Intorduction to Domain Wall Fermions

Derivation of Kaplans Fermions

Marcel Rodekamp
Faculty of Physics
University
Bielefeld
Germany

Abstract *Domain Wall fermions pave the way to simulate chiral fermions in Lattice QCD. By adding an artifitial extra dimension to 4-dimensional QCD, and introducing a space dependent mass, massless chiral zeromodes are found, localized at topological defects. These topological defects arise from points at which the mass changes its sign depending on the position in the artifitial* 5^{th} *dimension.*

e-mail: mrodekamp@physik.uni-bielefeld.de

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1 Chiral Symmetry

Chiral symmetry is an important symmetry when it comes to the low energy sector of QCD. In general parts of the chiral symmetry are broken by scalar density terms in the action, such as mass term. However, real world QCD is expected to govern chiral symmetry at least partially in the low energy sector. This comes from the fact that the quark masses of the lightest quarks, up, down and even strange quarks, are comparably small. Hence, symmetry breaking arising from mass terms are negligible small. Furthermore, a quantized theory breaks one part of the chiral symmetry, the axial symmetry, which leads to a anomalous broken symmetry explaining a various range of effects such as the magnitude of the proton mass.

1.1 Chiral Symmetry for Single Flavour free QCD

To understand the notion of chiral symmetry consider the single flavour, free QCD Lagrangian

$$\mathcal{L}\left[\bar{\psi},\psi\right] = \bar{\psi}\left[\gamma^{\mu}\partial_{\mu} + m\right]\psi. \tag{1.1}$$

Note the occurrence of the gamma matrices being defined by the Euclidean Clifford algebra

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2\delta^{\mu\nu},\ \forall \mu,\nu \in [3]. \tag{1.2}$$

The product of these form the Euclidean chiral matrix

$$\gamma^5 = i\prod_{\mu=0}^{3}\gamma^{\mu}. \tag{1.3}$$

The Lagrangian in (1.1), obeys a global $U(1)$ symmetry i.e. $\mathcal{L}\left[\bar{\psi}U^{\dagger},U\psi\right] = \mathcal{L}\left[\bar{\psi},\psi\right],\ \forall U \in U(1)$. To extend this further, the notion of left and right handed chiral fermions can be introduced by defining the chiral projector

$$P_{\pm} = \frac{1}{2}\left(\mathbb{1} \pm \gamma^5\right), \tag{1.4}$$

and with that the left and right handed spinors

$$\begin{array}{cc} \text{Right handed spinor} & \text{Left handed spinor} \\ \psi_+ = P_+\psi & \psi_- = P_-\psi \end{array} \tag{1.5}$$

Rewriting the Lagrangian in terms of these, shows that the kinetic terms decouples the two handiness parts while scalar densities, such as the mass term, mixes them

$$\mathcal{L}\left[\bar{\psi},\psi\right] = \bar{\psi}_+\gamma^\mu\partial_\mu\psi_+ + \bar{\psi}_-\gamma^\mu\partial_\mu\psi_- + m\left(\bar{\psi}_-\psi_+ + \bar{\psi}_+\psi_-\right). \tag{1.6}$$

Thus, if the quark mass vanishes, $m = 0$, the symmetry group has been enlarged such that (1.6) governs a $U(1)_+ \times U(1)_-$ symmetry.

1.2 Chiral Symmetry for N_f Flavour Free QCD

Now extend the Lagrangian in (1.1) to N_f flavours

$$\mathcal{L}\left[\bar{\psi},\psi\right] = \sum_{f=1}^{N_f} \bar{\psi}^f \left[\gamma^\mu\partial_\mu + m_f\right]\psi^f. \tag{1.7}$$

This now governs a $U(N_f)$ symmetry instead of a $U(1)$. Splitting the Lagrangian (1.7) into left and right handed spinors works similar to the single flavour case. And the extended symmetry group, for $m_f = 0$, becomes $U(N_f)_+ \times U(N_f)_-$.

Considering the isomorphism

$$\varphi : SU(N_f) \times U(1) \to U(N_f), (S, e^{i\alpha}) \mapsto S \cdot e^{i\alpha}$$

it can be shown that

$$U(N_f) \simeq SU(N_f) \times U(1). \tag{1.8}$$

Hence the full chiral symmetry group for mass less quarks is given by

$$\boxed{SU(N_f)_+ \times SU(N_f)_- \times U(1)_+ \times U(1)_-.} \tag{1.9}$$

However, that is not the full story yet. By considering the infinitesimal representation

$$\begin{aligned} SU(N_f)_\pm : \quad & e^{i\delta\alpha_{\pm,j}\tau^j_\pm} = 1 + i\delta\alpha_{\pm,j}\tau^j_\pm \\ U(1)_\pm : \quad & e^{i\delta\alpha_\pm} = 1 + i\delta\alpha_\pm, \end{aligned} \tag{1.10}$$

and adding and subtracting the transformation of left and right handed spinors shows that

$$\begin{aligned} SU(N_f)_+ \times SU(N_f)_- &\tilde{=} SU(N_f)_V \times SU(N_f)_A \\ U(1)_+ \times U(1)_- &\tilde{=} U(1)_V \times U(1)_A \end{aligned} \tag{1.11}$$

where the vector (V) and axial (A) symmetries can be defined by the representation

$$\begin{aligned} SU(N_f)_V : \quad & e^{i\alpha_j \tau^j} \quad \wedge \quad U(1)_V : \quad e^{i\alpha} \\ SU(N_f)_A : \quad & e^{i\alpha_j \gamma^5 \tau^j} \quad \wedge \quad U(1)_A : \quad e^{i\alpha\gamma^5} \end{aligned} \tag{1.12}$$

Hence the final form of the chiral symmetry is given by

$$\boxed{SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A.} \tag{1.13}$$

This symmetry group is also correct in the case of an interacting QCD since it acts on the spinors only and not in the gauge group. Even though, enforcing (1.7) to be gauge invariant the term

$$i\bar{\psi}\gamma^\mu A_\mu(x)\psi$$

has to be considered. Fortunately, it is invariant under vector and axial transformation as well. More general, any term containing an odd number of Dirac matrices will be invariant under chiral transformations.

However, quantizing QCD breaks the axial symmetry because the path integral measure, or fermion determinant, is not invariant, leaving the total chiral symmetry

$$SU(N_f)_R \times SU(N_f)_L \times U_V(1) \tag{1.14}$$

Note that also $SU(N_f)_A$ could be broken but there is no exhausting proof. Anyway it appears to be a useful modulation for QCDs low energy interactions. For example the three pions $\pi^\pm$ and π^0 would appear as Goldstone bosons of a broken $SU(2)_A$.

Introducing a degenerate mass which is non vanishing, $m_f = m \neq 0,\ \forall f \in [N_f]$, breaks the symmetry even further. It is important to note that the vector symmetry holds in this case while the axial symmetry is broken completely.[1] Thus the total chiral symmetry group is given by

$$SU(N_f)_V \times U(1)_V \tag{1.15}$$

If further the degeneracy is removed such that $m_f \neq m_{f'} \neq 0, \forall f, f' \in [N_f]$, the $SU(N_f)_V$ symmetry is broken into $U(1)_V$ for any flavour, leaving

$$\bigtimes_{f=1}^{N_f} U(1)_V \tag{1.16}$$

It is left to discuss the classical divergences of the axial symmetry. A focus lays on the $U(1)_A$ because it is broken by the fermion determinant and thus will be the symmetry of interest in the Nielsen Ninomiya No-Go theorem. The variation of the action, with a fermion matrix M, under the axial symmetry can be found to be

$$(\delta_A \bar{\psi}) \{\gamma^5, M\} \psi. \tag{1.17}$$

Which implies that the classical divergence is given by

$$\partial_\mu j^\mu(x) = i\bar{\psi}\{\gamma^5, M\}\psi, \tag{1.18}$$

such that a sufficient condition for the $U(1)_A$ symmetry to hold is given by

$$\boxed{\{\gamma^5, M\} = 0.} \tag{1.19}$$

[1] This can be seen by inserting the representation (1.12) into the Lagrangian.

2 Derivation of Domain Wall Fermions by Kaplan

In the following the derivation of Kaplans' Domain Wall Fermions (DWF) is reproduced. This section orientates at Kaplans lecture notes [2] supplemented with some details from one of his first papers on the matter [1].

The basic idea of DWF is to introduce an artificial dimension. On top of that a mass term is defined depending on this extra dimension. If the mass changes it's sign somewhere a topological defect arises at which chiral zeromodes are localized.

This enables an approximation to the overlap fermion which has been constructed as a solution to the Ginsparg-Wilson relation (GWR). Thus containing somewhat chiral properties. In fact, for simulations a finite extend of the 5^{th} dimension, L_s, has to be considered and in the continuum limit of the 5^{th} dimension the exact chiral symmetry in the sense of GWR is recovered.

In the following only the construction of these DWF is stated, there is unfortunately not enough time to cover the matter of approximating the GWR and further implications of the DWF.

2.1 Nilsen Ninomiya Theorem

The Nielsen Ninomiya No-Go theorem states that, for a lattice Dirac Operator (Fermion Matrix) $M(x|y)$ in an even dimensional quantum field theory at least one property,

1. M is local
2. M is doubler free
3. M transforms under chiral and gauge symmetry,

is broken. Property 1. can be expressed in two ways, ultralocality and locality. Ultralocality is defined by having only nearest neighbour interactions on the lattice. It is quite a strong restriction but ensures that in the continuum the correct locality is recovered. Fortunately, this can be mitigated while still keeping the correct continuum locality by enforcing

$$||M(x,y)|| \leq Ce^{-\gamma|x-y|} \tag{2.1}$$

for a constant $C \in \mathbb{R}^+$ and inverse interaction range $\gamma \in \mathbb{R}^+$ in lattice units.

The crucial point is to find chiral symmetry while keeping gauge invariance in place. As seen in the previous chapter chiral symmetry is ensured if the fermion matrix anti-commutes with γ^5. Ginsparg and Wilson have brought this property onto the lattice.

$$\left\{\gamma^5, M(x|y)\right\} = aM(x|y)\gamma^5 M(x|y) \tag{2.2}$$

This defines an exact chiral symmetry on the lattice! As the RHS of (2.2) is of $\mathcal{O}(a)$ the continuum chiral symmetry is restored in the limit $a \to 0$. This seems to be a nice loop hole when it comes to the Nilsen Ninomiya no go theorem because there is one less property to worry about. Unfortunately, it has been proven that solutions to (2.2) are non local in the sense of ultralocality. It is an ongoing investigation whether locality in the sense of (2.1) is fulfilled, but so far it looks promising with DWF. Furthermore, only the overlap operator has been found to be a solution of (2.2) but it involves the sign function of $\gamma^5 M(x|y)$ which is obviously an expensive expression and not really applicable for simulations. Thus a different formulation must be found and that is where DWF come into play.

2.2 Continuum Domain Wall Fermions

Fermions in even and odd dimensions look quite different. However, there are some interesting connections between them when the Dirac equation is considered with a space dependent mass. Such a mass can be thought of as arising from a (classical) Higgs field with a topological defect.

This picture will carry through the entire section as a mass is introduced depending on a 5$^{\text{th}}$ dimension

$$m : \mathbb{R} \to \mathbb{R},\ s \mapsto m(s) = \begin{cases} m & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -m & \text{if } s < 0 \end{cases} \tag{2.3}$$

In general any monotonic function with the limit $\lim_{s\to\pm\infty} m(s) = \pm m_\pm$, for $m_\pm > 0$ can do the job. For simplicity, (2.3) is used unless stated otherwise for the following consideration. A good starting point is the Dirac equation in five dimensions with a mass term corresponding to (2.3)

$$\left[\not{D} + \gamma^5 \partial_s + m(s)\right] \Psi(x,s) = 0. \tag{2.4}$$

Solutions of such an equation can always be expanded into modes $n \in \mathbb{Z}$, and factorized into factors dependent on the 5$^{\text{th}}$-direction, $f_n, b_n : \mathbb{R} \to \mathbb{C}$ and space time, spinor like ψ_n, factors. In particular f_n and b_n are functions for the left and right handed parts of the spinor $\psi_n(x)$ such that $\Psi(x,s)$ has the form

$$\Psi(x,s) = \sum_{n\in\mathbb{Z}} \left[b_n(s)P_+ + f_n(s)P_-\right] \psi_n(x) \tag{2.5}$$

In order for (2.5) to fulfill the Dirac equation (2.4) f_n and b_n have to satisfy

$$\begin{aligned} \left[\partial_s + m(s)\right] b_n(s) &= \mu_n f_n, \\ \left[-\partial_s + m(s)\right] f_n(s) &= \mu_n b_n, \end{aligned} \tag{2.6}$$

while the spinor part has to fulfill

$$\left[\not{D} + \mu_n\right] \psi_n(x) = 0. \tag{2.7}$$

Note that μ_n has been introduced as a coupling between the left and right handed parts of (2.5) the spinor. Regarding this it can be seen as a mode mass parameter. For now assume that a zeromode should exactly be chiral, thus left and right parts are not coupled enforcing $\mu_0 \overset{!}{=} 0$. With that at hand a generic solution, for a given normalization $N \in \mathbb{R}$, is given by

$$b_0(s) = Ne^{-\int_0^s m(s')ds'} \overset{(2.3)}{=} Ne^{-m|s|} \tag{2.8}$$

$$f_0(s) = Ne^{\int_0^s m(s')ds'} \overset{(2.3)}{=} Ne^{m|s|}. \tag{2.9}$$

The spinor on the other site is a linear combination of

$$\psi_0^\pm(x) = e^{ip_\mu x^\mu} u^\pm, \qquad \gamma^5 u^\pm = \pm u^\pm. \tag{2.10}$$

A closer look on the solutions (2.9) and (2.8) for $f_0, b_0 \in L_2$ shows that only one of them, b_0, is normalizable i.e. $N < \infty$. Thus f_0 is not part of the solution which in turn implies that there is no left handed, $\Psi_-(x,s)$, fermion in the spectrum. Nevertheless, there is nothing special about right handed fermions. This result is merely a convention. If the sign of the mass (2.3) would have been different the left handed fermion would arise in the spectrum. Generalizing the mass such that it is small around $s = 0$, for example $m(s) = m\tanh(s)$, will lead to a slightly distorted solutions for different modes and thus light modes will appear localized at $s = 0$.

2.2.1 A Finite 5$^{\text{TH}}$-Dimensional Extend

To understand the role of the extra dimension it is helpful to consider a finite extend and compactify it. This will also be an important application for lattice simulations since only a finite volume can be put on the computer. Assume that the 5$^{\text{th}}$ dimension has the extend L_s and is compactified with periodic boundary conditions. Then the mass term from (2.3) might be changed such that

$$m(s) = \begin{cases} +m & \text{if } s \in \left(0, \frac{L_s}{2}\right] \\ 0 & \text{if } s = 0 \\ -m & \text{if } s \in \left(-\frac{L_s}{2}, 0\right) \end{cases} \tag{2.11}$$

A schematic representation of this can be found in figure 2.1. Recall that the domain wall appears at points where the mass function has a defect i.e. changes sign. For finite L_s this appears not only at $s = 0$ but also at the boundary region $s = \pm\frac{L_s}{2}$. Therefore, a total of 3 Domain Walls can be seen in figure 2.1. Actually these refer to a domain wall anti domain wall pair since the $s = \pm\frac{L_s}{2}$ ones can be interpreted as anti domain walls and they coincide because of the boundary condition. Taking this setup, the solution for the left and right handed parts f_0, b_0 change slightly to

$$b_0(s) = Ne^{-\int_{-L_s/2}^{s} m(s')ds'} \tag{2.12}$$

$$f_0(s) = Ne^{+\int_{-L_s/2}^{s} m(s')ds'}, \tag{2.13}$$

which are both normalizable due to finite $s < L_s$. In order to obtain exactly massless zeromodes at finite L_s it is required to

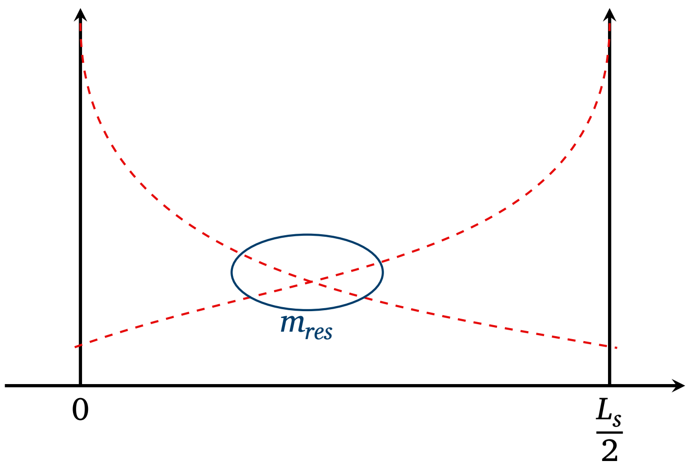

Figure 2.1: Schematic representation of the overlap of a left and right handed zero mode arising from a wall and anti wall for finite L_s resulting a residual mass.

have a mass term satisfying

$$\int_{-\frac{L_s}{2}}^{\frac{L_s}{2}} m(s)ds = 0 \tag{2.14}$$

which is depending on the choice of $m(s)$ and not a property of the theory itself. For instance, if a small interaction is turned on a small mass shift, of $\mathcal{O}(gm)$, appears breaking the property (2.14). Even in the non interacting case, if a asymmetric mass function of the form

$$m(s) = \begin{cases} -m & \text{if } s \in \left[-\frac{L_s}{2}, 0\right] \\ +\infty & \text{if } s \in \left(0, \frac{L_s}{2}\right) \end{cases}, \tag{2.15}$$

is used, (2.14) does not hold anymore. Thus, for any situation where (2.14) does not hold a small additive shift of the 4D-quark mass (μ_0) is induced which is called the residual mass m_{res}. An estimate can be found, in the case of the symmetric mass with an induced shift δm of a weakly coupled interaction, by considering that such a mass has to couple the left and right spinors

$$m_{res} = \delta\mu_0 \sim \delta m \int b_0(s) f_0(s) ds \sim gm \frac{L_s m}{\cosh\left(m\frac{L_s}{2}\right)} \tag{2.16}$$

This vanishes exponentially in the limit $mL_s \to \pm\infty$. For the asymmetric mass (2.15) even without interactions a residual mass can be found

$$m_{res} \sim me^{-mL_s} \tag{2.17}$$

Note that in full LQCD the computation of m_{res} is way more involved and requires the computation of correlators of currents. Current investigations hint that the residual mass rather behaves like

$$m_{res} \sim \frac{1}{L_s} + e^{-L_s} \tag{2.18}$$

Because this residual mass destroys the chiral properties of the theory, being an additive to the desired quark mass, it is considered as a measure of how well chiral symmetry is approximated. Some other implementations of the Domain Wall Fermion, such as the Möbius Domain Wall Fermion or the truncated Overlap, have been shown to improve the scaling of m_{res} at finite L_s.

2.3 Lattice Domain Wall Fermions

As pointed out previously the finite L_s is mandatory for any simulation, therefore, this section will assume a finite five-dimension extend. Putting everything on the lattice is straight forward since similar schemes as for standard 4D QCD can be used. Recalling the DWF action from the Dirac equation in (2.4)

$$S^{DWF}\left[\bar{\Psi},\Psi\right] = \int d^4x \int ds\; \bar{\Psi}(x,s)\left[\not{D} + \gamma^5\partial_s + m(s)\right]\Psi(x,s), \tag{2.19}$$

$\not{D}$ is the usual 4D Dirac operator, which can be transformed into one of the typical discretization schemes i.e. naive fermions, Wilson fermions, staggered fermions etc. It has been shown that the naive discretization will lead to doublers as will the staggered formulation. Since doubles should be avoided the Wilson discretization scheme is the usual choice.[1]

The $\gamma^5\partial_s$ term does not appear in the familiar four dimensional case. In free QCD one encounters $\gamma^\mu\partial_\mu$-like terms which are of the same kind as $\gamma^5\partial_s$. Hence $\not{D} + \gamma^5\partial_s$ can be merged into a five-dimensional Dirac operator. This works well in the free case, however, in the interacting case the gauge fields have to be handled with caution. Gauge invariance should be satisfied only

[1] However, it is a nice exercise to show that doublers appear in the naive case. For that please read page 4 of [1].

in the physical four dimensions. The fifth dimension does not acquire a gauge field as DWF are not describing a 5-dimensional QCD. On the lattice this would induce $U_\mu(x,s) = U_\mu(x),\ \forall \mu \in [4]$, and $U_5(x,s) = \mathbb{1}_{SU(3)}$. Therefore, the mass less part of (2.19) yields

$$\begin{aligned} &\mathcal{D}_{5D}^{Wilson}(x,r|y,s) = 4\delta_{x,y}\delta_{r,s} \\ &+\frac{1}{2a}\delta_{r,s}\sum_{\mu=1}^{4}\left[U_\mu(x,s)\left(\mathbb{1}+\gamma^\mu\right)\delta_{x,y+\hat{\mu}} + U_\mu^\dagger(x,s)\left(\mathbb{1}-\gamma^\mu\right)\delta_{x,y-\hat{\mu}}\right] \\ &+\delta_{x,y}\delta_{r,s} + \frac{1}{a_5}\delta_{x,y}\left[P_+\delta_{r,s+1} + P_-\delta_{r,s-1}\right] \end{aligned} \tag{2.20}$$

Hence the DWF matrix reads

$$M^{DWF}(x,r|y,s) = \mathcal{D}_{5D}^{Wilson}(x,r|y,s) + \delta_{x,y}m(s) \tag{2.21}$$

To understand some of the properties in a bit more detail consider the free lattice theory ($U_\mu(x) = \mathbb{1}_{SU(3)}$). The Dirac Fermion adds an additional spacial independent mass term which in turn depends on the 4-momentum. Therefore, the finite-difference zeromode solutions must obey

$$\left[\tilde{\partial}_s + m(s) + a_5\tilde{\partial}^s\tilde{\partial}_s + \frac{2}{a}\sum_{\mu=1}^{4}\left(\cos\left(ap_\mu\right) - 1\right)\right]b_0(s) = \mu_0 f_0(s) \tag{2.22}$$

$$\left[-\tilde{\partial}_s + m(s) + a_5\tilde{\partial}^s\tilde{\partial}_s + \frac{2}{a}\sum_{\mu=1}^{4}\left(\cos\left(ap_\mu\right) - 1\right)\right]f_0(s) = \mu_0 b_0(s) \tag{2.23}$$

$$\tag{2.24}$$

A abbreviation,

$$F(p) = \frac{2}{a}\sum_{\mu=1}^{4}\left(\cos\left(ap_\mu\right) - 1\right),$$

can be defined. Inserting the finite differences and rearranging (2.22) and (2.23), setting $\mu_0 = 0$, yields

$$b_0(s+1) = -m_{eff}(s)b_0(s) \tag{2.25}$$

$$f_0(s-1) = -m_{eff}(s)f_0(s), \tag{2.26}$$

$$m_{eff}(s) = a_5 m(s) - 1 - F(p) \tag{2.27}$$

Again, normalizability has to be checked for these in order to investigate the spectrum's content. It can be shown that for

$$\begin{aligned} b_0(s): \quad & |m_{eff}(s)| > 1, \ \forall s < 0 \quad && \wedge \quad |m_{eff}(s)| < 1, \ \forall s > 0 \\ f_0(s): \quad & |m_{eff}(s)| < 1, \ \forall s < 0 \quad && \wedge \quad |m_{eff}(s)| > 1, \ \forall s > 0 \end{aligned} \tag{2.28}$$

the zeromodes exist. Recalling from (2.11) that $am \geq 0$ and further $F(p) \geq 0$ it can be seen that the condition for f_0 can not be met. As a result, there is no right handed particle. A closer look into (2.27) reveals the more insightful constrain for the normalizability of b_0,

$$F(p) < m < F(p) + 2. \tag{2.29}$$

With that the derivation of Kaplans DWF is finished. It is, however, important to mention that 5D DWF theory coincide with the 4D overlap theory if the 5^{th} dimension is integrated out and the limit $L_s \to \infty$ and $a_5 \to 0$ is taken. Furthermore, it is possible to define an expectation value in the 5D theory, which coincides with the 4D QCD expectation value. This is notoriously difficult in the presented set up but becomes fairly easy in the Shamir formulation of DWF. In addition, it has been shown that Shamirs formulation contains the same physical content as Kaplans formulation. Other, formulations of the DWF, such as the Möbius DWF, Zolotarevs DWF, truncated Overlap and more, exists and improve certain aspects of DWF. However, state of the art is to use the Möbius fermions, which contains all other formulations as special cases, or Shamir fermions.

In summary, by introducing an artificial 5^{th} dimension into a 4-dimensional QCD, containing topological walls, coming from a 5^{th} dimension dependent mass, which changes sign at the topological wall, it is possible to find 4-dimensional chiral fermions even in a lattice regulated theory.