{B}$ directly, then $\mu/\mu_0 = 1 + \chi_H$.)

Continuum definition of the magnetic susceptibility

- How do we apply this to QCD matter?
- Naïvely applying a background field to QCD is by introducing $\mathbf{H}$.
- But the quarks in the Lagrangian interact with the total magnetic field $\mathbf{B}$ which already takes into account the magnetization of QCD matter itself.
- Counterintuitively the control parameter becomes $\mathbf{B}$ in QFT. If we can obtain $\mathbf{M}$ we can reconstruct $\mathbf{H}$.
- Generally the susceptibility is the second derivative of the free energy density w.r.t. to the control parameter:

$$\chi = -\frac{\partial^2 f}{\partial (eB)^2}\bigg|_{B=0} = \frac{1}{V_4}\frac{\partial^2 \log Z}{\partial (eB)^2}\bigg|_{B=0} .$$

- We can relate χ and χ_H using that $M = -\frac{\partial f}{\partial B}$ (generalizing $E = -MB$)

$$\chi_H = \frac{\partial M}{\partial H} \leftrightarrow \chi = -\frac{\partial^2 f}{\partial (eB)^2} = \frac{1}{e}\frac{\partial M}{\partial (eB)} = \frac{\chi_H}{e^2 \mu} .$$

Magnetic field on the lattice

UNIFORM MAGNETIC FIELD ON THE LATTICE

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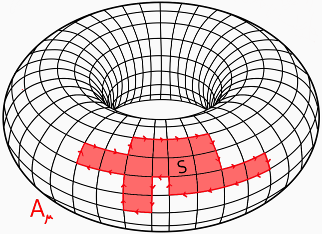
Stoke's theorem must hold in the torus.

$$\oint A_\mu dx_\mu = SB$$



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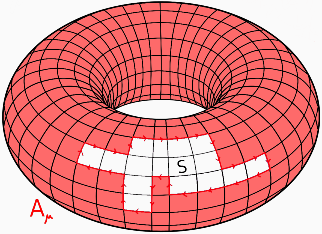

Stoke's theorem must hold in the torus.

$$\oint A_{\mu}dx_{\mu} = SB \qquad \oint A_{\mu}dx_{\mu} = (L_xL_y - S)B$$

UNIFORM MAGNETIC FIELD

The phase picked up by a particle winding around the path has to be unambiguous.

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Uniform magnetic field

The phase picked up by a particle winding around the path has to be unambiguous.

$$e^{iqBS} = e^{iqB(L_xL_y - S)} \qquad qB = \frac{2\pi N_b}{L_xL_y}, \quad N_b \in \mathbb{Z}$$

Lattice methods

Main problem: flux quantization

$$eB = \frac{6\pi}{L_x L_y} N_b \, .$$

$\log Z(B)$ is only defined at discrete points, differentiation w.r.t. B is ill-defined.

- In large volumes the flux quantum becomes small $\rightarrow$ approximate the derivative with finite differences.
  Bonati et al. PRL 2013, Bali et al. JHEP 2014

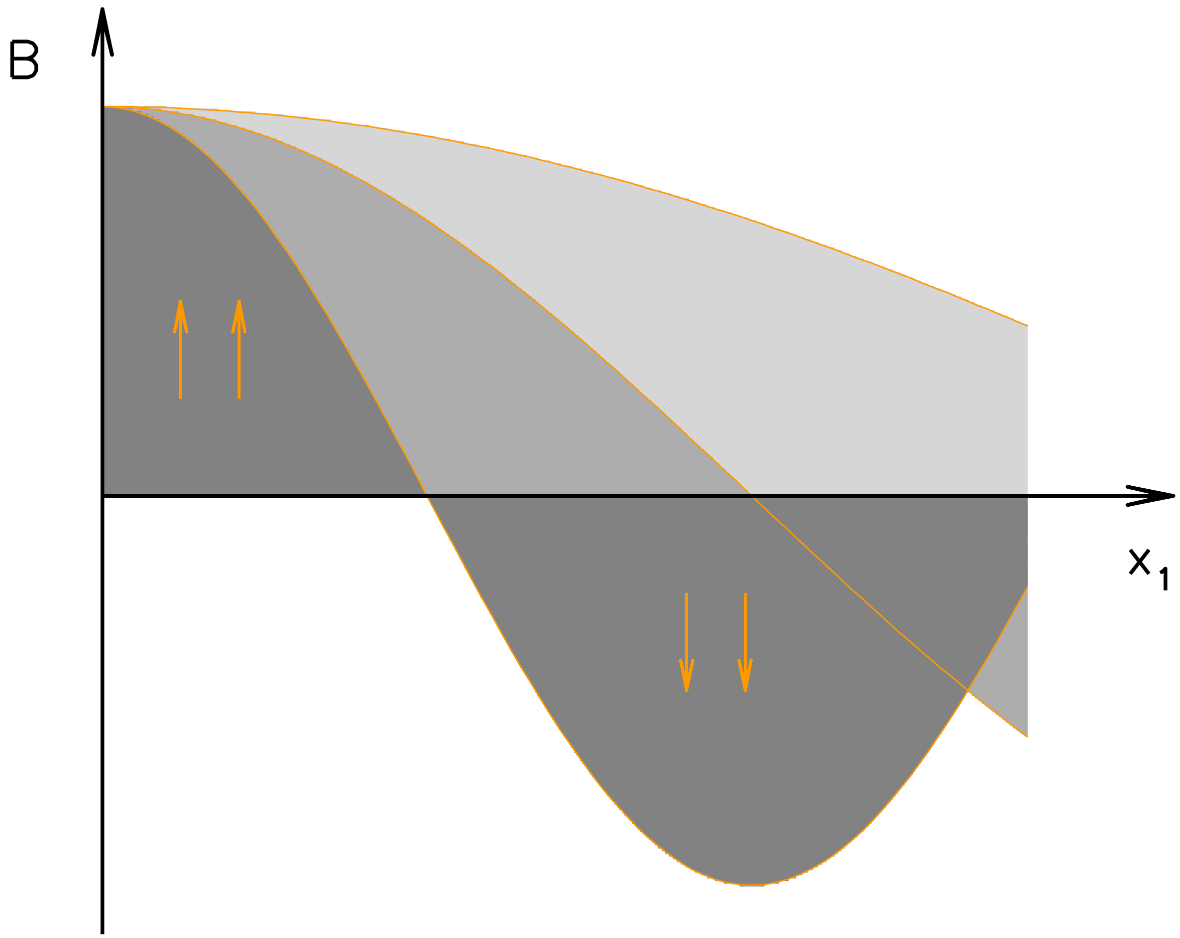
- "Half-half method", B and $-B$ in two halves of the lattice $\rightarrow$ overall flux is zero, B can be varied continuously. Finite volume effects are very strong because of the change in the middle $\rightarrow$ very large volumes again.
  Levkova, DeTar PRL 2014

- Isotropy is broken by $\mathbf{B}$ $\rightarrow$ hydrodynamic pressure becomes anisotropic if compressing the system while a constant magnetic flux is going through. This anisotropy can be related to the magnetization and hence the susceptibility. However anisotropic lattices have to be used.
  Bali et al. JHEP 2013

Current-current correlator method

Generalization of the half-half idea: introduce smoothly oscillating field with zero flux, and approach a constant field through increasing wavelength.

$$B(x) = B\cos(p_1 x_1), \quad p_1 = \frac{2\pi n}{L}$$



Current-current correlator method

The magnetic field enters through the term

$$S_{\mathrm{QED,int}} = i \int d^4x A_\mu(x) j_\mu(x) \,,$$

where $j_\mu = \sum_f q_f \bar{\psi}\gamma_\mu\psi$ is the electric current. For the oscillating field we need the vector potential

$$A_2(x_1) = B\frac{\sin(p_1 x_1)}{p_1} \,.$$

Taking the 2nd derivative of $\log Z$ w.r.t. B then gives a susceptibility with a certain p_1 (also use $V_4 = L^3/T$):

$$\chi^{(p_1)} = -\frac{T}{L^3}\int d^4x d^4y \frac{\sin(p_1 x_1)\sin(p_1 y_1)}{p_1^2}\langle j_2(x) j_2(y)\rangle \,.$$

Let's define the current-current correlator averaged over all the non-interesting directions:

$$G_{22}(z_1) = \int dz_2 dz_3 dz_4 \langle j_2(z) j_2(0)\rangle \,.$$

Current-current correlator method

Using a shifted box on the lattice (periodicity permits) the susceptibility can then be written as

$$\chi^{(p_1)} = -\frac{1}{L}\int_{-L/2}^{L/2} dx_1 dy_1 \frac{\sin(p_1 x_1)\sin(p_1 y_1)}{p_1^2} G_{22}(x_1 - y_1)\,.$$

Change the integration variable y_1 for every x_1 to $z_1 = y_1 - x_1$:

$$\chi^{(p_1)} = -\frac{1}{L}\int_{-L/2}^{L/2} dx_1 \int_{-L/2-x_1}^{L/2-x_1} dz_1 \frac{\sin(p_1 x_1)\sin(p_1(x_1+z_1))}{p_1^2} G_{22}(-z_1)\,.$$

Now use the trigonometric identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$:

$$\chi^{(p_1)} = -\frac{1}{L}\int_{-L/2}^{L/2} dx_1 \int_{-L/2-x_1}^{L/2-x_1} dz_1 \left[\frac{\sin^2(p_1 x_1)\cos(p_1 z_1)}{p_1^2} G_{22}(-z_1) + \frac{\sin(p_1 x_1)\cos(p_1 x_1)\sin(p_1 z_1)}{p_1^2} G_{22}(-z_1)\right]$$

Since $G_{22}(z_1)$ is periodic and even, the z_1 integral in the first term is **independent** of x_1 and the second term **vanishes**.

Current-current correlator method

$$\chi^{(p_1)} = -\frac{1}{L}\int_{-L/2}^{L/2} dx_1 \int_{-L/2}^{L/2} dz_1 \frac{\sin^2(p_1 x_1)\cos(p_1 z_1)}{p_1^2} G_{22}(z_1)\,.$$

Now the x_1 integral can be done independently

$$\int_{-L/2}^{L/2} dx_1 \sin^2(p_1 x_1) = \frac{L}{2} - \frac{\sin(Lp)}{2p} = \frac{L}{2}\,,$$

and then

$$\chi^{(p_1)} = -\frac{1}{2}\int_{-L/2}^{L/2} dz_1 \frac{\cos(p_1 z_1)}{p_1^2} G_{22}(z_1)\,.$$

What happens in the $L \to \infty$ limit? Write in $p_1 = 2\pi n/L$ and approximate the $\cos$ by its series:

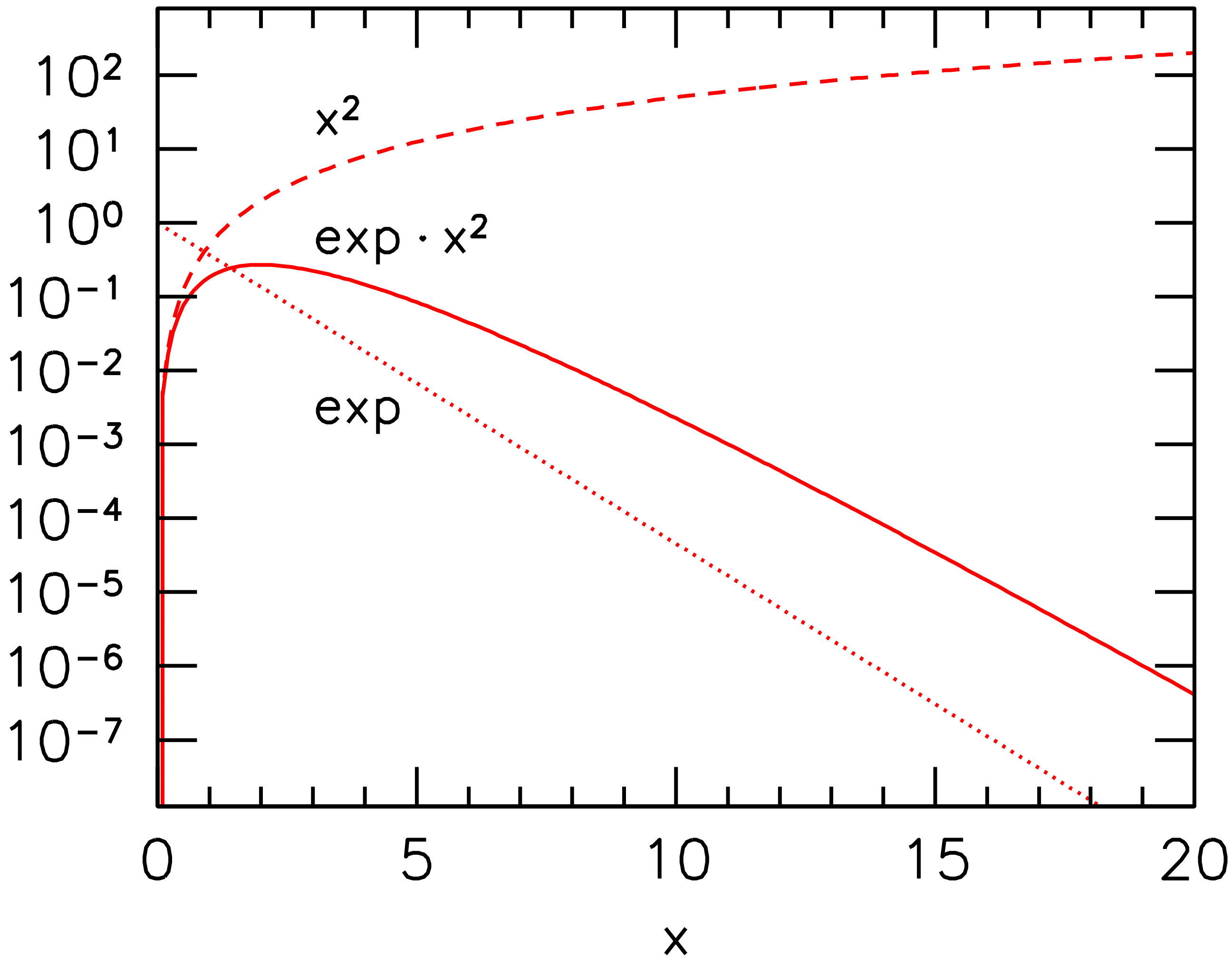
$$\chi^{(p_1)} = -\int_0^{L/2} dz_1 \sum_{k=0}^{\infty} (-1)^k \left(\frac{2\pi n}{L}\right)^{2k-2} \frac{z_1^{2k}}{(2k)!} G_{22}(z_1)\,.$$

To make sense of this series we have to know that $\int_0^{L/2} dz_1 G_{22}(z_1) \sim \mathrm{e}^{-L}$:

$$\chi^{(p_1)} \to \int_0^{L/2} dz_1 \frac{z_1^2}{2} G_{22}(z_1)\,.$$

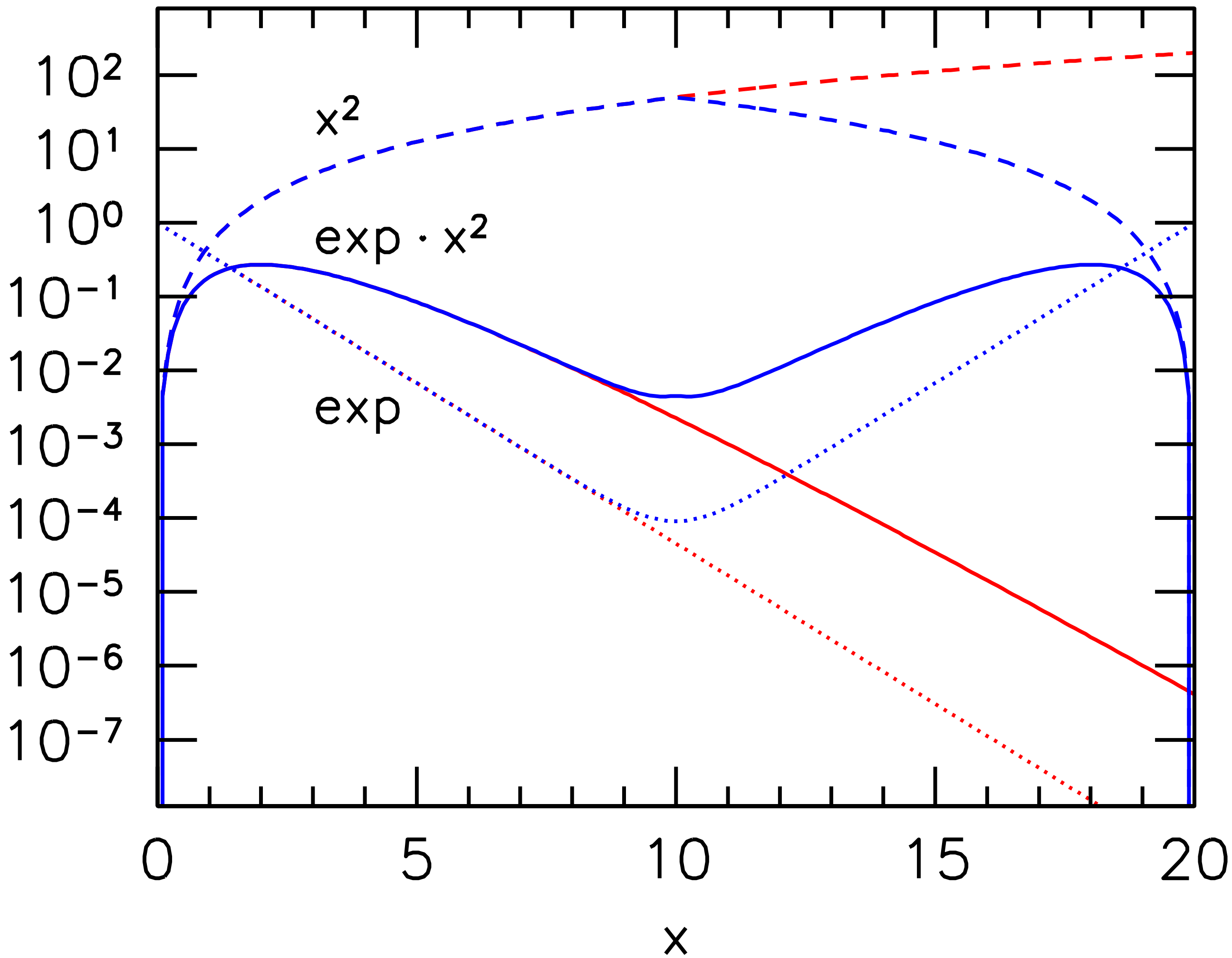
Current-current correlator method

$$\chi^{(p_1)} = \int_0^{L/2} dz_1 \frac{G_{22}(z_1)}{2} z_1^2 \,.$$



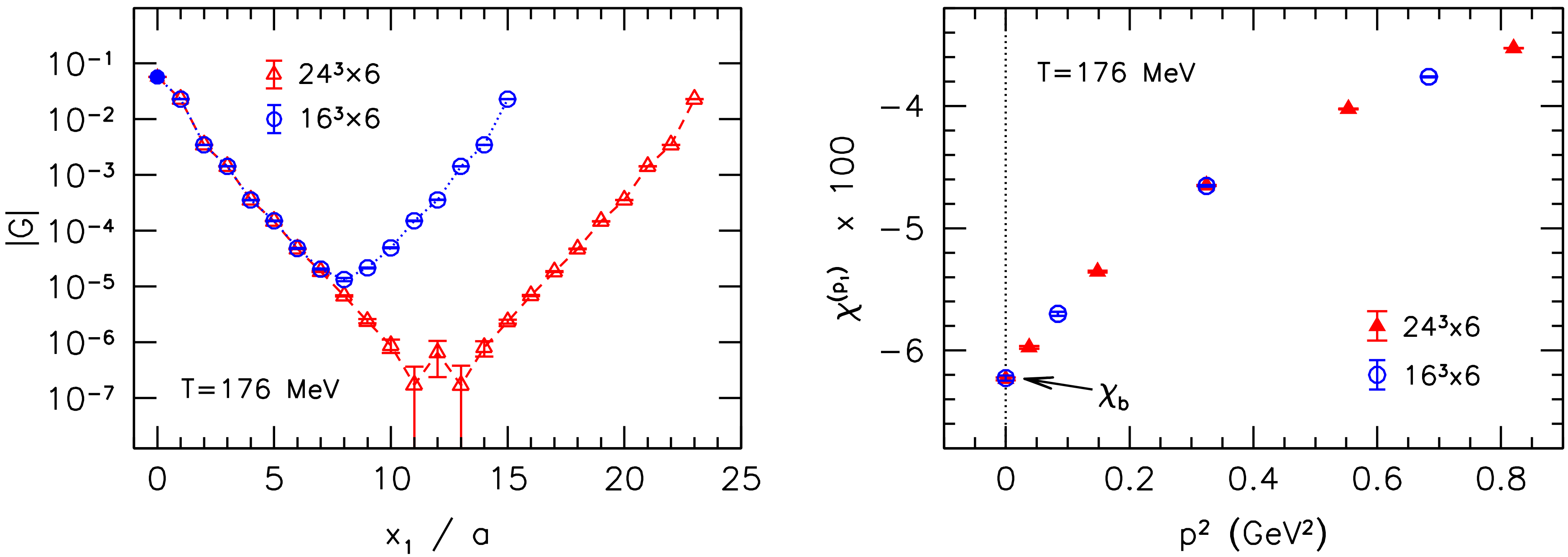
Current-current correlator method

$$\chi^{(p_1)} = \int_0^{L} dz_1 \frac{G_{22}(z_1)}{2} \begin{cases} z_1^2, & z_1 \leq L/2 \\ (z_1 - L)^2, & z_1 > L/2 \end{cases} .$$



Current-current correlator method

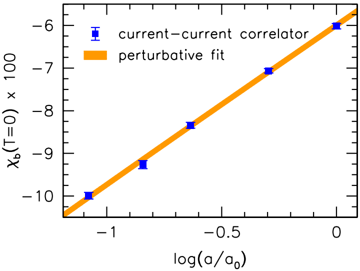
G_{22} and $\chi^{(p_1)}$ at a high temperature ($T > T_c$) show almost no L dependence:



Results

Zero temperature

- susceptibility contains additive divergence $\propto \log a$ due to charge renormalization Schwinger '51 Bali et al. '14



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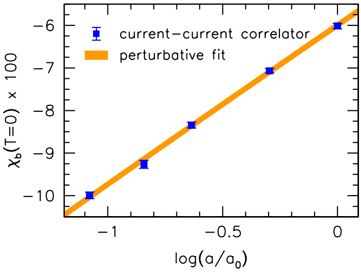

- renormalize as $\chi(T) = \chi_b(T) - \chi_b(T=0)$

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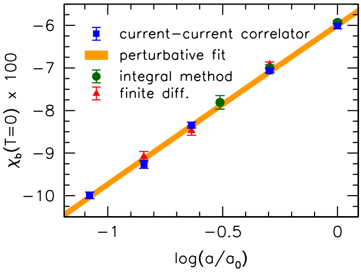
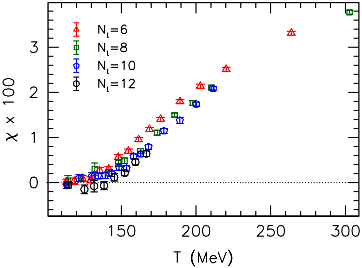


- renormalize as $\chi(T) = \chi_b(T) - \chi_b(T=0)$
- different methods in the literature agree with each other

Nonzero temperature

- continuum extrapolation using four lattice spacings



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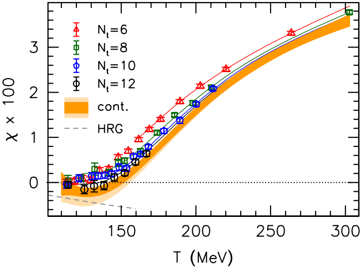

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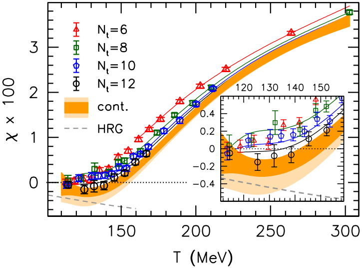

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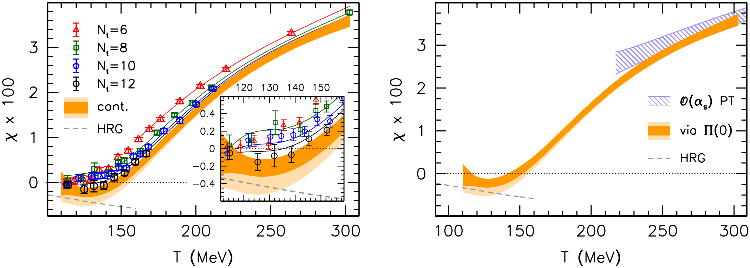

- comparison to HRG model (low T) Endrődi '13
  and to perturbation theory (high T) Bali et al. '14

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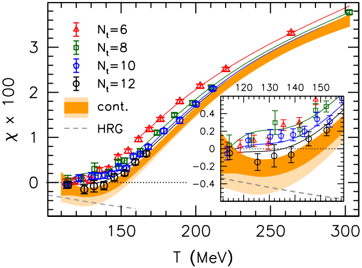

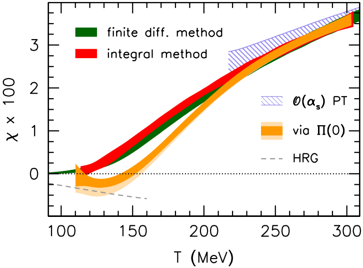

- comparison to HRG model (low T) Endrődi '13
  and to perturbation theory (high T) Bali et al. '14
- taste splitting lattice artefacts severe at low T; careful continuum extrapolation required Bali, Endrődi, Piemonte '20