Overlap fermions: An Introduction

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Overview

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General context

- The $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ chiral symmetry is of central importance for the study of the non-perturbative low-energy dynamics of quarks.
- Yet, the implementation of chiral symmetry on the lattice is a nontrivial issue, no less because of a famous no-go-theorem by Nielsen and Ninomiya. For Wilson fermions this requires delicate fine-tuning of the bare fermion mass.
- We'll see there exists a way-out of this dilemma by introducing a lattice version of chiral symmetry.

Wilson's Formulation of lattice QCD

- The Naive fermion action is given by

$$S_F[\psi, \overline{\psi}, U] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a} + m\psi(n) \right)$$

- Wilson's complete Dirac operator reads

$$S_F[\psi, \overline{\psi}, U] = \sum_{f=1}^{N_f} a^4 \sum_{n,m\in\Lambda} \overline{\psi}^{(f)}(n) D^{(f)}(n|m) \psi^{(f)}(m)$$

with

$$D^{(f)}(n|m)_{\alpha,\beta,a,b} = \left(m^{(f)} + \frac{4}{a} \right) \delta_{\alpha\beta}\delta_{ab}\delta_{nm} - \frac{1}{2a} \sum_{\mu=\pm1}^{\pm4} \left(\mathbb{1} - \gamma_\mu \right)_{\alpha\beta} U_\mu(n)_{ab} \delta_{n+\hat{\mu},m}.$$

- Note: D is γ_5-hermitian: $D^\dagger = \gamma_5 D \gamma_5$.
  (Exception for non-vanishing chemical potential, θ-term ...)

Excursion: Chiral symmetry on the Lattice

- Reminder: Chiral symmetry in the continuum case

$$S_F[\psi, \overline{\psi}, A] = \int \mathrm{d}^4 x \mathcal{L}(\psi, \overline{\psi}, A), \quad \mathcal{L}(\psi, \overline{\psi}, A) = \overline{\psi} D \psi.$$

- $\mathcal{L}$ is invariant under a chiral rotation

$$\psi \longrightarrow \psi' = e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi} e^{i\alpha\gamma_5} \qquad (*)$$

- Chiral symmetry acts differently for left-handed and the right-handed components

$$P_{R/L} = \frac{\mathbb{1} \pm \gamma_5}{2} \Longrightarrow \psi_{R/L} \equiv P_{R/L}\psi, \ \gamma_5 \psi_{R/L} = \pm \psi_{R/L}.$$

$$\Longrightarrow \mathcal{L}(\psi, \overline{\psi}, A) = \overline{\psi}_L D \psi_L + \overline{\psi}_R D \psi_R, \quad (*) \ \psi_{R/L} \longrightarrow \psi_{R/L} e^{\pm i\alpha}$$

Excursion: Chiral symmetry on the Lattice

- A mass term $m\overline{\psi}\psi = m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R)$ breaks the symmetry.
- Chiral symmetry can be summarized by the condition

$$\gamma_5 D + D\gamma_5 = \{\gamma_5, D\} = 0. \quad (Note : \partial_\mu j_5^\mu(x) = i\overline{\psi}\{\gamma^5, D\}\psi).$$

- On the Lattice: The Wilson term breaks chiral symmetry explicitly even for massless quarks.
  - Difficult to explore the regime of small quark masses in numerical simulations and especially to study spontaneous chiral symmetry breaking.
  - Hard to test the Witten-Veneziano mass formula, relating the eta-prime mass to the topological susceptibility.

Excursion: Chiral symmetry on the lattice

- Question: Can we remove the doublers in a different way without breaking chiral symmetry?

Theorem (Nielsen-Ninomiya, 1981)

For a lattice Dirac operator D in a 2d-dimensional QFT at least one of the statements 1) D is local, 2) D is doubler free, 3) D respects chiral symmetry is broken.

Ginsparg-Wilson equation

- Ginsparg and Wilson (1982) proposed to replace the continuum condition for chiral symmetry with

$$\gamma_5 D + D\gamma_5 = \{\gamma_5, D\} = aD\gamma_5 D. \quad (**)$$

- Extra term constitutes a contact term

$$\gamma_5 D^{-1}(n|m) + D^{-1}(n|m)\gamma_5 = a\gamma_5\delta(n-m).$$

- Observe that every operator of the form $D = \frac{1}{a}(1 - V)$ with $V^\dagger V = 1$ and $V^\dagger = \gamma_5 V \gamma_5$ solves (**).

The Overlap Dirac Operator

- Particular solution to the Ginsparg-Wilson equation.

Definition (Neuberger & Narayanan)

$$D_{ov} = \frac{1}{a}(\mathbb{1} - \underbrace{A\,(A^\dagger A)^{-1/2}}_{\equiv\ \mathrm{sign}(A^\dagger A)\ =\ \mathrm{sign}[\sum_i \lambda_i |i\rangle\langle i|]\ =\ \sum_i \mathrm{sign}(\lambda_i)|i\rangle\langle i|})\,,\ \gamma_5 A \gamma_5 = A^\dagger.$$

- Explicit choice $A = \mathbb{1} + s - aD_W$, $|s| < 1$.
- D_{ov} is not ultra-local! (Horváth, 1998)
- A natural generalization of locality is

$$|D(n|m)| \leq C \exp\left(-\gamma ||n - m||\right).$$

- Established for the Overlap operator in 1999 by *Hernández*, *Jansen* and *Lüscher* (arXiv:hep-lat/9808010).

The Overlap Dirac Operator

- A fermionic action involving a Dirac operator solving (**) possesses an exact chiral symmetry, differing from the continuum symmetry by $\mathcal{O}(a)$ artefacts:

$$\psi \longrightarrow \psi' = e^{i\alpha\gamma_5\left(1-\frac{a}{2}D\right)}\psi, \quad \overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi}e^{i\alpha\gamma_5\left(1-\frac{a}{2}D\right)} \qquad (*')$$

$\Longrightarrow$ Exact symmetry of the action for m $=0$!

Spectrum of the Overlap Operator

- $Dv_\lambda = \lambda v_\lambda \Longleftrightarrow P(\lambda) = \det\left[D - \lambda \mathbb{1}\right].$
- Assume γ_5-hermiticity: $\gamma_5 D \gamma_5 = D^\dagger \Longrightarrow P(\lambda) = P(\lambda^*)^*$.
- Only eigenvectors v_λ with real eigenvalue λ can have non-vanishing chirality

$$\lambda(v_\lambda, \gamma_5 v_\lambda) = \lambda^*(v_\lambda, \gamma_5 v_\lambda) \Longrightarrow \mathrm{Im}(\lambda)(v_\lambda, \gamma_5 v_\lambda) = 0.$$

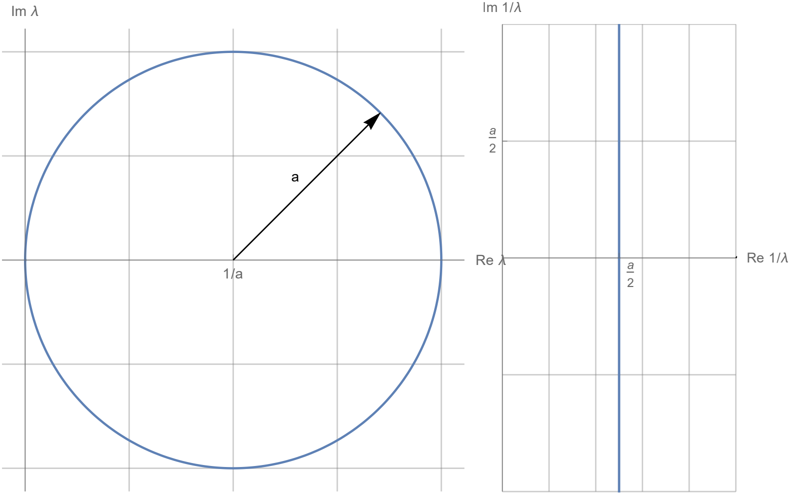
- If in addition D fulfills the Ginsparg-Wilson equation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \Longrightarrow D^\dagger + D = aD^\dagger D,\ D + D^\dagger = aDD^\dagger.$$

- The eigenvalues of D_{ov} are restricted to a circle in the complex plane

$$\lambda^* + \lambda = a\lambda^*\lambda \overset{\lambda = x + \mathrm{i}y}{\Longleftrightarrow} \left(x - \frac{1}{a}\right)^2 + y^2 = \frac{1}{a^2}.$$

Spectrum of the Overlap Operator



Spectrum of the Overlap Operator

- A useful parametrization is $\lambda = \frac{1}{a}\left(1 - e^{i\varphi}\right), \varphi \in (-\pi, \pi]$.
- The eigenvalues of the quark propagator D^{-1} lie on the vertical $x = \frac{1}{\lambda}$ with

$$\frac{1}{\lambda} = \frac{a}{2} + \mathrm{i}\frac{a}{2}\,\frac{\sin\left(\varphi\right)}{1 - \cos\left(\varphi\right)}.$$

- For the potential zero modes v_0 one finds $\gamma_5 D v_0 = D\gamma_5 v_0$
  $\implies$ Zero modes can be chosen as chiral γ_5: $\gamma_5 v_0 = \pm v_0$.
- Similarly eigenmodes with real eigenvalue $2/a$ are chiral (doubler partners of the zero-modes), but decouple in the continuum limit!

Numerical evaluation

- Many methods are based on polynomial approximations of the inverse square root $\left(A^\dagger A\right)^{-1/2}$ (mostly Legendre and Chebyshev).
- Rational approximations of $1/\sqrt{x^2}$.

$$\text{Zolotarev:}\quad \frac{1}{\sqrt{x^2}} \approx d \prod_{n=1}^{m} \frac{x^2 + c_{2n}}{x^2 + c_{2n-1}}, \quad c_n = \frac{\mathrm{sn}^2(nK(k')/(2m+1); k')}{1 - \mathrm{sn}^2(nK(k')/(2m+1); k')}$$

with $k' = \sqrt{1 - \alpha/\beta}$ and $x \in \left[1, \beta^2/\alpha^2\right]$.

$$\Longrightarrow \quad \mathrm{sign}\left[H\right] \approx d \frac{H}{|\alpha|} \left(\frac{H^2}{\alpha^2 + c_{2m}}\right) \sum_{n=1}^{m} b_n \left(\frac{H^2}{\alpha^2 + c_{2n-1}}\right)^{-1}.$$

Thanks for listening!

References I

Pilar Hernández, Karl Jansen, and Martin Lüscher. *Locality properties of Neuberger's lattice Dirac operator*. Vol. 98-250. CERN-TH. Hamburg: DESY, 1998.

Thomas DeGrand and Urs M. Heller. "Witten-Veneziano relation, quenched QCD, and overlap fermions". In: *Physical Review D* 65.11 (2002), p. 269. ISSN: 0556-2821. DOI: \url{10.1103/PhysRevD.65.114501}.

S. J. Dong et al. "Chiral properties of pseudoscalar mesons on a quenched 204 lattice with overlap fermions". In: *Physical Review D* 65.5 (2002), p. 67. ISSN: 0556-2821. DOI: \url{10.1103/PhysRevD.65.054507}.

References II

Heinz J. Rothe. *Lattice gauge theories: An introduction*. 3. ed. Vol. 74. World Scientific lectures in physics. Hackensack, NJ: World Scientific, 2005. ISBN: 9812560629.

Christof Gattringer and Christian B. Lang. *Quantum chromodynamics on the lattice: An introductory presentation*. Vol. 788. Lecture notes in physics. Berlin: Springer, 2010. ISBN: 978-3-642-01850-3.

Backup: Overlap operator and DWF

- Kaplan proposed a novel method to preserve chirality on the lattice in the early 90's by introducing a Wilson-Dirac operator in five dimensions

$$S_F\left[\overline{\psi},\psi,U\right]=a^4a_5\sum_{x,x_5,y,y_5}\overline{\psi}_{x,x_5}D_{DW}[U]_{x,x_5,y,y_5}\psi_{x,x_5}.$$

- Mass term changes sign and creates a "Domain-wall" at the points where it vanishes.
- Neuberger found an analytical formula for an effective Dirac operator that describes the massless chiral mode of the DWF for which one finds

$$\lim_{L_5\to\infty}D_{L_5}[U]=\frac{1}{2a}\left[\mathbb{1}+\gamma_5\frac{\tilde{H[U]}}{\sqrt{\tilde{H}[U]^2}}\right]\stackrel{a_5\to 0}{\longrightarrow}D_{ov}[U].$$

with $\tilde{H}[U]=\gamma_5\frac{a_5D^{||}[U]}{2+a_5D^{||}[U]}$. More details: arXiv:hep-lat/0405024v1.