Hadronic Correlation and Spectral Functions in Lattice QCD

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HIC for FAIR Workshop
Dense QCD Phases in Heavy Ion Collisions and Supernovae
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Dilepton rate directly related to vector spectral function:

\[
\frac{dW}{d\omega d^3p} = \frac{5\alpha^2}{27\pi^2} \frac{1}{\omega^2(e^\omega/T - 1)} \sigma_V(\omega, \vec{p}, T)
\]
Hard Probes in Heavy Ion Collisions

electromagnetic observables
-hard(er) probes

hadronic observables
-soft probes

\( \pi, K, p, \ldots \)

freeze-out

Hadron gas

Mixed phase?

QGP

production thermalization

\[ A, E_b, b \]

Photonrate directly related to vector spectral function (at finite momentum):

\[
\omega \frac{dN_\gamma}{d^4 x d^3 q} = \frac{5 \alpha}{6\pi^2} \frac{1}{e^{\omega/T} - 1} \sigma_V (\omega = |\vec{p}|, T)
\]
Transport Coefficients

The small $\omega$ limit of $\sigma_V$ is related to transport (Kubo-Formulas)

Light quark sector $\rightarrow$ electrical conductivity:

$$\sigma_{el} = \lim_{\omega \to 0} \frac{\sigma^{ii}_V(\omega, \vec{p} = 0, T)}{6\omega}$$

Heavy quark sector $\rightarrow$ heavy quark diffusion constant:

$$D = \frac{1}{3 \chi^{00}} \lim_{\omega \to 0} \frac{\sigma^{ii}_V(\omega, \vec{p} = 0, T)}{\omega}$$

Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

$$\sigma_H = \frac{N_c}{8\pi^2} \Theta(\omega^2 - 4m^2) \frac{\omega^2}{2m} \frac{\omega}{4T} \tanh\left(\frac{\omega}{4T}\right)$$

$$\times \sqrt{1 - \left(\frac{2m}{\omega}\right)^2 \left[a_H + \left(\frac{2m}{\omega}\right)^2 b_H\right]}$$

$$+ \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega)$$

$$\delta(\omega) \rightarrow \frac{1}{\pi} \frac{\eta}{\omega^2 + \eta^2}$$ [Petreczky+Teany 06, Aarts et al. 05]
Thermal hadronic correlation functions

\[ J_H = \bar{q}(\tau, \vec{x}) \Gamma_H q(\tau, \vec{x}) \]

\[ G_H(\tau, T, \vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \langle J_H(0, 0) J_H^\dagger(\tau, \vec{x}) \rangle \]

O(a)-improved Clover improved fermionic action

on large quenched lattice configurations up to 128^3 \times 16 and 32

includes all the physics

how to extract it?

directly from the correlators?

spectral functions using MEM?
Temporal Correlators:

- Splitting below $T_c$ due to chiral and axial U(1) symmetry breaking
- Symmetry restoration above $T_c$
splitting below $T_c$ due to chiral and axial U(1) symmetry breaking

symmetry restoration above $T_c$
Light Quark Screening Masses – Definition

use spatial correlators

\[ G_H(z, T, \vec{p}_\perp) = \sum_{\tau, \vec{x}_\perp} e^{-i\vec{p}_\perp \cdot \vec{x}_\perp} \langle J_H(0, 0) J_H^\dagger(\tau, \vec{x}) \rangle \]

correlation function depends on the same spectral density, but the relation is more involved

however, \( m_{\text{screen}}(T) \neq m_{\text{pole}}(T) \) in general:

look for zeros of \( G^{-1}(p) = p_0^2 + \vec{p}^2 + m_0^2 + \Pi(p_0, \vec{p}, T) \)

\begin{align*}
\vec{p} = 0 : & \quad -p_0^2 = m_0^2 + \Pi(p_0, \vec{0}, T) = (m_{\text{pole}}(T))^2 \\
p_0 = 0 : & \quad -\vec{p}^2 = m_0^2 + \Pi(0, \vec{p}, T) = (m_{\text{screen}}(T))^2
\end{align*}

\[ \Rightarrow m_{\text{screen}}(T) = \frac{m_{\text{pole}}(T)}{A(T)} \]
large collection of lattices ranging from $16^3 \times 8$ to $128^3 \times 16$
allowing for thermodynamic limit $V \to \infty$ at $N_t=8,12$ and 16

\[
T \leq T_c : \quad m_{\text{screen}}(L, a) = m_{\text{screen}}(a) \left[ 1 + \gamma_V \left( \frac{N_T}{N_\sigma} \right)^3 \right]
\]

\[
T > T_c : \quad m_{\text{screen}}(L, a) = m_{\text{screen}}(a) \left[ 1 + \gamma_V \left( \frac{N_T}{N_\sigma} \right)^p \right]
\]

\[
T = \infty : \quad m_{\text{screen}}(L, a) = m_{\text{screen}}(a) \left[ 1 + \gamma_V \left( \frac{N_T}{N_\sigma} \right)^1 \right]
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combined fit:

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<tr>
<th>$p$</th>
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<th>V</th>
</tr>
</thead>
<tbody>
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<td>1.5 $T_c$</td>
<td>2.22(10)</td>
<td>2.18(13)</td>
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<td>3.0 $T_c$</td>
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![Graph 1](image1)

![Graph 2](image2)
Continuum Limit

lattice spacing $a \to 0$

Non-perturbatively improved action

⇒ discretization errors $O(a^2)$

$$T = \infty : \frac{m_z(a)}{T} = \frac{m_z}{T} - \lambda \left( \frac{1}{N_t} \right)^2$$
Thermodynamic and Continuum Limit:

\[ V \rightarrow \infty \quad \text{and} \quad a \rightarrow 0 \]

- Weak temperature dependence below \( T_c \)
- Data still below free limit \( (2\pi T) \) at 3 \( T_c \), vector closer to free case
- Perturbative limit reached from above [Laine, Vepsäläinen]
- Need higher temperatures
Light Quark Correlators vs Free Correlators

comparison with free high temperature lattice correlator

vector and axial-vector close to free case

Cut-off effects are well described by free lattice correlator
Light Quark Correlators vs Free Correlators

T=1.5 \ T_c \ 128^3 \times 32

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still strong correlations in scalar and pseudo-scalar channel
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Spectral Functions – Maximum Entropy Method

How to obtain continuous spectral function \( \sigma(\omega, T) \)
from discrete (and small) number of correlators?

\[
G(\tau, T) = \int_0^{\infty} d\omega K(\tau, \omega, T) \sigma(\omega, T)
\]

\[
K(\tau, \omega, T) = \frac{\cosh \left( \omega \left( \tau - \frac{1}{2T} \right) \right)}{\sinh \left( \frac{\omega}{2T} \right)}
\]

Best method on the market: Maximum Entropy Method (MEM)

based on Bayesian theorem [Asakawa et al. 01] \( \rightarrow \) most probable spectral function

prior knowledge needed as input \( \rightarrow \) default model \( m(\omega) \)

result should be independent of default model \( \leftarrow \) usually not the case
Large $\omega$ behaviour well described by free lattice SPF.

Cut-Off effects are under control and well separated from physical interesting region.

Vector SPF close to free case except at small $\omega$.

Still large correlations in the Pseudoscalar sector.

Small $\omega$ region accessible $\rightarrow$ hope to extract transport properties.

Next step: Dilepton rate (in the continuum limit).
screening masses at 1.50 $T_c$ already close to the free case $m^{scr\text{ free}}(T) = 2\sqrt{(\pi T)^2 + m_c^2}$.

does this tell us anything about dissociation?

need to understand the momentum dependence of $G_H(\tau,T,p)$ and $\sigma(\omega,T,p)$ in detail

thermodynamic and continuum limit not performed yet
non-degenerate states still at 1.50 $T_c$

(almost) close to free correlators at (very) small separations

largest distance 0.25 fm due to compact temporal direction

only small distance regime (0.1-0.25 fm) relevant

for thermal effects

for bound state effects
Charmonium Correlators vs Free Correlators

non-degenerate states still at $1.50 \, T_c$

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Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

\[ \sigma_H = \frac{N_c}{8\pi^2} \Theta(\omega^2 - 4m^2) \omega^2 \tanh\left(\frac{\omega}{4T}\right) \]
\[ \times \sqrt{1 - \left(\frac{2m}{\omega}\right)^2} \left[ a_H + \left(\frac{2m}{\omega}\right)^2 b_H \right] \]
\[ + \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega) \]
MEM – Free spectral function

Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

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+ \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega)
\]

Lattice cut-off effects

\[
\omega_{max} = 2 \log(7 + ma)
\]

[Graphs showing \(\sigma(\omega,T)/\omega^2\) for different values of \(T\)]
MEM – Free spectral function

Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

\[ \sigma_H = \frac{N_c}{8\pi^2}\Theta(\omega^2 - 4m^2) \omega^2 \tanh\left(\frac{\omega}{4T}\right) \]

\[ \times \sqrt{1 - \left(\frac{2m}{\omega}\right)^2} \left[ a_H + \left(\frac{2m}{\omega}\right)^2 b_H \right] \]

\[ + \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega) \]

zero mode contribution at \( \omega \simeq 0 \) [Umeda 07]

\[ \delta(\omega) \to \frac{1}{\pi} \frac{\eta}{\omega^2 + \eta^2} \]

[Petreczky+Teany 06
Aarts et al. 05]
N_σ=128 and N_τ=64 → cut-off effects well separated

Pronounced ground state peak close to J/ψ mass

no zero mode contributions observed below T_c
MEM – Spectral function below $T_c$

$V \beta = 7.457$

$\sigma(\omega,T)/\omega^2$

$T=0.75 \, T_c$

$\omega [\text{GeV}]$

$N_\sigma = 128$ and $N_\tau = 64 \, \rightarrow \, \text{cut-off effects well separated}$

pronounced ground state peak close to $J/\psi$ mass

no zero mode contributions observed below $T_c$

first peak independent of default model
Use $0.75T_c$ SPF as default model at $1.50T_c$
Use free SPF as default model at $1.50T_c$.
Charmonium Correlators – Zero Mode Contributions

\[
G_{\text{rec}}(\tau, T) = \int_{2m_c}^\infty \sigma_0(\omega, 0.75T_c) K(\omega, \tau, T) \\
G_{\text{low}}(\tau, T) = \int_0^{2m_c} \sigma_T(\omega, T) K(\omega, \tau, T) \\
G_{\mu\mu}(\tau, T) = G_{ii}(\tau, T) + G_{44}(\tau, T)
\]

- main T-effect due to zero-mode contribution
- well described by small $\omega$-part of $\sigma_T(\omega, T)$
- smaller than $G_{44}(\tau, T)\chi(T)T$
- no zero-mode contribution in PS-channel

(similar to discussions by Umeda, Petreczky)
Charmonium Correlators – Zero Mode Contributions

\[ G_{\text{rec}}(\tau, T) = \int \sigma_0(\omega, 0.75T_c)K(\omega, \tau, T) \]

\[ G_{\text{low}}(\tau, T) = \int_0^{2m_c} \sigma_T(\omega, T)K(\omega, \tau, T) \]

- larger zero-mode contribution in S-wave
- larger T-effect in the S-wave states

systematic uncertainties in reconstruction and low-\(\omega\) part of spectral function
high quality data and small lattice spacing + large momenta (volume) needed
Conclusions and Outlook

Hadronic correlators for light quarks ($m_q=0$)

- Thermodynamic and Continuum Limit of screening masses!
- Spectral functions $\rightarrow$ Dilepton rates, Transport coefficients?
- Momentum dependence needs to be analyzed $\rightarrow$ Photon rates
- Comparison with HTL calculations and experiment

Charmonium hadronic correlators ($m_q=m_c$)

- Can we really trust any of the spectral functions obtained with MEM?
- What can we learn from Hadronic correlators on Dissociation?
- Momentum dependence vs. Spatial correlators/screening masses?
- Spectral functions $\rightarrow$ Dilepton rates, Transport coefficients?
- Momentum dependence needs to be analyzed $\rightarrow$ Photon rates
- Comparison with HTL/NRQCD calculations and experiment
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