Spectral and transport properties of the QGP from Lattice QCD calculations

Olaf Kaczmarek

University of Bielefeld

KITPC program “sQGP and Extreme QCD“
Beijing
20.05.2015
Lattice calculations of hadronic correlation functions

... and how we try to

extract transport properties \textit{and} spectral properties \textit{from} them

1) Vector meson correlation functions for light quarks

continuum extrapolation \hspace{1cm} \text{with H-T.Ding, F.Meyer, et al.}

correlation to perturbation theory \hspace{1cm} \text{with J.Ghiglieri, M.Laine, F.Meyer}

$\rightarrow$ Electrical conductivity

$\rightarrow$ Thermal dilepton rates \textit{and} thermal photon rates

2) Color electric field correlation function \hspace{1cm} \text{with A.Francis, M. Laine, T.Neuhaus, H.Ohno}

Heavy quark momentum diffusion coefficient $\kappa$

3) Vector meson correlation functions for heavy quarks \hspace{1cm} \text{with H-T.Ding, H.Ohno et al.}

Heavy quark diffusion coefficients

Charmonium and Bottomonium dissociation patterns
Transport Coefficients are important ingredients into hydro/transport models for the evolution of the system. Usually determined by matching to experiment (see right plot)

Need to be determined from QCD using first principle lattice calculations!

here heavy flavour:

Heavy Quark Diffusion Constant $D$
[H.T.Ding, OK et al., PRD86(2012)014509]

Heavy Quark Momentum Diffusion $\kappa$
[OK, arXiv:1409.3724]

or for light quarks:

Light quark flavour diffusion

Electrical conductivity
[A.Francis, OK et al., PRD83(2011)034504]
Heavy Ion Collision                      QGP             Expansion+Cooling                      Hadronization

Charmonium+Bottmonium is produced (mainly) in the early stage of the collision

Depending on the **Dissociation Temperature**

- remain as bound states in the whole evolution
- release their constituents in the plasma

---

Motivation - Quarkonium in Heavy Ion Collisions

**Sequential suppression for bottomonium observed at CMS**
Charmonium+Bottmonium is produced (mainly) in the early stage of the collision

Depending on the Dissociation Temperature

- remain as bound states in the whole evolution
- release their constituents in the plasma

Motivation - Quarkonium in Heavy Ion Collisions

First estimates on Dissociation Temperatures

from detailed knowledge of Heavy Quark Free Energies and Potential Models
Motivation - Quarkonium in Heavy Ion Collisions

Heavy quark potential complex valued at finite temperature

\[ V(r) = \lim_{t \to \infty} \frac{i \partial_t W(t, r)}{W(t, r)} \iff V(r) = \lim_{t \to \infty} \int d\omega e^{-i\omega t} \frac{\rho(\omega, r)}{\int d\omega e^{-i\omega t} \rho(\omega, r)} \]
Transport coefficients from Lattice QCD – Flavour Diffusion

Transport coefficients usually calculated using correlation function of conserved currents

\[ G(\tau, p, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, p, T) K(\tau, \omega, T) \]

\[ K(\tau, \omega, T) = \frac{\cosh \left( \omega (\tau - \frac{1}{2T}) \right)}{\sinh \left( \frac{\omega}{2T} \right)} \]

Lattice observables:

\[ G_{\mu\nu}(\tau, \vec{x}) = \langle J_\mu(\tau, \vec{x}) J_\nu^\dagger(0, \vec{0}) \rangle \]

\[ J_\mu(\tau, \vec{x}) = 2\kappa Z_V \bar{\psi}(\tau, \vec{x}) \Gamma_\mu \psi(\tau, \vec{x}) \]

\[ G_{\mu\nu}(\tau, \vec{p}) = \sum_{\vec{x}} G_{\mu\nu}(\tau, \vec{x}) e^{i\vec{p}\cdot\vec{x}} \]

only correlation functions calculable on lattice but

Transport coefficient determined by slope of spectral function at \( \omega=0 \) (Kubo formula)

\[ D = \frac{\pi}{3\chi_{00}} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T} \]
Quarkonium spectral function – What do we expect!?

\[ \rho(\omega) = \frac{1}{\omega^2} \]

+ zero-mode contribution at \( \omega=0 \):

\[ \rho(\omega) = 2\pi \chi_{00} \ \omega \delta(\omega) \]
Quarkonium spectral function – What do we expect!? 

+ zero-mode contribution at \( \omega = 0 \): 
  \[
  \rho(\omega) = 2\pi \chi_{00} \omega \delta(\omega)
  \]

+ transport (narrow) peak at small \( \omega \): 
  \[
  \rho(\omega \ll T) \approx 2\chi_{00} \frac{T}{M} \frac{\omega \eta}{\omega^2 + \eta^2}, \quad \eta = \frac{T}{MD}
  \]
Quarkonium spectral function – What do we expect!?

+ zero-mode contribution at $\omega=0$:
  $$\rho(\omega) = 2\pi\chi_{00} \omega \delta(\omega)$$

+ transport (narrow) peak at small $\omega$:
  $$\rho(\omega \ll T) \simeq 2\chi_{00} \frac{T}{M} \frac{\omega\eta}{\omega^2 + \eta^2} , \quad \eta = \frac{T}{MD}$$
+ zero-mode contribution at $\omega=0$: $\rho(\omega) = 2\pi \chi_{00} \omega \delta(\omega)$

+ transport (narrow) peak at small $\omega$: $\rho(\omega \ll T) \approx 2\chi_{00} \frac{T}{M} \frac{\omega \eta}{\omega^2 + \eta^2}$, $\eta = \frac{T}{MD}$
Quarkonium spectral function – What do we expect!? 

\[
\rho(\omega) \sim \begin{cases} 
2\pi\chi_{00} \omega \delta(\omega) & \text{zero-mode contribution at } \omega=0 \\
2\chi_{00} \frac{T}{M} \frac{\omega\eta}{\omega^2 + \eta^2} & \text{transport (narrow) peak at small } \omega 
\end{cases}
\]

\[\eta = \frac{T}{MD}\]
Quarkonium spectral function – What do we expect!? 

- zero-mode contribution at $\omega = 0$: $\rho(\omega) = 2\pi \chi_{00} \omega \delta(\omega)$
- transport (narrow) peak at small $\omega$: $\rho(\omega \ll T) \simeq 2\chi_{00} \frac{T}{M} \frac{\omega \eta}{\omega^2 + \eta^2}$, $\eta = \frac{T}{MD}$
Vector spectral function – hard to separate different scales

Different contributions and scales enter in the spectral function

- continuum at large frequencies
- possible bound states at intermediate frequencies
- transport contributions at small frequencies
- in addition cut-off effects on the lattice

notoriously difficult to extract from correlation functions

\[ G(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, \vec{p}, T) K(\tau, \omega, T) \]

transport (narrow) peak at small \( \omega \):

\[ \rho(\omega \ll T) \simeq 2\chi_{00} \frac{T}{M} \frac{\omega \eta}{\omega^2 + \eta^2}, \quad \eta = \frac{T}{MD} \]
Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

\[\sigma_H = \frac{N_c}{8\pi^2} \Theta(\omega^2 - 4m^2) \omega^2 \tanh\left(\frac{\omega}{4T}\right) \]
\[\times \sqrt{1 - \left(\frac{2m}{\omega}\right)^2} \left[ a_H + \left(\frac{2m}{\omega}\right)^2 b_H \right] \]
\[+ \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega)\]
Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

\[
\sigma_H = \frac{N_c}{8\pi^2} \Theta(\omega^2 - 4m^2) \omega^2 \tanh(\frac{\omega}{4T}) \\
\times \sqrt{1 - \left(\frac{2m}{\omega}\right)^2 \left[a_H + \left(\frac{2m}{\omega}\right)^2 b_H\right]} \\
+ \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega)
\]

Lattice cut-off effects:

\[
\omega_{max} = 2 \log(7 + ma)
\]

we will perform the continuum extrapolation to get rid of the cut-off effects
Free (non-interacting) spectral function [Karsch et al. 03, Aarts et al. 05]

\[
\sigma_H = \frac{N_c}{8\pi^2} \Theta(\omega^2 - 4m^2) \omega^2 \tanh\left(\frac{\omega}{4T}\right) \\
\times \sqrt{1 - \left(\frac{2m}{\omega}\right)^2 \left[a_H + \left(\frac{2m}{\omega}\right)^2 b_H\right]} \\
+ \frac{N_c}{3} \frac{T^2}{2} f_H \omega \delta(\omega)
\]

zero mode contribution at \( \omega \approx 0 \) [Umeda 07]

with interactions:

\[
\delta(\omega) \rightarrow \frac{1}{\pi} \frac{\eta}{\omega^2 + \eta^2}
\]

[Petreczky+Teaney 06
Aarts et al. 05]
What are the contribution of the different scales to the correlator?

Illustrative example:

- Continuum contribution dominates at small $\tau T$
- Bound state contribution over a wide range, only one state in this example
- Only small contribution from the transport peak at large $\tau T$
Heavy Quark Effective Theory (HQET) in the large quark mass limit

**for a single quark in medium**

leads to a (pure gluonic) “color-electric correlator”


Heavy quark (momentum) diffusion:

\[ G_E(\tau) \equiv -\frac{1}{3} \sum_{i=1}^{3} \frac{\langle \text{Re} \, \text{Tr} \left[ U(\frac{1}{T};\tau) gE_i(\tau,0) U(\tau;0) gE_i(0,0) \right] \rangle}{\langle \text{Re} \, \text{Tr}[U(\frac{1}{T};0)] \rangle} \]

\[ \kappa = \lim_{\omega \to 0} \frac{2T \rho_E(\omega)}{\omega} \quad D = \frac{2T^2}{\kappa} \]
Heavy Quark Momentum Diffusion Constant – Perturbation Theory

can be related to the thermalization rate:

\[ \eta_D = \frac{\kappa}{2 M_{kin} T} \left( 1 + O \left( \frac{\alpha_s^{3/2} T}{M_{kin}} \right) \right) \]


very poor convergence

\(\rightarrow\) Lattice QCD study required in the relevant temperature region

in contrast to a narrow transport peak, from this a smooth limit

\[ \frac{\kappa}{T^3} = \lim_{\omega \to 0} \frac{2T \rho_E(\omega)}{\omega} \]

is expected

Qualitatively similar behaviour also found in AdS/CFT [S.Gubser, Nucl.Phys.B790 (2008)175]
due to the gluonic nature of the operator, signal is extremely noisy

→ multilevel combined with link-integration techniques to improve the signal


[Parisi, Petronzio, Rapuano PLB 128 (1983) 418, and de Forcrand PLB 151 (1985) 77]
normalized by the LO-perturbative correlation function:

$$G_{\text{norm}}(\tau T) \equiv \frac{G_{\text{cont}}^{\text{LO}}(\tau T)}{g^2 C_F} = \pi^2 T^4 \left[ \frac{\cos^2(\pi\tau T)}{\sin^4(\pi\tau T)} + \frac{1}{3\sin^2(\pi\tau T)} \right]$$

and renormalized using NLO renormalization constants $Z(g^2)$

$$C_F \equiv \frac{N_c^2 - 1}{2N_c}$$
lattice cut-off effects visible at small separations (left figure)

→ **tree-level improvement** (right figure) to reduce discretization effects

\[ G^{\text{LO}}_{\text{cont}}(\tau \bar{T}) = G^{\text{LO}}_{\text{lat}}(\tau T) \]

leads to an effective reduction of cut-off effect for all \( \tau \bar{T} \)
Quenched Lattice QCD on large and fine isotropic lattices at $T \approx 1.4 \, T_c$

- standard Wilson gauge action
- algorithmic improvements to enhance signal/noise ratio
- fixed aspect ratio $N_s/N_t = 4$, i.e. fixed physical volume $(2\text{fm})^3$
- perform the continuum limit, \( a \to 0 \iff N_t \to \infty \)
- determine $\kappa$ in the continuum using an Ansatz for the spectral fct. $\rho(\omega)$

<table>
<thead>
<tr>
<th>$N_\sigma$</th>
<th>$N_\tau$</th>
<th>$\beta$</th>
<th>$1/a[\text{GeV}]$</th>
<th>$a[\text{fm}]$</th>
<th>#Confcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>16</td>
<td>6.872</td>
<td>7.16</td>
<td>0.03</td>
<td>172</td>
</tr>
<tr>
<td>80</td>
<td>20</td>
<td>7.035</td>
<td>8.74</td>
<td>0.023</td>
<td>180</td>
</tr>
<tr>
<td>96</td>
<td>24</td>
<td>7.192</td>
<td>10.4</td>
<td>0.019</td>
<td>160</td>
</tr>
<tr>
<td>144</td>
<td>36</td>
<td>7.544</td>
<td>15.5</td>
<td>0.013</td>
<td>693</td>
</tr>
<tr>
<td>192</td>
<td>48</td>
<td>7.793</td>
<td>20.4</td>
<td>0.010</td>
<td>223</td>
</tr>
</tbody>
</table>
Heavy Quark Momentum Diffusion Constant – Lattice results

finest lattices still quite noisy at large $\tau T$ but only small cut-off effects at intermediate $\tau T$

cut-off effects become visible at small $\tau T$

need to extrapolate to the continuum

perturbative behavior in the limit $\tau T \rightarrow 0$

allows to perform continuum extrapolation, $a \rightarrow 0 \leftrightarrow N_t \rightarrow \infty$, at fixed $T=1/a N_t$

<table>
<thead>
<tr>
<th>$N_\sigma$</th>
<th>$N_\tau$</th>
<th>$\beta$</th>
<th>$1/a$ [GeV]</th>
<th>$a$ [fm]</th>
<th>#Confis</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>16</td>
<td>6.872</td>
<td>7.16</td>
<td>0.03</td>
<td>172</td>
</tr>
<tr>
<td>80</td>
<td>20</td>
<td>7.035</td>
<td>8.74</td>
<td>0.023</td>
<td>180</td>
</tr>
<tr>
<td>96</td>
<td>24</td>
<td>7.192</td>
<td>10.4</td>
<td>0.019</td>
<td>160</td>
</tr>
<tr>
<td>144</td>
<td>36</td>
<td>7.544</td>
<td>15.5</td>
<td>0.013</td>
<td>693</td>
</tr>
<tr>
<td>192</td>
<td>48</td>
<td>7.793</td>
<td>20.4</td>
<td>0.010</td>
<td>223</td>
</tr>
</tbody>
</table>
Heavy Quark Momentum Diffusion Constant – Continuum extrapolation

The finest lattices still quite noisy at large $\tau T$, but only small cut-off effects at intermediate $\tau T$.

Cut-off effects become visible at small $\tau T$, so we need to extrapolate to the continuum.

Perturbative behavior in the limit $\tau T \to 0$.

Well behaved continuum extrapolation for $0.05 \leq \tau T \leq 0.5$.

The finest lattice already close to the continuum.

Coarser lattices at larger $\tau T$ are close to the continuum.

How to extract the spectral function from the correlator?
Model spectral function: transport contribution + NNLO \cite{Y.Burnier et al. JHEP 1008 (2010) 094}

Some contribution at intermediate distance/frequency seems to be missing.

\[
G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh \left( \frac{1}{2} - \tau T \right) \omega}{\sinh \frac{\omega}{2T}}
\]

\[
\rho_{\text{model}}(\omega) \equiv \max \left\{ \rho_{\text{NNLO}}(\omega), \frac{\omega \kappa}{2T} \right\}
\]
Model spectral function: transport contribution + NNLO + correction

\[ \rho_{\text{model}}(\omega) \equiv \max \left\{ A\rho_{\text{NNLO}}(\omega) + B\omega^3, \frac{\omega \kappa}{2T} \right\} \]

\[ G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh \left( \frac{1}{2} - \tau T \right) \frac{\omega}{T}}{\sinh \frac{\omega}{2T}} \]

used to fit the continuum extrapolated data

\[ \kappa / T^3 \sim \lim_{\omega \to 0} \frac{2T \rho_{E}(\omega)}{\omega} \approx 2.5(4) \]

result of the fit to \( \rho_{\text{model}}(\omega) \)

with three parameters: \( \kappa, A, B \)

NNLO (vacuum) perturbation theory

\[ \chi / T^3 = 2.5(4) \]
Model spectral function: transport contribution + NNLO + correction

\[ \rho_{\text{model}}(\omega) \equiv \max \left\{ A \rho_{\text{NNLO}}(\omega) + B \omega^3, \frac{\omega \kappa}{2T} \right\} \]

used to fit the continuum extrapolated data

\[ G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh \left( \frac{1}{2} - \tau T \right) \omega}{\sinh \frac{\omega}{2T}} \]

\[ \kappa/T^3 = \lim_{\omega \to 0} \frac{2T \rho_E(\omega)}{\omega} \approx 2.5(4) \]

\[ \omega \kappa \]

small but relevant contribution at \( \tau T > 0.2 \)!

Heavy Quark Momentum Diffusion Constant – Model Spectral Function

result of the fit to \( \rho_{\text{model}}(\omega) \)

\[ A \rho_{\text{NNLO}}(\omega) + B \omega^3 \]

NNLO (vacuum) perturbation theory
Conclusions and Outlook

\[ G_{E}(\tau) \equiv -\frac{1}{3} \sum_{i=1}^{3} \frac{\left\langle \text{Re} \text{Tr} \left[ U(\frac{1}{T}; \tau) gE_{i}(\tau, 0) U(\tau; 0) gE_{i}(0, 0) \right] \right\rangle}{\left\langle \text{Re} \text{Tr} [U(\frac{1}{T}; 0)] \right\rangle} \]

→ Continuum extrapolation for the color electric correlation function extracted from quenched Lattice QCD

- using noise reduction techniques to improve signal
- and an Ansatz for the spectral function

→ first continuum estimate for the Heavy Quark Momentum Diffusion Coefficient \( \kappa \)

- still based on a simple Ansatz for the spectral function

→ More detailed analysis of the systematic uncertainties needed

- different Ansätze for the spectral function
- using contributions from thermal perturbation theory
- other techniques to extract the spectral function

Other Transport coefficients from Effective Field Theories?
In non-relativistic QCD the Lagrangian is expanded in terms of $v = |p|/M$

$$\mathcal{L} = \mathcal{L}_0 + \delta \mathcal{L}$$

with

$$\mathcal{L}_0 = \psi^\dagger \left( D_\tau - \frac{D^2}{2M} \right) \psi + \chi^\dagger \left( D_\tau + \frac{D^2}{2M} \right) \chi$$

and

$$\delta \mathcal{L} = -\frac{c_1}{8M^3} \left[ \psi^\dagger (D^2)^2 \psi - \chi^\dagger (D^2)^2 \chi \right]$$

$$+ c_2 \frac{ig}{8M^2} \left[ \psi^\dagger (D \cdot E - E \cdot D) \psi + \chi^\dagger (D \cdot E - E \cdot D) \chi \right]$$

$$- c_3 \frac{g}{8M^2} \left[ \psi^\dagger \sigma \cdot (D \times E - E \times D) \psi + \chi^\dagger \sigma \cdot (D \times E - E \times D) \chi \right]$$

$$- c_4 \frac{g}{2M} \left[ \psi^\dagger \sigma \cdot B \psi - \chi^\dagger \sigma \cdot B \chi \right]$$

which is correct up to order $O(v^4)$  

NRQCD is more sensitive to the bound state region

Kernel is $T$-independent

$\rightarrow$ contributions at $\omega < 2M$ absent

$\rightarrow$ no small-$\omega$ contribution

$\rightarrow$ no information on transport properties

\[
G(\tau) = \int_{-2M}^{\infty} \frac{d\omega'}{2\pi} \exp(-\omega' \tau) \rho(\omega')
\]

\[
\omega' = \omega - 2M
\]

requires anisotropic lattices with $a_sM \gg 1$

no continuum limit in NRQCD

only small energy region accessible
Lattice cut-off effects – free spectral functions

[G.Aarts et al., JHEP1407(2014)097]
gauge configurations from $n_f=2+1$
dynamical Wilson fermion action

$a_s \approx 0.16 \text{ fm}$ \hspace{1cm} $a_s \approx 0.13 \text{ fm}$

$1/a_t \approx 7.35 \text{ GeV}$ \hspace{1cm} $1/a_t \approx 5.63 \text{ GeV}$

anisotropic NRQCD

[see also F.Karsch et al., PRD68 (2003) 014504]

[H.T.Ding, OK et al., arXiv:1204.4945]
gauge configurations from quenched action

$a \approx 0.01 \text{ fm}$

$1/a \approx 19 \text{ GeV}$

isotropic Wilson

cut-off effects and energy resolution determined by spatial lattice spacing

no continuum limit in NRQCD, $a_s M \gg 1$

only small energy region accessible

continuum limit straight forward, but expensive

transport properties accessible

[see also F.Karsch et al., PRD68 (2003) 014504]
Correlations of conserved charges – open charm sector

[A.Bazavov, H.T.Ding, P.Hegde, OK et al., PLB737 (2014) 210]

2+1 flavor HISQ with almost physical quark masses
32^3 \times 8 and 24^3 \times 6 lattices with m_l = m_s/20 and physical m_s and quenched charm quarks

generalized susceptibilities of conserved charges

\[ \chi^{BQSC}_{klmn} = \frac{\partial^{(k+l+m+n)}[P(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S, \hat{\mu}_C)/T^4]}{\partial \hat{\mu}_B \partial \hat{\mu}_Q \partial \hat{\mu}_S \partial \hat{\mu}_C} |_{\bar{\mu}=0} \]

are sensitive to the underlying degrees of freedom

\[ \chi_{13}/\chi_{22} \]

\[ \chi_{BC}/\chi_{BC} \]

\[ \chi_{11}/\chi_{13} \]

\[ \chi_{C}/\chi_{2} \]

\[ (\chi_{2}/\chi_{22})/(\chi_{4}/\chi_{13}) \]

\[ \text{charmed baryon sector} \]

\[ \text{open charm meson sector} \]

\[ \Rightarrow \] indications that open charm hadrons start to dissolve already close to the chiral crossover
In the following: **Meson Correlation Functions**

\[
G(\tau, p, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, p, T) K(\tau, \omega, T)
\]

\[
K(\tau, \omega, T) = \frac{\cosh \left( \omega (\tau - \frac{1}{2T}) \right)}{\sinh \left( \frac{\omega}{2T} \right)}
\]

**Lattice observables:**

\[
G_{\mu\nu}(\tau, \vec{x}) = \langle J_\mu(\tau, \vec{x}) J_\nu^\dagger(0, \vec{0}) \rangle
\]

\[
J_\mu(\tau, \vec{x}) = 2\kappa Z_V \bar{\psi}(\tau, \vec{x}) \Gamma_\mu \psi(\tau, \vec{x})
\]

\[
G_{\mu\nu}(\tau, \vec{p}) = \sum_{\vec{x}} G_{\mu\nu}(\tau, \vec{x}) e^{i\vec{p}\cdot\vec{x}}
\]

related to a conserved current in the vector channel

<table>
<thead>
<tr>
<th>Channel</th>
<th>$\Gamma_H$</th>
<th>$2S+1 L_J$</th>
<th>$J^{PC}$</th>
<th>Quarkonia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pseudoscalar (PS)</td>
<td>$\gamma_5$</td>
<td>$^1S_0$</td>
<td>0--</td>
<td>$\eta_c, \eta_b$</td>
</tr>
<tr>
<td>Vector (V)</td>
<td>$\gamma_i$</td>
<td>$^3S_1$</td>
<td>1--</td>
<td>$J/\psi, \Upsilon$</td>
</tr>
<tr>
<td>Scalar (S)</td>
<td>1</td>
<td>$^1P_0$</td>
<td>0++</td>
<td>$\chi_{c0}, \chi_{b0}$</td>
</tr>
<tr>
<td>Axialvector (AV)</td>
<td>$\gamma_i\gamma_5$</td>
<td>$^3P_1$</td>
<td>1++</td>
<td>$\chi_{c1}, \chi_{b1}$</td>
</tr>
</tbody>
</table>
Lattice observables:

\[ G_{\mu\nu}(\tau, \vec{x}) = \langle J_{\mu}(\tau, \vec{x}) J_{\nu}^\dagger(0, \vec{0}) \rangle \]

\[ J_{\mu}(\tau, \vec{x}) = 2\kappa Z_V \bar{\psi}(\tau, \vec{x}) \Gamma_{\mu} \psi(\tau, \vec{x}) \]

\[ G_{\mu\nu}(\tau, \vec{p}) = \sum_{\vec{x}} G_{\mu\nu}(\tau, \vec{x}) e^{i\vec{p}\cdot\vec{x}} \]

only correlation functions calculable on lattice but

Transport coefficient determined by slope of spectral function at \( \omega=0 \) (Kubo formula)

\[ D = \frac{\pi}{3\chi_{00}} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T} \]
Charmonium Spectral function

from Maximum Entropy Method analysis on a fine but finite lattice:

statistical error band from Jackknife analysis

no clear signal for bound states at and above $1.46 \ T_c$

study of the continuum limit and quark mass dependence required!
Charmonium Spectral function – Transport Peak

\[ D = \frac{\pi}{3\chi_{00}} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T} \]

Perturbative estimate (\( \alpha_s \approx 0.2\), \( g \approx 1.6 \)):

- LO: \( 2\pi TD \approx 71.2 \)
- NLO: \( 2\pi TD \approx 8.4 \)


Strong coupling limit:

\( 2\pi TD = 1 \)

Charmonium Spectral function – Transport Peak

\[ D = \frac{\pi}{3\chi_{00}} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T} \]

Perturbative estimate (\(\alpha_s \sim 0.2, g \sim 1.6\)):

- LO: \(2\pi TD \simeq 71.2\)
- NLO: \(2\pi TD \simeq 8.4\)


Strong coupling limit:

\(2\pi TD = 1\)

Charmonium and Bottomonium correlators

[H.T.Ding, H.Ohno, OK, M.Laine, T.Neuhaus, work in progress]

- standard plaquette gauge & O(a)-improved Wilson quarks
- quenched gauge field configurations
- on fine and large isotropic lattices
- $T = 0.7 - 1.4 T_c$
- 2 different lattice spacing
  - analysis of cut-off effects
  - continuum limit (in the future)
- both charm & bottom
  - tuned close to their physical masses

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$N_\sigma$</th>
<th>$N_\tau$</th>
<th>$T/T_c$</th>
<th># confs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.192</td>
<td>96</td>
<td>48</td>
<td>0.7</td>
<td>259</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>1.1</td>
<td></td>
<td>476</td>
</tr>
<tr>
<td></td>
<td>28</td>
<td>1.2</td>
<td></td>
<td>336</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>1.4</td>
<td></td>
<td>336</td>
</tr>
<tr>
<td>7.793</td>
<td>192</td>
<td>96</td>
<td>0.7</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>56</td>
<td>1.2</td>
<td></td>
<td>66</td>
</tr>
<tr>
<td></td>
<td>48</td>
<td>1.4</td>
<td></td>
<td>217</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a$ [fm]</th>
<th>$a^{-1}$ [GeV]</th>
<th>$\kappa_{\text{charm}}$</th>
<th>$\kappa_{\text{bottom}}$</th>
<th>$m_{J/\Psi}$ [GeV]</th>
<th>$m_\Upsilon$ [GeV]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.192</td>
<td>0.0190</td>
<td>10.4</td>
<td>0.13194</td>
<td>0.12257</td>
<td>3.105(3)</td>
<td>9.468(3)</td>
</tr>
<tr>
<td>7.793</td>
<td>0.00968</td>
<td>20.4</td>
<td>0.13221</td>
<td>0.12798</td>
<td>3.092(5)</td>
<td>9.431(5)</td>
</tr>
</tbody>
</table>

Experimental values: $m_{J/\Psi} = 3.096.916(11)$ GeV, $m_\Upsilon = 9.46030(26)$ GeV
Spatial correlation function and screening masses


Correlation functions along the **spatial direction**

\[ G(z, T) = \int dx dy \int_0^{1/T} d\tau \langle J(x, y, z, \tau) J(0, 0, 0, 0) \rangle \]

are related to the meson spectral function at **non-zero spatial momentum**

\[ G(z, T) = \int_{-\infty}^{\infty} dp_z e^{ip_z z} \int_0^\infty d\omega \frac{\sigma(\omega, p_z, T)}{\omega} \]

exponential decay defines **screening mass** \( M_{scr} \):

\[ G(z, T) \xrightarrow{z \gg 1/T} e^{-M_{scr} z} \]

**bound state contribution**

\[ \sigma(\omega, p_z, T) \sim \delta(\omega^2 - p_z^2 - M^2) \]

**high-T limit (non-interacting free limit)**

\[ \sigma(\omega, p_z, T) \sim \sigma_{free}(\omega, p_z, T) \]

\[ M_{scr} = M \]

indications for medium modifications/dissociation

\[ M_{scr} = 2\sqrt{(\pi T)^2 + m_c^2} \]
Spatial Correlation Function and Screening Masses

\[ m_{eff}(zT) = \log \left( \frac{G(zT)}{G((z + 1)T)} \right) \]
exponential decay defines screening mass $M_{\text{scr}}$:
$$G(z, T) \xrightarrow{z \gg 1/T} e^{-M_{\text{scr}}z}$$

**bound state contribution**
$$\sigma(\omega, p_z, T) \sim \delta(\omega^2 - p_z^2 - M^2)$$

**high-T limit (non-interacting free limit)**
$$\sigma(\omega, p_z, T) \sim \sigma_{\text{free}}(\omega, p_z, T)$$

$M_{\text{scr}} = M$

indications for medium modifications/dissociation

$M_{\text{scr}} = 2\sqrt{(\pi T)^2 + m_c^2}$
Spatial correlation function and screening masses

2+1 flavor HISQ with almost physical quark masses
48³ × 12 lattices with m_l = m_s/20 and physical m_s

“sş and sč possibly dissolve close to crossover temperature”
“cč in line with the sequential melting of charmonia states”

[A.Bazavov, F.Karsch, Y.Maezawa et al., PRD91 (2015) 054503]
Reconstructed correlation function

\[
G_{\text{rec}}(\tau, T; T') \equiv \int_{0}^{\infty} d\omega \rho(\omega, T') K(\omega, \tau, T)
\]

\[
\frac{G(\tau, T)}{G_{\text{rec}}(\tau, T; T')} \quad \text{equals to unity at all } \tau
\]

if the spectral function doesn’t vary with temperature

S. Datta et al., PRD 69 (2004) 094507

can be calculated directly from correlation function for suitable ratios of \( N'_{\tau} / N_{\tau} \) without knowledge of spectral function:

\[
\frac{\cosh[\omega(\tau - N_{\tau}/2)]}{\sinh[\omega N_{\tau}/2]} = \frac{N'_{\tau} - N_{\tau} + \tau}{\cosh[\omega(\tau' - N'_{\tau}/2)]} \sum_{\tau' = \tau; \Delta \tau' = N_{\tau}} m N'_{\tau} = m N_{\tau} \quad m = 1, 2, 3, \ldots
\]

\[
T = 1 / (N_{\tau} a)
\]

H.-T. Ding et al., PRD 86 (2012) 014509
Charmonium and Bottomonium correlators

\[ G_{\text{rec}}(\tau, T; T') \equiv \int_0^\infty d\omega \rho(\omega, T') K(\omega, \tau, T) \]

\[ G(\tau, 0.7T_c)/T^3 \]

\[ G_{\text{rec}}(\tau, 1.4T_c; 0.7T_c)/T^3 \]
Charmonium and Bottomonium correlators – S-wave channels

\[ G_{\text{rec}}(\tau, T; T') \equiv \int_{0}^{\infty} d\omega \rho(\omega, T') K(\omega, \tau, T) \]

different behavior in pseudo-scalar (left) and vector (right) channel

strong modification at large \( \tau \) in vector channel

stronger for charm compared to bottom

related to transport contribution
Charmonium and Bottomonium correlators – P-wave channels

\[ G_{\text{rec}}(\tau, T; T') \equiv \int_0^\infty d\omega \rho(\omega, T') K(\omega, \tau, T) \]

comparable behavior in scalar (left) and axial-vector (right) channel

strong modification at large \( \tau \) in both channels

stronger for bottom compared to charm

related to transport/constant contribution
Mid-point subtracted correlators – P-wave channels

$$G'(\tau) \equiv G(\tau) - G(1/2T)$$

Small $\omega$ region gives (almost) constant contribution to correlators

effectively removed by mid-point subtraction

Pseudo-scalar (left) and vector (right) very comparable

Need to understand cut-off effects and quark-mass effects
Outlook

work is still in progress

continuum extrapolation for the quarkonium correlators still needed
detailed analysis of the systematic uncertainties

Extract spectral properties (on continuum extrapolated correlators) by
- comparing to perturbation theory
- Fits using Ansätze for the spectral function
- Bayesian techniques to extract the spectral function

Final goal:
understand the temperature and quark mass dependence of
heavy quark diffusion coefficient
dissociation temperatures for different states