UNIVERSITÄT BIELEFELD

Domain Wall Fermions

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OUTLINE

Definition

Domain wall fermions consist in a lattice theory of massive interacting fermions in $2n+1$ dimensions to simulate the behavior of massless chiral fermions in $2n$ dimensions if the fermion mass has a step function shape in the extra dimension. [1]

Motivation

CHIRAL SYMMETRY

Chiral symmetry

Chiral symmetry is an important concept in QCD

$$S_E = \int d^4x \, \overline{\Psi} D \Psi + S_g \qquad D = \gamma_\mu \partial_\mu + ig\gamma^\mu A_\mu + m$$

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$$m\bar{\Psi}\Psi = m\bar{\Psi}_R \Psi_R + m\bar{\Psi}_L \Psi_L + m(\bar{\Psi}_R \Psi_L + \bar{\Psi}_L \Psi_R)$$

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The theory respects chiral symmetry if:

$$\{\gamma^5, D\} = 0$$

$$\{\gamma^5, D\} = \{\gamma^5, \gamma^\mu\}\partial_\mu + \{\gamma^5, m\mathbb{1}\} = 2m\gamma^5$$

m has to be zero.

A NO-GO THEOREM

According to the Nilsen-Ninomiya theorem:

In a translationally invariant lattice theory of fermions in even dimensions, at least one of the following properties is violated:

- D is local;
- D is invertible everywhere except at $p = 0$;
- The theory respects chiral symmetry.

The Nilsen-Ninomiya theorem is a necessary consequence of anomalies.

Some available techniques

Examples of methods to handle the doubling problem in vector theories:

- (1975) Wilson fermions;
- (1975) Kogut-Susskind fermions;
- (1992) Domain wall fermions;
- (1992) Infinitely many fields;
- (1992) Overlap fermions;
- (1997) Neuberger fermions;
- (1997) Perfect action fermions;
- (1982) Ginsparg-Wilson fermions;
- (1999) Molecule chains;
- (2000) Topological QFT in 5 dimensions.

Wilson fermions

$$\widetilde{D}(\mathbf{p}) = m\mathbb{1} + \frac{i}{a}\sum_{\mu}\gamma^{\mu}\sin(p_{\mu}a) + \mathbb{1}\frac{1}{a}\sum_{\mu}\left[1-\cos(p_{\mu}a)\right]$$

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$$\delta\Psi = \gamma^5\left(\mathbb{1} - \frac{1}{2}aD\right)\Psi \qquad \delta\overline{\Psi} = \overline{\Psi}\left(\mathbb{1} - \frac{1}{2}aD\right)\gamma^5$$

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The overlap operator was one of the solutions found.

$$D = \frac{\mathbb{1} + \epsilon(D_w)}{2} \qquad \epsilon(D_w) = \frac{D_w}{\sqrt{D_w^\dagger D_w}}$$

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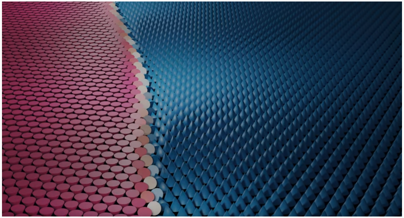
What else can we do to achieve chiral symmetry?

Introduction to domain wall fermions

DOMAIN WALLS IN PHYSICS

Domain walls in physics

Domain walls are topological defects that appear when a discrete symmetry is spontaneously broken.



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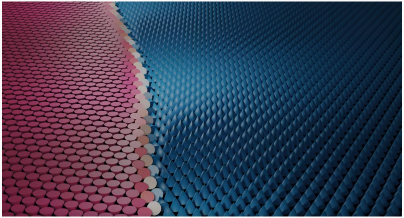
- Optics
- String theory



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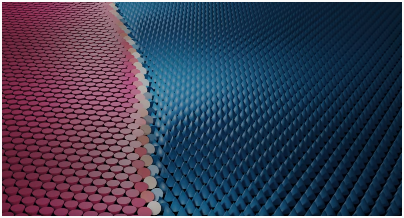

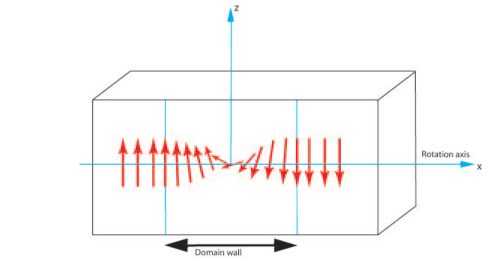

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Domain wall fermions

The 4D action in the free case is

$$S_E = \sum_x \overline{\Psi}_x D \Psi_x \qquad D = \gamma^\mu \partial_\mu + m$$

in Kaplan's formulation [1], the DWF action is

$$S_E = \sum_x \sum_s \overline{\Psi}_{x,s} D \Psi_{x,s} \qquad D = \gamma^\mu \partial_\mu + \gamma^5 \partial_s + m(s)$$

The mass dependence has to be such that $\lim_{s \to \pm\infty} m(s) = \pm m$.

$$m(s) = \begin{cases} -m & s < 0 \\ 0 & s = 0 \\ +m & s > 0 \end{cases} \qquad m(s) = m \tanh(s)$$

This corresponds to a mass defect in the extra dimension (domain wall!)

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- In the limit above, chiral symmetry is exact even for non-zero a;
- When $L_5 \to \infty$, domain wall fermions turn into overlap fermions;
- Kaplan's formulation of DWF restores the full SU(N)$_L \times$SU(N)$_R$ in the continuum limit.

Domain wall fermions on the lattice

We still have doublers in the theory, so we can add a Wilson term:

$$S = -\sum_{x,y,s,r} \overline{\Psi}(D_{x,y}\delta_{s,r} + D_{s,r}\delta_{x,y})\Psi \qquad x,y \in \Lambda \quad s,r \in \Lambda_5$$

$$D_{x,y} = \frac{1}{2}\sum_{\mu}[(1+\gamma^{\mu})U_{x,\mu}\delta_{x+\hat{\mu},y} + (1-\gamma^{\mu})U^{\dagger}_{y,\mu}\delta_{x-\hat{\mu},y}] + (M-4)\delta_{x,y}$$

$$D_{s,r} = \begin{cases} \frac{(1+\gamma^5)}{2}\delta_{1,r} - m\frac{(1-\gamma^5)}{2}\delta_{N_5-1,r} - \delta_{0,r}, & s=0, \\ \frac{(1+\gamma^5)}{2}\delta_{s+1,r} + \frac{(1-\gamma^5)}{2}\delta_{s-1,r} - \delta_{s,r}, & 1 \le s \le N_5-2, \\ -m\frac{(1+\gamma^5)}{2}\delta_{0,r} + \frac{(1-\gamma^5)}{2}\delta_{N_5-2,r} - \delta_{N_5-1,r}, & s = N_5-1 \end{cases}$$

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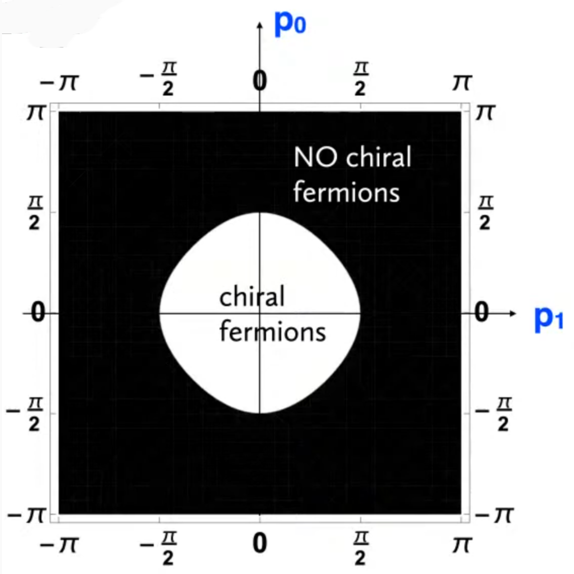

Lattice simulations

SIMULATIONS

We need to have the chiral modes decouple from the walls.

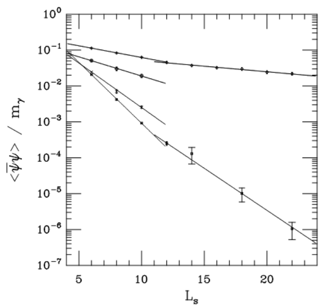

Figure 1: Chiral condensate in units of the photon mass m_γ in the Schwinger model ((1+1)-dimensional QED) as a function of L_5 [3].

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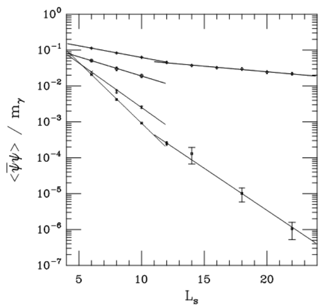

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The chiral modes decouple from the walls exponentially fast $\sim e^{-\alpha L_5}$.

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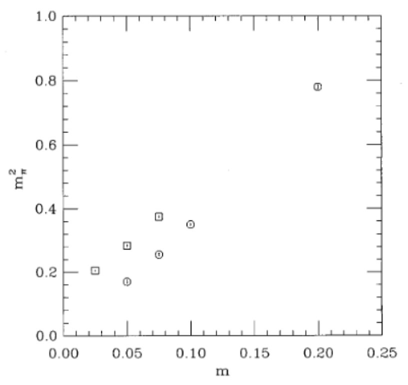

Figure 2: Pion mass squared as a function of m for $N_5 = 4$ and $N_5 = 10$.

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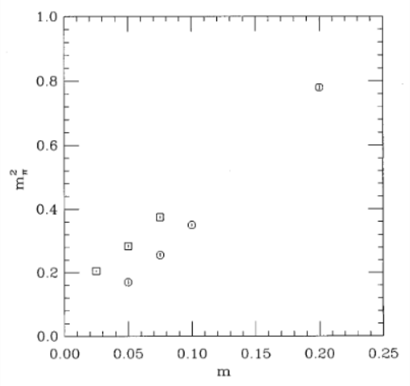

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From χPT $m_\pi^2 \propto \sqrt{m}$ or m. In the plot $m_\pi^2 = 0.0002 \pm 0.0160$.

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- Even for a moderate size of L_5, chiral symmetry is preserved to a high degree of accuracy;
- In the limit of $L_5 \to \infty$, the domain wall operator becomes the overlap one;
- Domain wall fermions is also applicable in condensed matter physics.

Bibliography I

References

[1] David B Kaplan. “A method for simulating chiral fermions on the lattice”. In: *Physics Letters B* 288.3-4 (1992), pp. 342–347.

[2] Martin Lüscher. “Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation”. In: *Physics Letters B* 428.3-4 (1998), pp. 342–345.

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