Staggered Fermions

by

Pratitee Pattanaik

Reference:

Quantum Chromodynamics on the lattice
C. Gattringer, C.B. Lang

Seminar writeup by Alessandro Sciarra

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- Basic idea of the staggered formulation.
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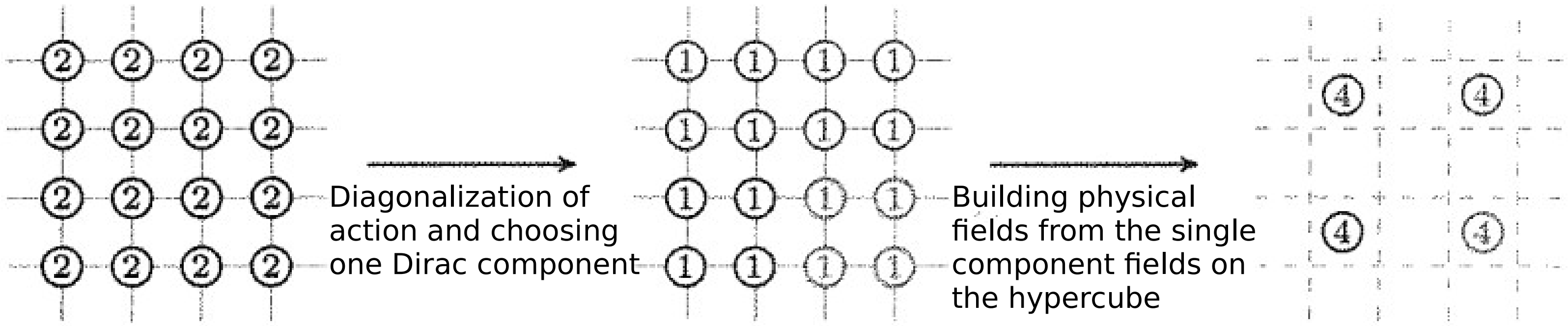
Motivation

- Naive discretization of QCD action leads to the fermion doubling problem.
- The quark propagator has 16 poles instead of one.
- The additional poles are at the edges of the Brillouin zone.
- A solution to this is to decrease the extent of the Brillouin zone by increasing the lattice spacing to remove the doublers.
- This is the basic idea of staggered fermions. The lattice spacing in the staggered formulation becomes b=2a instead of a.
- The advantage is that a non-trivial part of the chiral symmetry (U(1) X U(1)) is still intact in this formulation in the massless case.
- This is useful to study spontaneous chiral symmetry breaking and the associated Goldstone phenomenon in theory like QCD with massless quarks.

Idea of the staggered formulation



Staggering procedure in two (four) dimensions.

(The number inside each circle indicates how many degrees of freedom belong to each site.)

In the beginning, there is 2(4) degrees of freedom per site due to the Dirac components.

The Dirac matrices are diagonalized (and hence the action) by a change of variables. Then, only one of the 2(4) Dirac components of the fields is chosen per site. This reduces the degrees of freedom per site to 1(1).

Finally, the 4(16) fields at the vertices of each lattice unit hypercube are combined to build 2(4) physical fields, each with 2(4) components and the degree of freedom per site becomes 4(16). But the lattice spacing gets doubled.

Diagonalization of the γ matrices

- The naive lattice action is

$$\mathcal{S} = \frac{1}{2} \sum_{n,\mu} \left[\hat{\bar{\psi}}(n)\, \gamma_\mu\, \hat{\psi}(n+\hat{\mu}) - \hat{\bar{\psi}}(n)\, \gamma_\mu\, \hat{\psi}(n-\hat{\mu}) \right] + \hat{M} \sum_n \hat{\bar{\psi}}(n)\, \hat{\psi}(n)$$

- If T(n) are unitary 4x4 matrices, such that it diagonalizes the γ matrices

$$T^\dagger(n)\, \gamma_\mu\, T(n \pm \hat{\mu}) = \eta_\mu(n)\, \mathbb{1}$$

then the transformation

$$\hat{\psi}(n) = T(n)\, \chi(n)$$
$$\hat{\bar{\psi}}(n) = \bar{\chi}(n)\, T^\dagger(n)$$

will make S diagonal in the Dirac space

Diagonalization of the γ matrices

- We can see this by substituting the transformation in the action to get

$$\mathcal{S} = \frac{1}{2} \sum_{n,\mu} \left[\bar{\chi}(n)\, T^\dagger(n) \gamma_\mu T(n+\hat{\mu})\, \chi(n+\hat{\mu}) - \bar{\chi}(n)\, T^\dagger(n) \gamma_\mu T(n-\hat{\mu})\, \chi(n-\hat{\mu}) \right] + \hat{M} \sum_n \bar{\chi}(n)\, T^\dagger(n) T(n)\, \chi(n)$$

$$= \sum_{n,\mu,\alpha,\beta} \delta_{\alpha,\beta}\, \eta_\mu(n)\, \bar{\chi}_\alpha(n)\, \hat{\partial}^S_\mu\, \chi_\beta(n) + \hat{M} \sum_{n,\alpha,\beta} \delta_{\alpha,\beta}\, \bar{\chi}_\alpha(n)\, \chi_\beta(n)\,.$$

- This is a rewriting of the original action but the γ-matrices are replaced by $\delta_{\alpha\beta}$. α,β run from 1 to 4. But we can fix $\alpha=\beta=1$, since the terms in the action have the same form for other values.
- The Dirac indices are omitted and the fields have only one component per site

$$\mathcal{S}_F^{(stag.)} = \frac{1}{2} \sum_{n,\mu} \eta_\mu(n) \left[\bar{\chi}(n)\, \chi(n+\hat{\mu}) - \bar{\chi}(n)\, \chi(n-\hat{\mu}) \right] + \hat{M} \sum_n \bar{\chi}(n)\, \chi(n)$$

- The degrees of freedom per site gets reduced from 4 to 1.

Diagonalization of the γ matrices

- The complex numbers η_μ are called the staggered phases
- The standard choice of T(n) that satisfies the diagonalization condition is

$$T(n) = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4}$$

- We can check that for a particular direction say μ=3 the Dirac matrices indeed get diagonalized:

$$\begin{aligned} T^\dagger(n)\gamma_3 T(n\pm\hat{3}) &= \gamma_4^{n_4}\gamma_3^{n_3}\gamma_2^{n_2}\gamma_1^{n_1}.\gamma_3.\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3\pm1}\gamma_4^{n_4} \\ &= (-1)^{n_1+n_2}\gamma_4^{n_4}\gamma_3^{n_3}\gamma_3\gamma_2^{n_2}\gamma_1^{n_1}\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3\pm1}\gamma_4^{n_4} \\ &= (-1)^{n_1+n_2}\gamma_4^{n_4}\gamma_3^{n_3+1}\gamma_3^{n_3\pm1}\gamma_4^{n_4} = (-1)^{n_1+n_2}\mathbb{1} \end{aligned}$$

- The staggered phases are

$$\begin{aligned} &\eta_1(n) = 1 \\ &\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu} \quad \text{if } \mu \neq 1 \end{aligned}$$

Building the physical fields

- We need to build fermionic fields with a Dirac structure from the 16 single component fields at the vertices of a hypercube.
- The staggered action is

$$\mathcal{S}_F^{(stag.)} = \frac{1}{2} \sum_{n,\mu} \eta_\mu(n) \left[\bar{\chi}(n)\, \chi(n+\hat{\mu}) - \bar{\chi}(n)\, \chi(n-\hat{\mu}) \right] + \hat{M} \sum_n \bar{\chi}(n)\, \chi(n)$$

- The coordinates of each site can be rewritten as

$$n = 2N + \rho$$

Here N are the coordinates of the hypercube which the site n belongs to.

ρ is a vector with components either 0 or 1, so that we can select a vertex of the hypercube. From now on, all the vectors with components restricted to 0 or 1 will be called as a vector of type ρ.

Building the physical fields

- With the new notation, the action becomes

$$\mathcal{S}_F^{(stag.)} = \frac{1}{2} \sum_{N,\mu,\rho} \eta_\mu(\rho)\, \bar{\chi}(2N+\rho) \left[\chi(2N+\rho+\hat{\mu}) - \chi(2N+\rho-\hat{\mu}) \right] + \hat{M} \sum_{N,\rho} \bar{\chi}(2N+\rho)\, \chi(2N+\rho)$$

- We have used the fact

$$\eta_\mu(2N+\rho) = \eta_\mu(\rho)$$

- We can relabel the fields as

$$\chi_\rho(N) \equiv \chi(2N+\rho)$$

- N labels sites of a lattice with lattice spacing b=2a and ρ is still a vector with components restricted to 0 and 1.

Building the physical fields

- Let us consider the field $\chi(2N+\rho+\hat{\mu})$ that appears in the kinetic term.
- If ρ+μ is a vector of type ρ, then the lattice site 2N+ρ+μ still belongs to a hypercube whose origin is at site 2N and

$$\chi(2N+\rho+\hat{\mu}) \equiv \chi_{\rho+\hat{\mu}}(N)$$

- If ρ+μ is not a vector of type ρ, then ρ-μ is such a vector and

$$\chi(2N+\rho+\hat{\mu}) \equiv \chi_{\rho-\hat{\mu}}(N+\hat{\mu})$$

- The fields become

$$\chi(2N+\rho+\hat{\mu}) = \sum_{\rho'} \left[\delta_{\rho+\hat{\mu},\rho'}\, \chi_{\rho'}(N) + \delta_{\rho-\hat{\mu},\rho'}\, \chi_{\rho'}(N+\hat{\mu})\right]$$

$$\chi(2N+\rho-\hat{\mu}) = \sum_{\rho'} \left[\delta_{\rho-\hat{\mu},\rho'}\, \chi_{\rho'}(N) + \delta_{\rho+\hat{\mu},\rho'}\, \chi_{\rho'}(N-\hat{\mu})\right]$$

Building the physical fields

- Using this we get the staggered action as

$$\mathcal{S}_F^{(stag.)} = \sum_{N,\mu,\rho,\rho'} \frac{1}{2}\,\eta_\mu(\rho)\,\bar{\chi}_\rho(N)\left[\left(\delta_{\rho+\hat{\mu},\rho'} + \delta_{\rho-\hat{\mu},\rho'}\right)\hat{\partial}_\mu^S + \frac{1}{2}\left(\delta_{\rho-\hat{\mu},\rho'} - \delta_{\rho+\hat{\mu},\rho'}\right)\hat{\Box}_\mu\right]\chi_{\rho'}(N) + \hat{M}\sum_{N,\rho}\bar{\chi}_\rho(N)\,\chi_\rho(N)$$

$$= \frac{1}{2}\sum_{N,\rho,\rho'}\bar{\chi}_\rho(N)\left[\sum_\mu\left(\Gamma^\mu_{\rho\rho'}\,\hat{\partial}_\mu^S + \frac{1}{2}\,\Gamma^{5\mu}_{\rho\rho'}\,\hat{\Box}_\mu\right) + 2\,\hat{M}\,\delta_{\rho,\rho'}\right]\chi_\rho(N)$$

  where the symmetric derivative and four dimensional Laplacean operator are used

- The Γ-matrices are introduced to simplify the equation

$$\Gamma^\mu_{\rho\rho'} \equiv \eta_\mu(\rho)\left[\delta_{\rho-\hat{\mu},\rho'} + \delta_{\rho+\hat{\mu},\rho'}\right]$$

$$\Gamma^{5\mu}_{\rho\rho'} \equiv \eta_\mu(\rho)\left[\delta_{\rho-\hat{\mu},\rho'} - \delta_{\rho+\hat{\mu},\rho'}\right]$$

Building the physical fields

- The physical fields can be formed by the linear combination of the single component fields

$$\hat{\psi}_{\alpha\beta}(N) = \mathcal{N}_0 \sum_{\rho'} U_{\alpha\beta,\rho'}\, \chi_{\rho'}(N)$$

$$\hat{\bar{\psi}}_{\alpha\beta}(N) = \mathcal{N}_0 \sum_{\rho'} \bar{\chi}_{\rho'}(N)\, (U^\dagger)_{\rho',\alpha\beta}$$

where $U_{\alpha\beta,\rho} = \frac{1}{2}(T_\rho)_{\alpha\beta}$ with $T_\rho \equiv \gamma_1^{\rho_1}\gamma_2^{\rho_2}\gamma_3^{\rho_3}\gamma_4^{\rho_4}$

- There are 16 different ρ vectors and the indices α and β run from 1 to 4. So U is a 16x16 matrix and it's rows are identified by a double index.

- Inverting this and using it in the action, we get

$$\mathcal{S}_F^{(stag.)} = \frac{1}{2} \sum_{\substack{N,\rho,\rho' \\ \alpha,\beta,\alpha',\beta'}} \frac{1}{\mathcal{N}_0} \hat{\bar{\psi}}_{\alpha\beta}(N)\, U_{\alpha\beta,\rho} \cdot \left[\sum_{\mu} \left(\Gamma^{\mu}_{\rho\rho'}\, \hat{\partial}_\mu + \frac{1}{2} \Gamma^{5\mu}_{\rho,\rho'}\, \hat{\Box}_\mu \right) + 2\,\hat{M}\, \delta_{\rho,\rho'} \right] \cdot \frac{1}{\mathcal{N}_0} (U^\dagger)_{\rho',\alpha'\beta'} \hat{\psi}_{\alpha'\beta'}(N)$$

Building the physical fields

- With some algebra, the action becomes

$$\mathcal{S}_F^{(stag.)} = \frac{1}{2\mathcal{N}_0^2} \sum_{\substack{N,\alpha,\beta \\ \alpha',\beta'}} \hat{\bar{\psi}}_{\alpha\beta}(N) \left[\sum_{\mu} (\gamma_\mu)_{\alpha\alpha'} \, \delta_{\beta,\beta'} \, \hat{\partial}_\mu + \frac{1}{2} (\gamma_5)_{\alpha\alpha'} \, (\gamma_\mu^* \gamma_5)_{\beta\beta'} \, \hat{\Box}_\mu + 2 \, \hat{M} \, \delta_{\alpha,\alpha'} \, \delta_{\beta,\beta'} \right] \hat{\psi}_{\alpha'\beta'}(N)$$

- In the continuum limit, the Laplacean operator term (second term) vanishes and we can compare the remaining terms with the staggered action of 4 degenerate fermions on the lattice.

$$\mathcal{S}_F^{(stag.)} = \sum_f \sum_N \hat{\bar{\psi}}^f(N)(\gamma_\mu \hat{\partial}_\mu + \hat{m}) \hat{\psi}^f(N)$$

- Then we can see that α and β should be identified with the Dirac and flavor quark-degrees of freedom respectively.
- It does not correspond to the flavors of the quark but is called the taste degree of freedom. So a single staggered fermion corresponds to 4 different tastes of the continuum fermion.

Spin ⊗ taste basis

- We have seen that α and β of the physical fields can be identified with the Dirac and taste degrees of freedom respectively.
- We can think of a spin ⊗ taste basis for the staggered fermions.
- We have seen that the action was

$$\mathcal{S}_F^{(stag.)} = \frac{1}{2\mathcal{N}_0^2} \sum_{\substack{N,\alpha,\beta \\ \alpha',\beta'}} \hat{\bar{\psi}}_{\alpha\beta}(N) \left[\sum_\mu (\gamma_\mu)_{\alpha\alpha'} \, \delta_{\beta,\beta'} \, \hat{\partial}_\mu + \frac{1}{2} (\gamma_5)_{\alpha\alpha'} \, (\gamma_\mu^* \gamma_5)_{\beta\beta'} \, \hat{\Box}_\mu + 2 \hat{M} \, \delta_{\alpha,\alpha'} \, \delta_{\beta,\beta'} \right] \hat{\psi}_{\alpha'\beta'}(N)$$

- This can be written in the spin ⊗ taste basis as

$$2\mathcal{N}_0^2 \cdot \mathcal{S}_F^{(stag.)} = \sum_{N,\mu} b^4 \bar{\psi}(Nb) \left[(\gamma_\mu \otimes \mathbb{1}) \, \partial_\mu + \frac{1}{2} b \, (\gamma_5 \otimes \gamma_\mu^* \gamma_5) \, \Box_\mu \right] \psi(Nb) + \sum_N b^4 \, \bar{\psi}(Nb) \left[2 \, M \, (\mathbb{1} \otimes \mathbb{1}) \right] \psi(Nb)$$

- The physical units are used and b=2a.

Propagator in the continuum limit

- In the staggered formulation there are no terms in the quark propagator without an analogue in the continuum.
- The quark propagator becomes

$$\mathcal{J}^{-1}(p) = \frac{\sum_\mu \left[-\imath(\gamma_\mu \otimes \mathbb{1}) \frac{1}{b} \sin(p_\mu b) + \frac{2}{b} (\gamma_5 \otimes \gamma_\mu^* \gamma_5) \sin^2\left(\frac{p_\mu b}{2}\right)\right] + M_0 \cdot (\mathbb{1} \otimes \mathbb{1})}{\sum_\mu \frac{4}{b^2} \sin^2\left(\frac{p_\mu b}{2}\right) + M_0^2}$$

- This shows that there are no contributions from the edges of the Brillouin zone due to the presence of the ½ factor in the sine term of the denominator.
- In the continuum limit, we have

$$\lim_{b\to 0} \mathcal{J}^{-1}(p) = \frac{-\imath \sum_\mu (\gamma_\mu \otimes \mathbb{1}) \cdot p_\mu + M_0 \cdot (\mathbb{1} \otimes \mathbb{1})}{p^2 + M_0^2}$$

- This is the propagator of a continuum system composed of four degenerate fermionic particles.

Symmetry of the staggered fermions

- The U(4) X U(4) symmetry (present in the continuum for M=0) is explicitly broken in the staggered formulation.

$$\mathcal{S}_F^{(stag.)} = \sum_N b^4\, \bar{\psi}(Nb) \sum_\mu \left[(\gamma_\mu \otimes \mathbb{1})\, \partial_\mu + \frac{b}{2} (\gamma_5 \otimes \gamma_\mu^* \gamma_5)\, \Box_\mu \right] \psi(Nb)$$

- A part of chiral symmetry survives whose generator is $\gamma_5 \otimes \gamma_5$
- Under the action of this subgroup, the physical fields transform as

$$\psi(Nb) \to \psi'(Nb) = e^{\imath\theta(\gamma_5 \otimes \gamma_5)}\, \psi(Nb)$$
$$\bar{\psi}(Nb) \to \bar{\psi}'(Nb) = \bar{\psi}(Nb)\, e^{\imath\theta(\gamma_5 \otimes \gamma_5)}$$

where θ is a real parameter not depending on N

- It can be shown that the action is invariant under this transformation

Symmetry of the staggered fermions

- Let us see how the remnant chiral symmetry generator acts on the one component staggered fields.
- The U matrices are the connection between the one-component fields and the physical fields. So we can check the term

$$\left(U^{\dagger}.(\gamma_5 \otimes \gamma_5).U\right)_{\rho\rho'} = \sum_{\substack{\alpha,\alpha' \\ f,f'}} (U^{\dagger})_{\rho,\alpha f} (\gamma_5)_{\alpha\alpha'} (\gamma_5)_{ff'} U_{\alpha' f',\rho'}$$
$$= \frac{1}{4} \sum_{\substack{\alpha,\alpha' \\ f,f'}} \left[\gamma_1^{\rho_1} \gamma_2^{\rho_2} \gamma_3^{\rho_3} \gamma_4^{\rho_4}\right]^*_{\alpha f} (\gamma_5 \otimes \gamma_5)_{\alpha f,\alpha' f'} \left[\gamma_1^{\rho'_1} \gamma_2^{\rho'_2} \gamma_3^{\rho'_3} \gamma_4^{\rho'_4}\right]_{\alpha' f'}$$
$$= (-1)^{\rho 1+\rho 2+\rho 3+\rho 4} \delta_{\rho,\rho'} \equiv \Gamma^{55}_{\rho\rho'}$$

- To get a convenient expression we recall n=2N+ρ which implies

$$(-1)^{\rho 1+\rho 2+\rho 3+\rho 4} = (-1)^{n1+n2+n3+n4}$$

- We can conclude that $\Gamma^{55}(n) = (-1)^{n1+n2+n3+n4} \mathbb{1}_{16\times 16}$

Symmetry of the staggered fermions

- The axial transformation of the one-component fields is

$$\chi(n) \longrightarrow \mathrm{e}^{\mathrm{i}\alpha\eta_5(n)}\,\chi(n) \qquad \overline{\chi}(n) \longrightarrow \overline{\chi}(n)\,\mathrm{e}^{\mathrm{i}\alpha\eta_5(n)}$$

  where $\eta_5(n) = (-1)^{n_1+n_2+n_3+n_4}$

- The symmetry group of the staggered fermions is $U(1)_V$ X $U(1)_\varepsilon$
- The axial symmetry $U(1)_\varepsilon$ is not identical to the $U(1)_A$ axial symmetry due to the extra $\eta_5(n)$ term. It is a subgroup of $SU(4)_A$.
- Hence, $U(1)_\varepsilon$ is denoted as the residual chiral symmetry.
- The $U(1)_V$ still corresponds to the baryon number conservation.

Conclusion

- The single component staggered fermions are formed by the diagonalization of the action and keeping only one component of the fermions per site.
- The physical staggered fermion fields are built from the single component ones. The physical fields corresponds to four tastes of the fermions each with a Dirac structure.
- There are no lattice artifacts in the propagator of the staggered fermions but it is similar to the propagator of four degenerate fermions.
- The staggered action can be expressed in a spin $\otimes$ taste basis.
- The U(4) X U(4) symmetry is broken in the staggered formulation. However, a residual chiral symmetry survives.

Thank You