

Spectral and transport properties from Lattice QCD

From correlators to spectral functions – case studies in the quenched approximation

- I) Light quark vector meson spectral function
- thermal photon, dilepton rates, electrical conductivity
[J.Ghiglieri, OK, M.Laine, F.Meyer, PRD94(2016)016005]
[H-T.Ding, F.Meyer, OK, PRD94 (2016) 034504]
- II) Thermal quarkonium physics in the pseudoscalar channel
[Y. Burnier, H.-T. Ding, OK et al. JHEP11 (2017) 206]
- III) Thermal quarkonium physics in the vector channel
[A.-L. Kruse, H.T. Shu, OK et al. work in progress]
- IV) Heavy quark momentum diffusion coefficient
[A.Francis, OK, et al., PRD92(2015)116003]
- V) Correlation functions under gradient flow – towards full QCD
[L. Altenkort, H.T. Shu, OK et al., work in progress]

Spectral and transport properties in the QGP

Thermal dilepton rate

$$\frac{dW}{d\omega d^3p} = \frac{5\alpha^2}{54\pi^3} \frac{1}{\omega^2 (e^{\omega/T} - 1)} \rho_V(\omega, \mathbf{T})$$

Thermal photon rate

$$\omega \frac{dN_\gamma}{d^4x d^3q} = \frac{5\alpha}{6\pi^2} \frac{1}{e^{\omega/T} - 1} \rho_V(\omega = |\vec{k}|, T)$$

Transport coefficients are encoded
in the same spectral function

→ Kubo formulae

Diffusion coefficients:

$$DT = \frac{T}{2\chi_q} \lim_{\omega \rightarrow 0} \frac{\rho_{ii}(\omega)}{\omega}$$

On the lattice only correlation functions can be calculated

$$G(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, \vec{p}, T) K(\tau, \omega, T) \quad \rightarrow \text{spectral reconstruction required}$$

$$K(\tau, \omega, T) = \frac{\cosh(\omega(\tau - \frac{1}{2T}))}{\sinh(\frac{\omega}{2T})}$$

In this talk: continuum extrapolated lattice correlation functions compared to perturbation theory

for a comparison of Bayesian and stochastic reconstructions of spectral functions see

[H.-T. Ding, OK, S. Mukherjee, H. Ohno, H.-T. Shu, PRD97(2018)094503]

$$G(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, \vec{p}, T) K(\tau, \omega, T)$$

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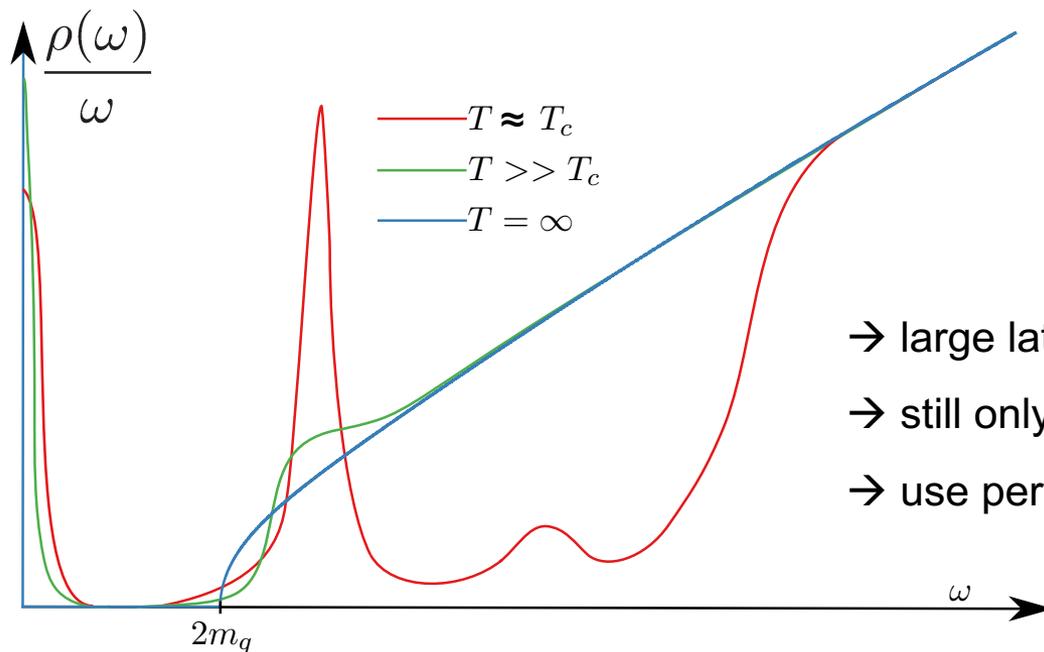
Different contributions and scales enter

in the spectral function

- **continuum at large frequencies**
- **possible bound states at intermediate frequencies**
- **transport contributions at small frequencies**
- **in addition cut-off effects on the lattice**

Spectral functions in the QGP

notoriously difficult to extract from correlation functions



$$G_{\mu\nu}(\tau, \vec{x}) = \langle J_\mu(\tau, \vec{x}) J_\nu^\dagger(0, \vec{0}) \rangle$$

$$J_\mu(\tau, \vec{x}) = 2\kappa Z_V \bar{\psi}(\tau, \vec{x}) \Gamma_\mu \psi(\tau, \vec{x})$$

- large lattices and continuum extrapolation needed
- still only possible in the quenched approximation
- use perturbation theory to constrain the UV behavior

(narrow) transport peak at small ω : $\rho(\omega \ll T) \simeq 2\chi_{00} \frac{T}{M} \frac{\omega\eta}{\omega^2 + \eta^2}, \quad \eta = \frac{T}{MD}$

Photonrate directly related to vector spectral function (at finite momentum):

$$\omega \frac{dN_\gamma}{d^4x d^3q} = \frac{5\alpha}{6\pi^2} \frac{1}{e^{\omega/T} - 1} \rho_V(\omega = |\vec{k}|, T)$$

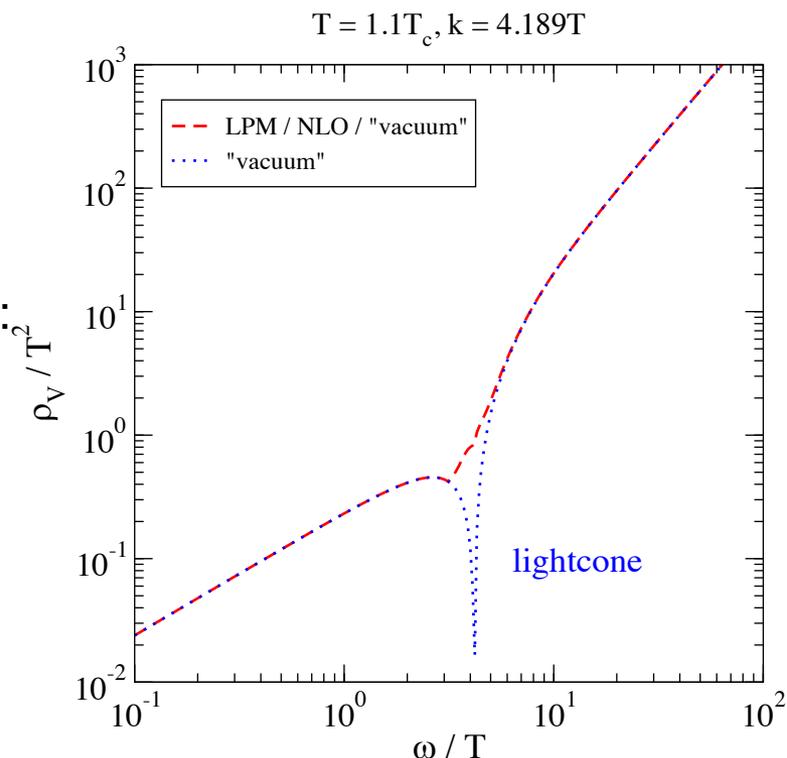
pQCD spectral function used to constrain the UV

interpolation between different (perturbative) regimes:

$3T < \omega < 10T$: [J.Ghiglieri, G.D.Moore, JHEP 1412 (2014) 029]

$\omega > 10T$: [I. Ghisoiu, M.Laine, JHEP 10 (2014) 84]

$\omega \gg 10T$: [M.Laine, JHEP 1311 (2013) 120]



vector spectral function in the hydrodynamic regime for $\omega, k \lesssim \alpha_s^2 T$:

$$\frac{\rho_V(\omega, \mathbf{k})}{\omega} = \left(\frac{\omega^2 - k^2}{\omega^2 + D^2 k^4} + 2 \right) \chi_q D$$

we model the infrared behavior assuming smoothness at the light cone and fit to continuum extrapolated lattice correlators

Continuum lattice correlators vs. perturbation theory

[J.Ghiglieri, OK, M.Laine, F.Meyer, PRD94(2016)016005]

Fixed aspect ratio used to perform continuum extrapolation at finite p

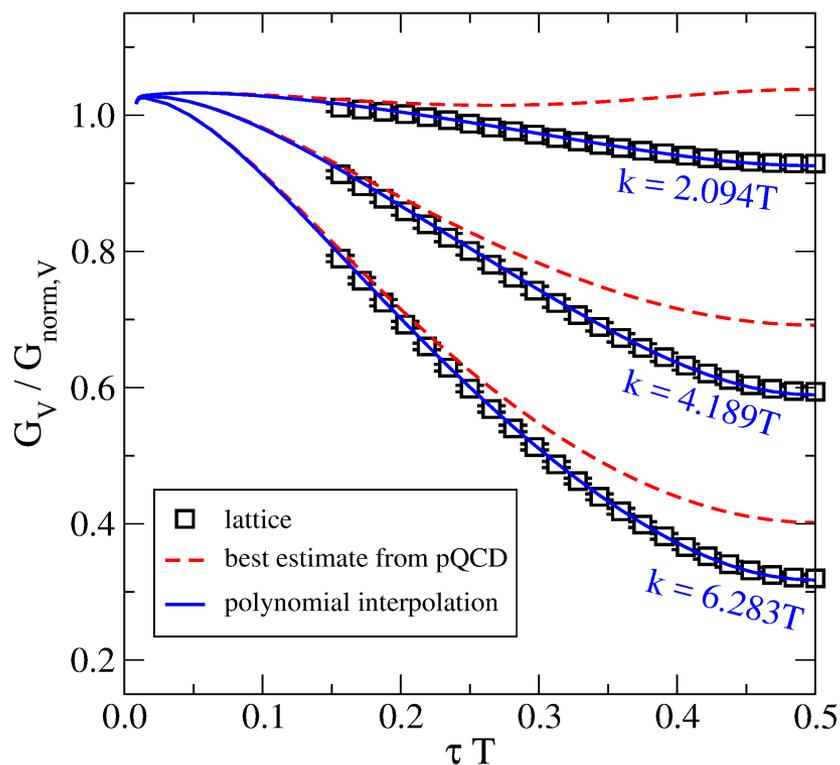
$$\frac{\vec{p}}{T} = 2\pi \vec{k} \frac{N_\tau}{N_\sigma}$$

use perturbation theory at large ω

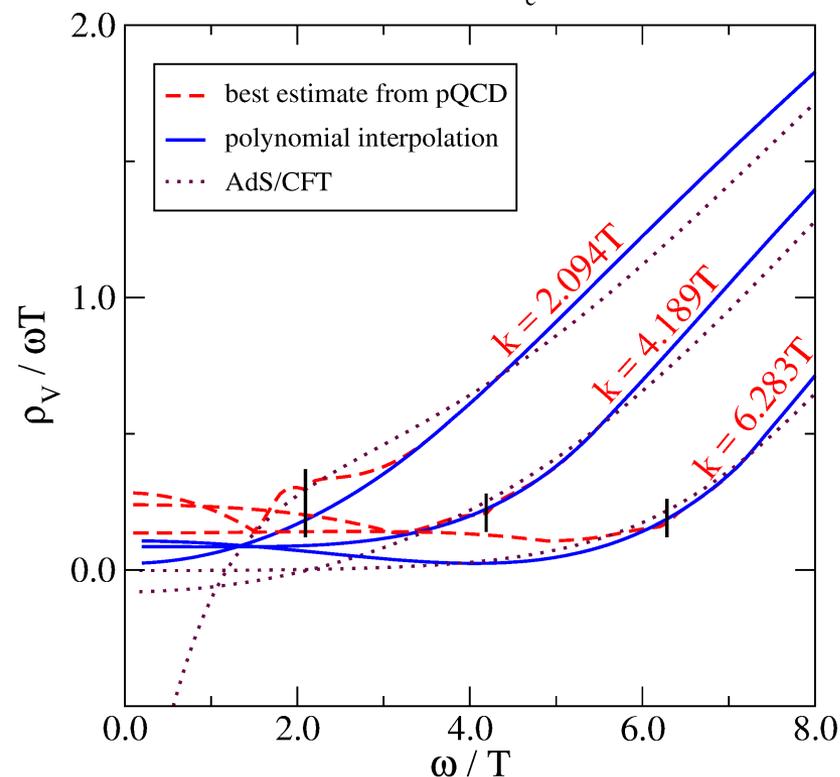
and fit a polynomial at small ω to extract the spectral function

β_0	$N_s^3 \times N_\tau$	confs	$T\sqrt{t_0}$	$T/T_c _{t_0}$	Tr_0	$T/T_c _{r_0}$
7.192	$96^3 \times 32$	314	0.2796	1.12	0.816	1.09
7.544	$144^3 \times 48$	358	0.2843	1.14	0.817	1.10
7.793	$192^3 \times 64$	242	0.2862	1.15	0.813	1.09
7.192	$96^3 \times 28$	232	0.3195	1.28	0.933	1.25
7.544	$144^3 \times 42$	417	0.3249	1.31	0.934	1.25
7.793	$192^3 \times 56$	273	0.3271	1.31	0.929	1.25

$T = 1.1T_c$



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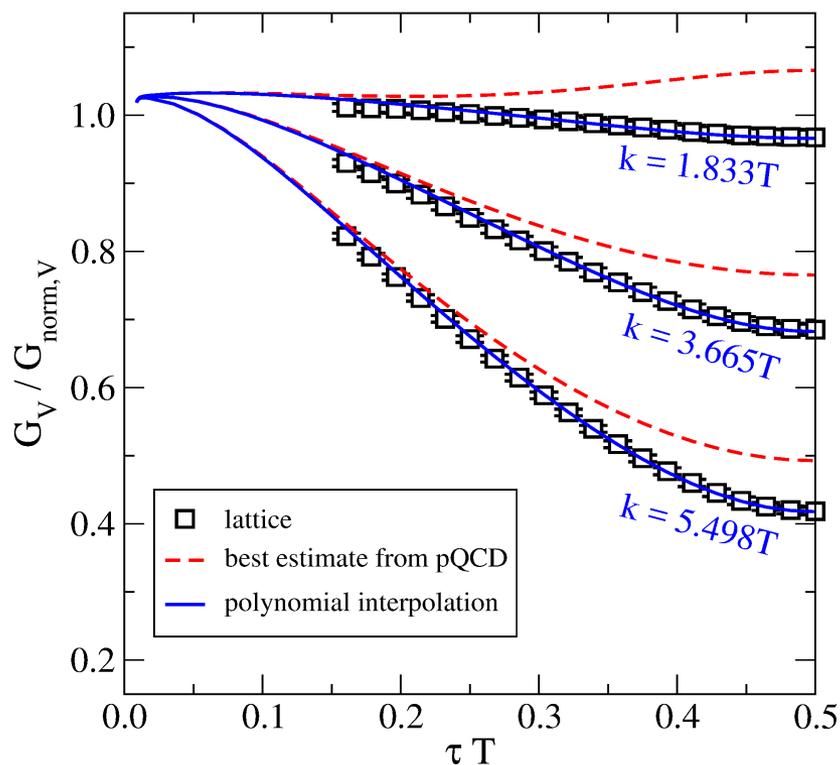
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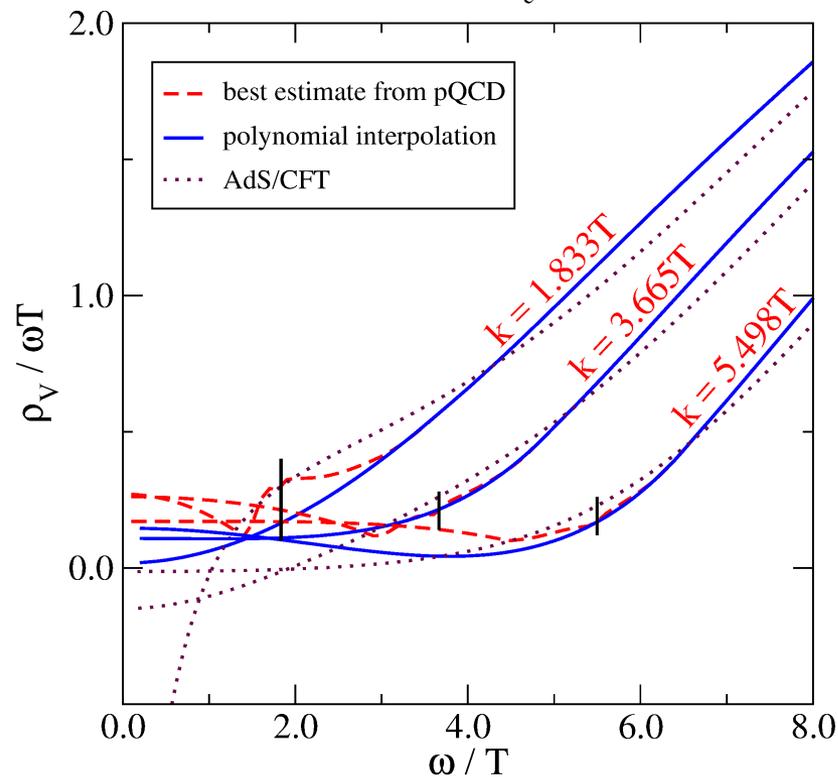
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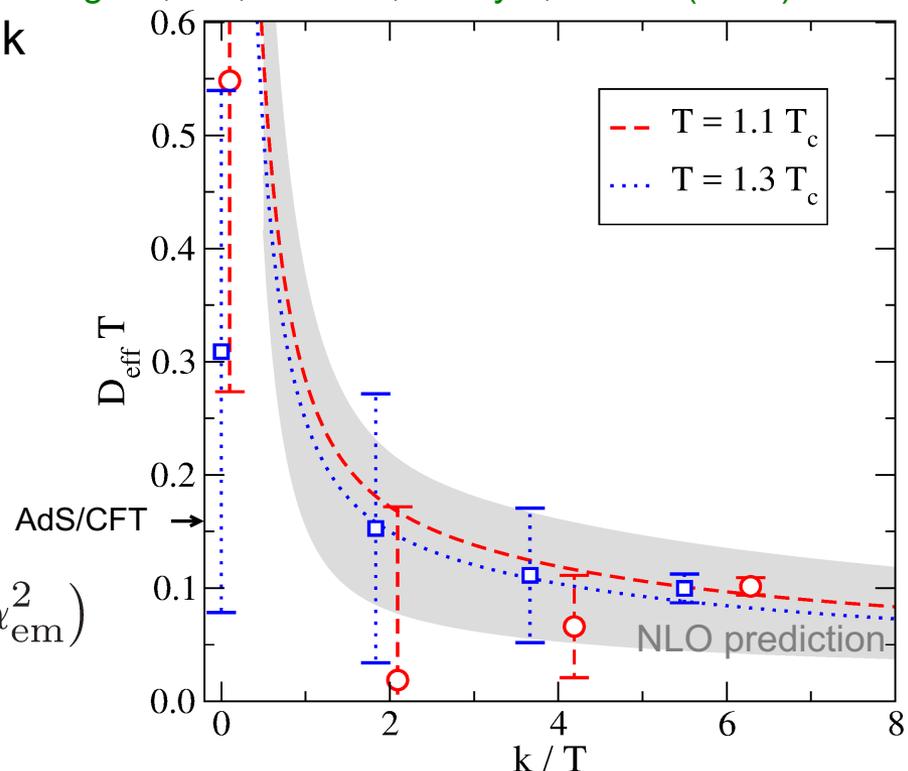
[J.Ghiglieri, OK, M.Laine, F.Meyer, PRD94(2016)016005]

The spectral function at the photon point $\omega = k$

$$D_{\text{eff}}(k) \equiv \begin{cases} \frac{\rho_v(k, \mathbf{k})}{2\chi_q k} & , \quad k > 0 \\ \lim_{\omega \rightarrow 0^+} \frac{\rho^{ii}(\omega, \mathbf{0})}{3\chi_q \omega} & , \quad k = 0 \end{cases} .$$

can be used to calculate the photon rate

$$\frac{d\Gamma_\gamma(\mathbf{k})}{d^3\mathbf{k}} = \frac{2\alpha_{\text{em}}\chi_q}{3\pi^2} n_B(k) D_{\text{eff}}(k) + \mathcal{O}(\alpha_{\text{em}}^2)$$



becomes more perturbative at larger k , approaching the NLO prediction (valid for $k \gg gT$)

[J. Ghiglieri, G.D. Moore, JHEP12 (2014) 029]

but non-perturbative for $k/T < 3$

Electrical conductivity obtained in the limit $k \rightarrow 0$ between the results from

AdS/CFT: $DT = \frac{1}{2\pi}$

[G.Policastro, D.T.Son, A.O.Starinets, JHEP09(2002)043]

LO perturbation theory [Arnold, Moore Yaffe, JHEP 05 (2003)]

using lattice value for χ_q/T^2 : $DT = 2.9 - 3.1$

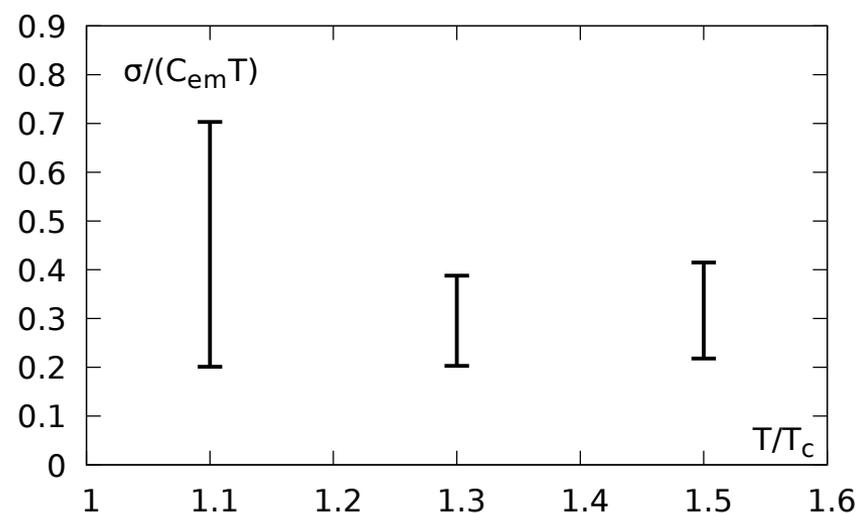
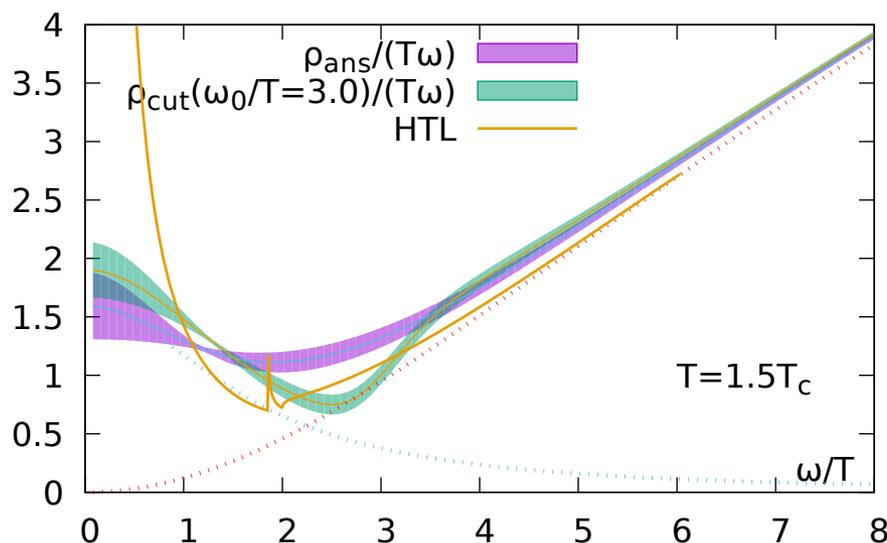
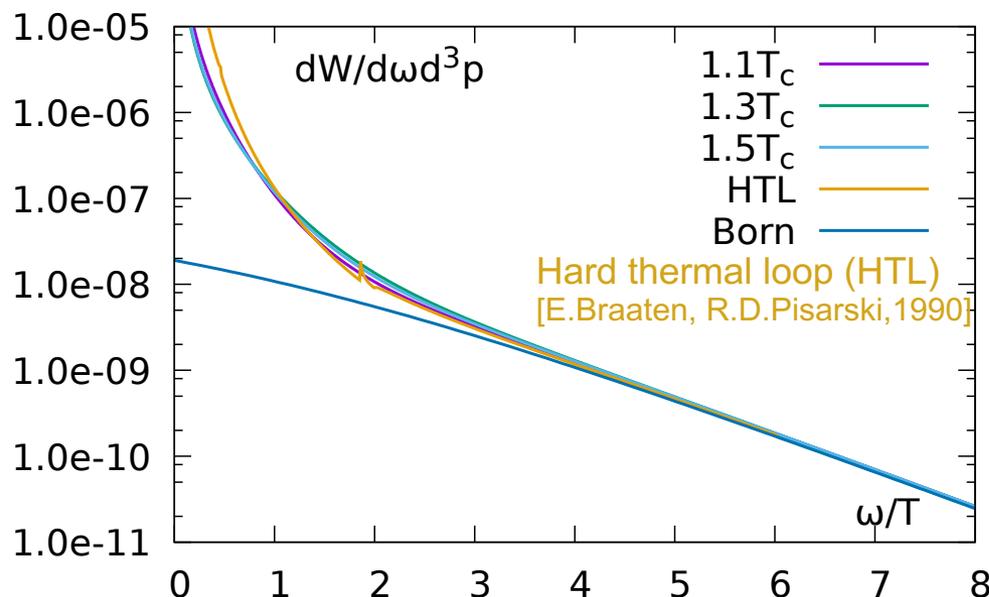
[H-T.Ding, F.Meyer, OK, PRD94(2016)034504]

Dileptonrate directly related to vector spectral function:

$$\frac{dW}{d\omega d^3p} = \frac{5\alpha^2}{54\pi^3} \frac{1}{\omega^2 (e^{\omega/T} - 1)} \rho_{\mathbf{V}}(\omega, \mathbf{T})$$

**continuum estimate for the
of the electrical conductivity**

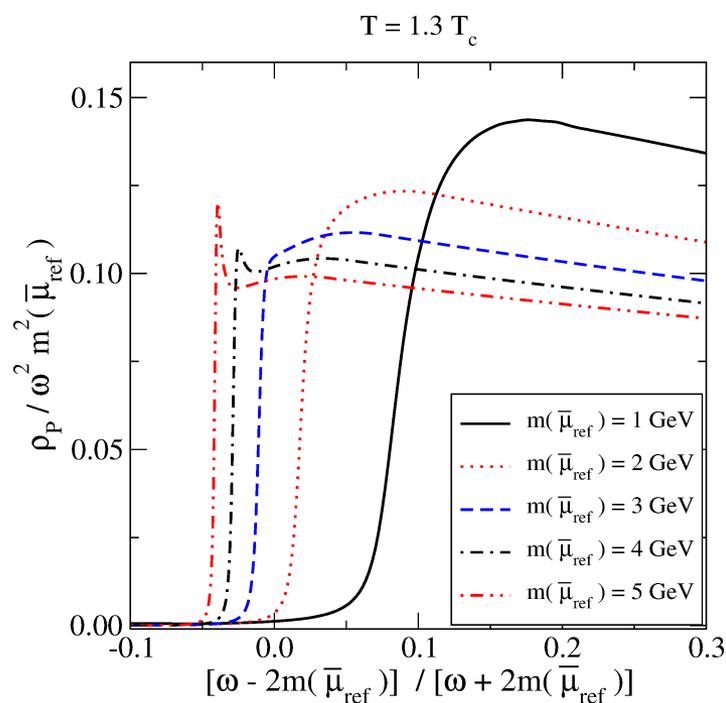
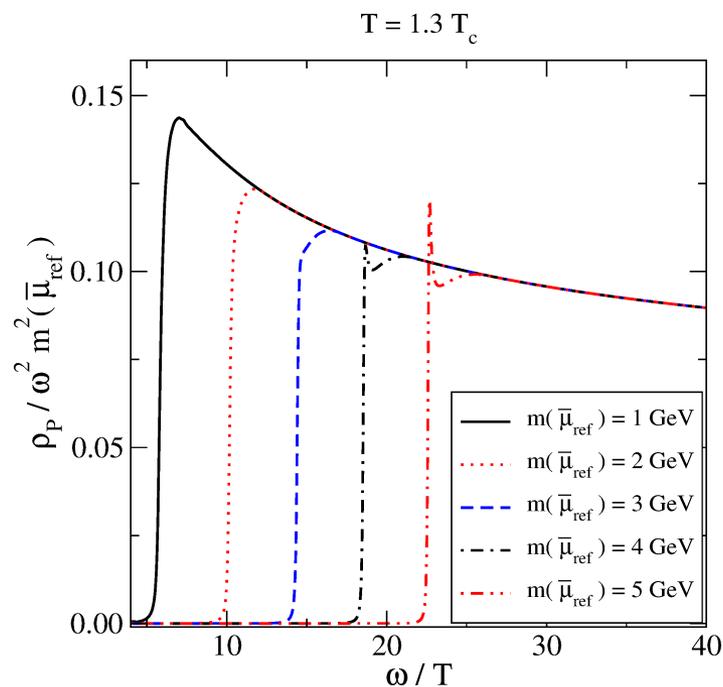
$$\frac{\sigma_{el}}{C_{em}T} = \frac{1}{6} \lim_{\omega \rightarrow 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T}$$



Using **continuum extrapolated correlation functions** from Lattice QCD

$$G_P(\tau) \equiv M_B^2 \int_{\vec{x}} \left\langle (\bar{\psi} i \gamma_5 \psi)(\tau, \vec{x}) (\bar{\psi} i \gamma_5 \psi)(0, \vec{0}) \right\rangle_c, \quad 0 < \tau < \frac{1}{T},$$

and best knowledge on the spectral function from **perturbation theory and pNRQCD** interpolated between different regimes



we will focus on the pseudo-scalar channel (no transport contribution in this channel)

quenched SU(3) gauge configurations (separated by 500 updates)

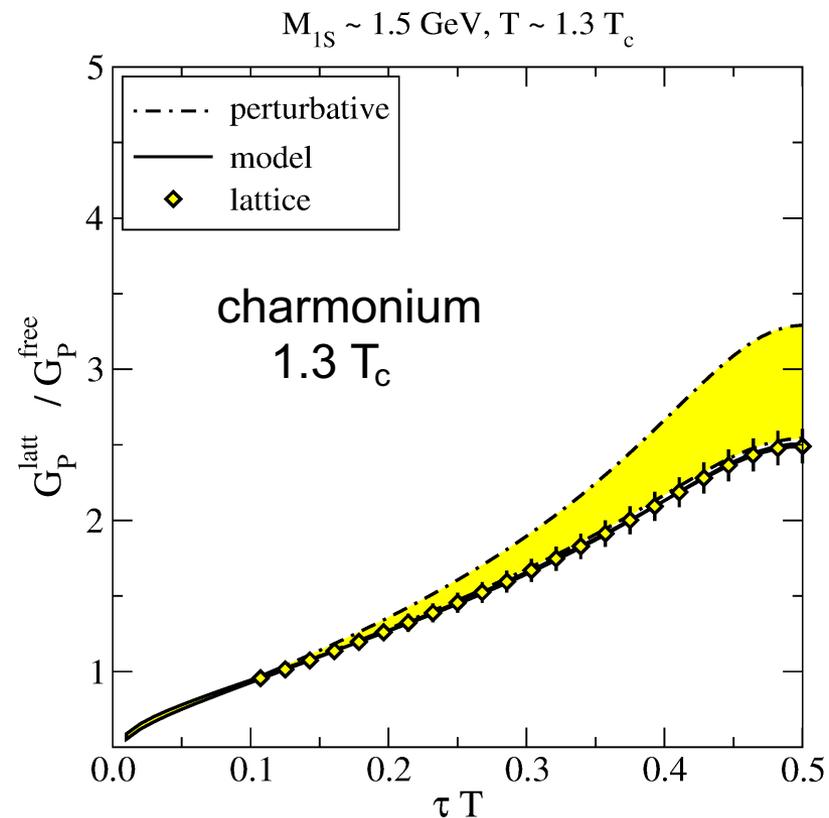
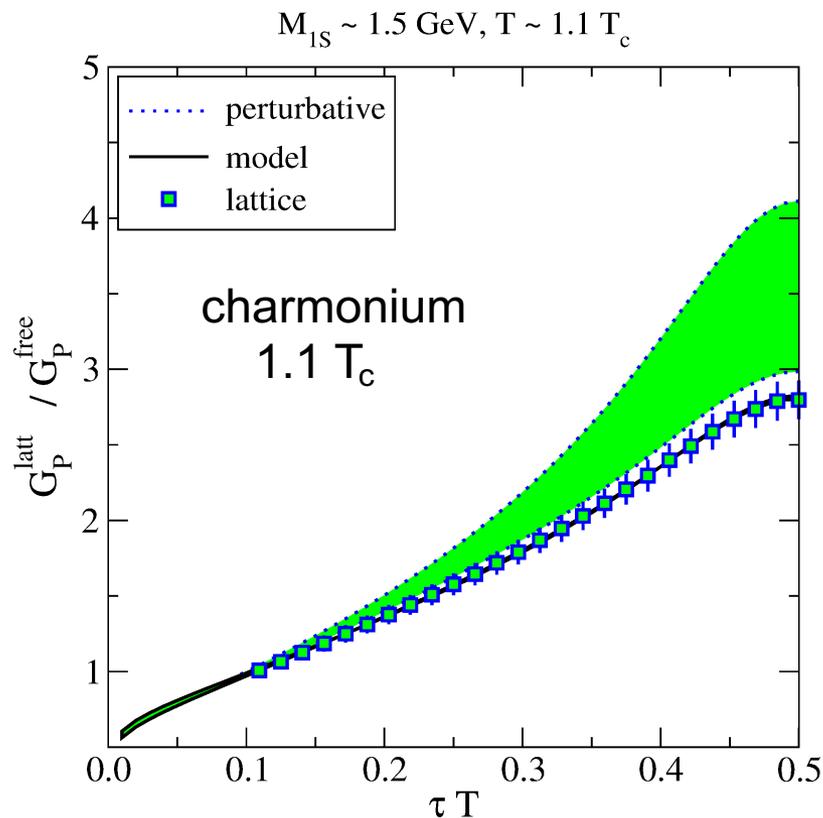
non-perturbatively O(a) clover improved Wilson fermion valence quarks

6 quark masses between charm and bottom \rightarrow interpolate to physical c and b mass

β	N_s	N_τ	confs	r_0/a	T/T_c	c_{SW}	κ_c	κ	$\frac{m^2(1/a)}{m^2(\bar{\mu}_{ref})}$
7.192	96	48	237	26.6	0.74	1.367261	0.13442	0.12257, 0.12800, 0.13000, 0.13100, 0.13150, 0.13194	0.6442
		32	476		1.12				
		28	336		1.27				
		24	336		1.49				
		16	237		2.23				
7.394	120	60	171	33.8	0.76	1.345109	0.13408	0.124772, 0.12900, 0.13100, 0.13150, 0.132008, 0.132245	0.6172
		40	141		1.13				
		30	247		1.51				
		20	226		2.27				
7.544	144	72	221	40.4	0.75	1.330868	0.13384	0.12641, 0.12950, 0.13100, 0.13180, 0.13220, 0.13236	0.5988
		48	462		1.13				
		42	660		1.29				
		36	288		1.51				
		24	237		2.26				
7.793	192	96	224	54.1	0.76	1.310381	0.13347	0.12798, 0.13019, 0.13125, 0.13181, 0.13209, 0.13221	0.5715
		64	249		1.13				
		56	190		1.30				
		48	210		1.51				
		32	235		2.27				

In this talk: only results based on continuum extrapolated correlation functions

See [Y. Burnier, H.-T. Ding, OK et al. JHEP11 (2017) 206] for more details



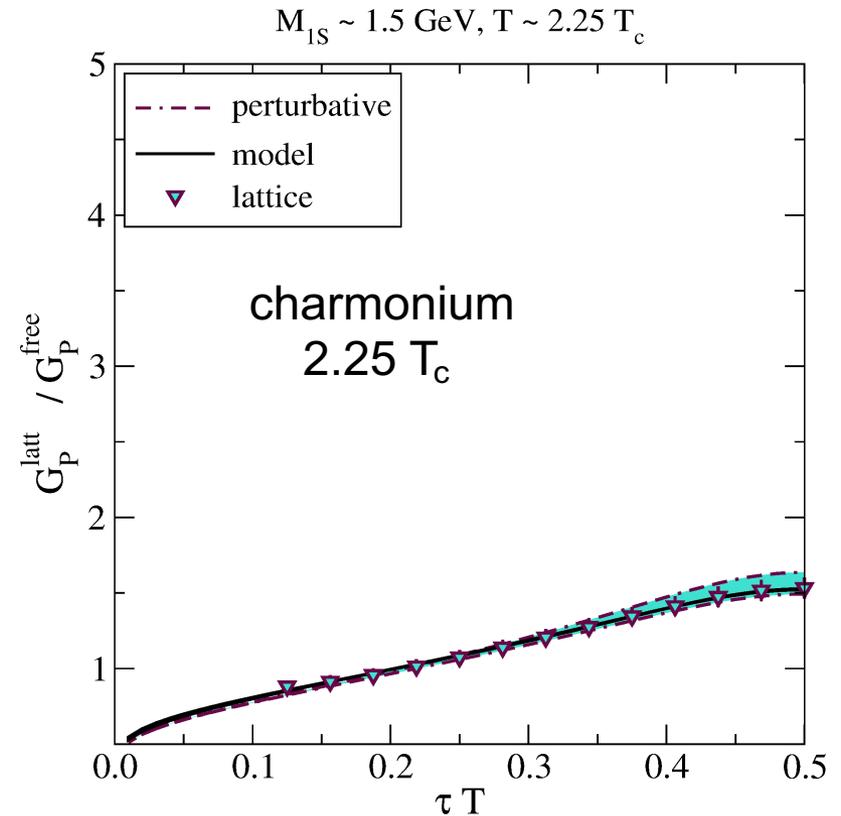
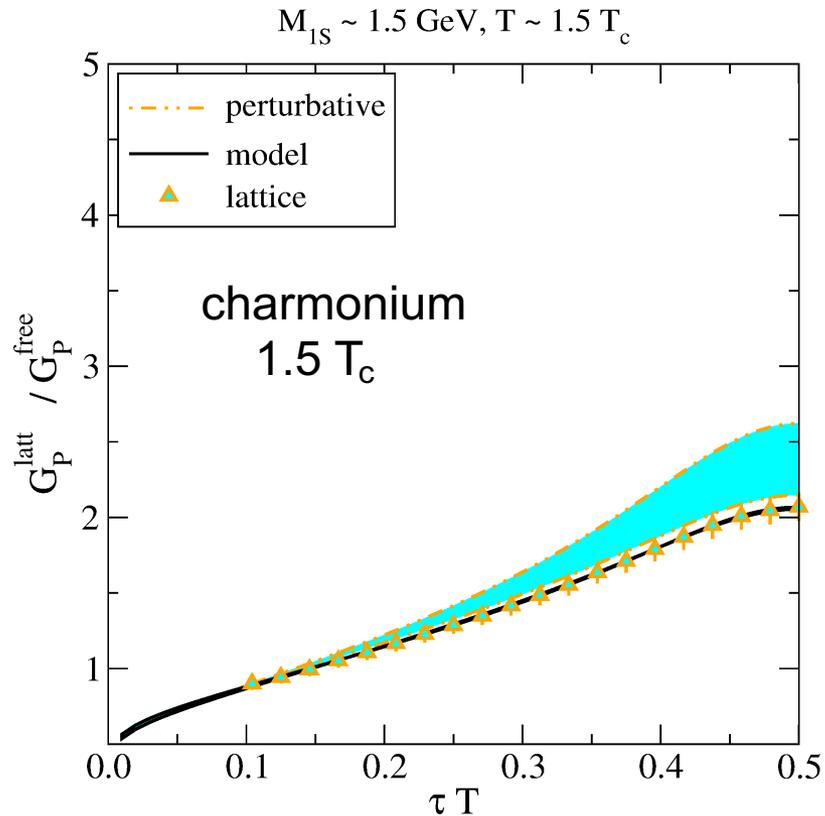
differences between lattice and perturbation theory may have a simple explanation

A: uncertainties related to the perturbative renormalization factors

B: non-perturbative mass shifts

$$\rho_P^{\text{model}}(\omega) \equiv A \rho_P^{\text{pert}}(\omega - B) .$$

→ continuum lattice data well described by this model with $\chi^2/\text{d.o.f} < 1$



differences between lattice and perturbation theory may have a simple explanation

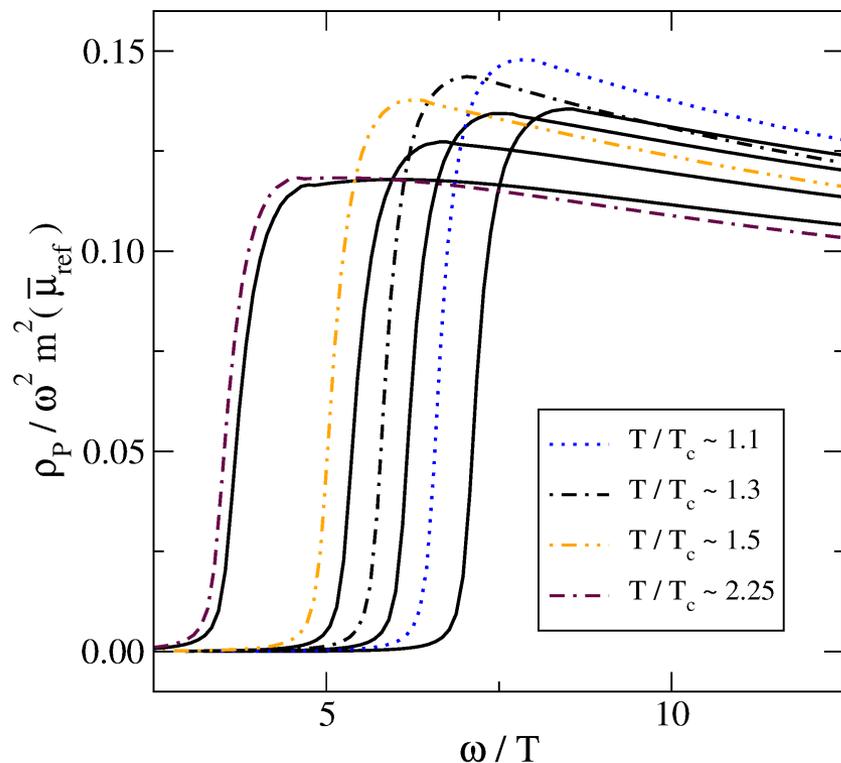
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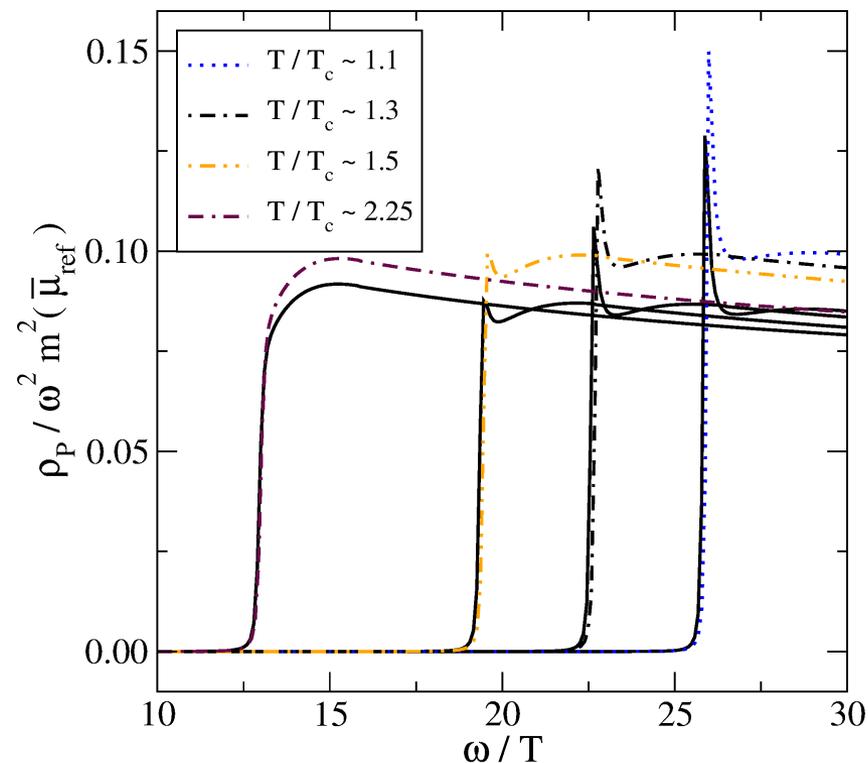
[Y. Burnier, H.-T. Ding, OK et al. JHEP11 (2017) 206]

charmonium: $m(\bar{\mu}_{\text{ref}}) = 1 \text{ GeV}$ **charmonium:**

no resonance peaks are needed for representing the lattice data even for $1.1 T_c$
 modest threshold enhancement sufficient in the analyzed temperature region

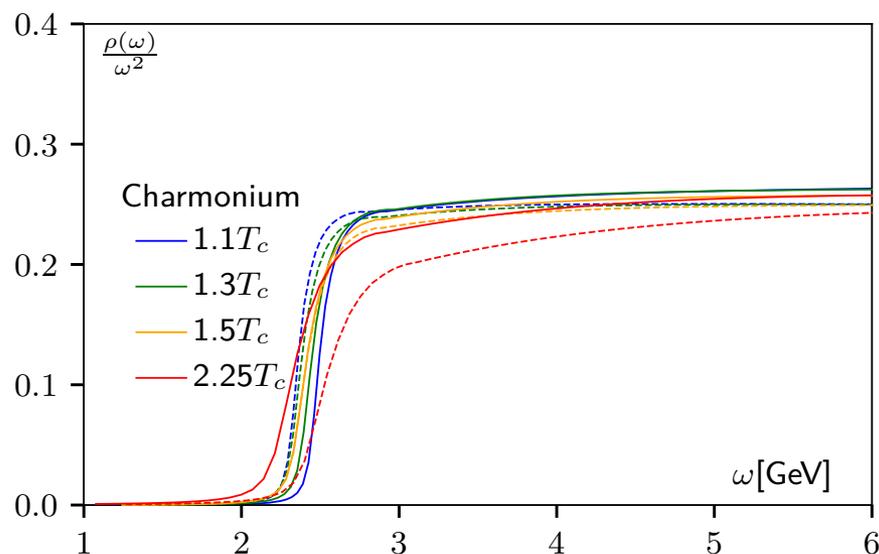
bottomonium:

thermally broadened resonance peak
 present up to temperatures around $1.5 T_c$

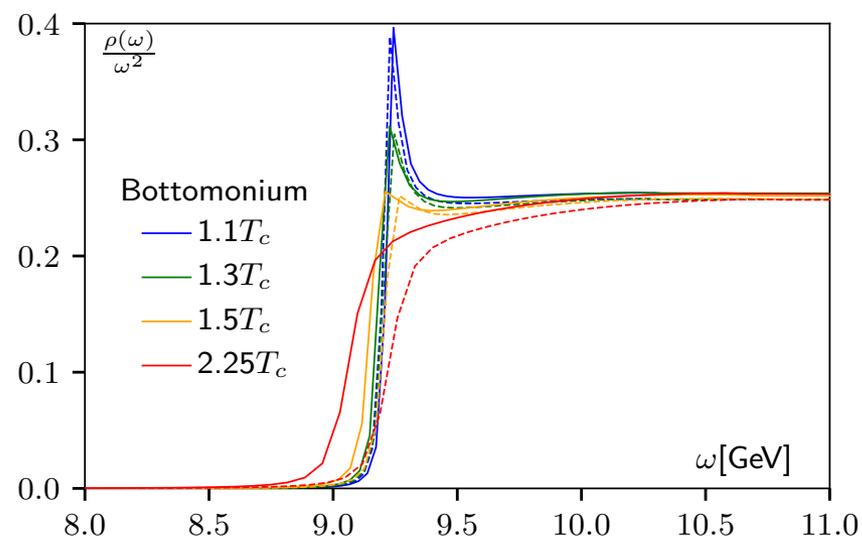
bottomonium: $m(\bar{\mu}_{\text{ref}}) = 5 \text{ GeV}$ 

[H.T. Ding, O. Kaczmarek, A.-L. Lorenz, R. Larsen, Swagato Mukherjee, H. Ohno, H. Sandmeyer, H.-T. Shu, paper in preparation]

charmonium:



bottomonium:



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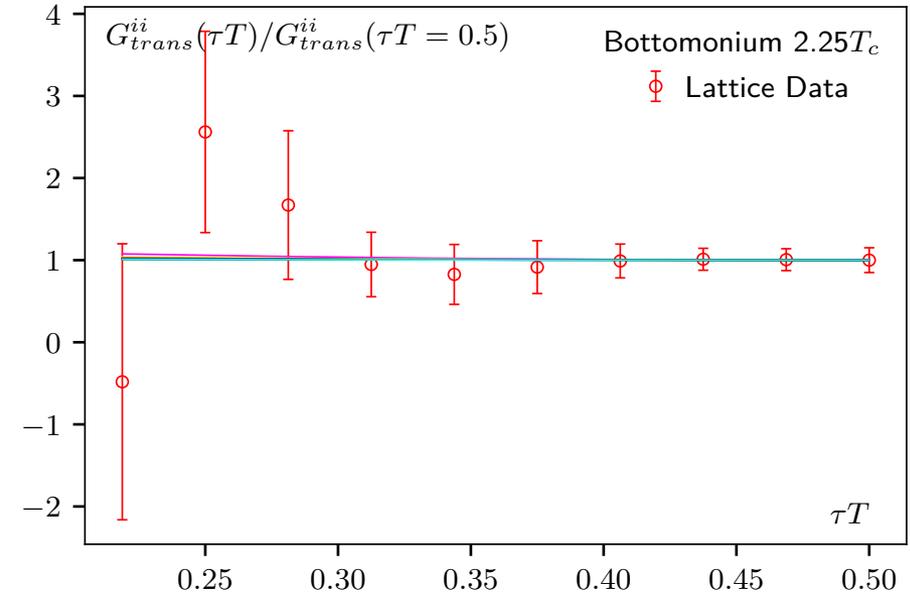
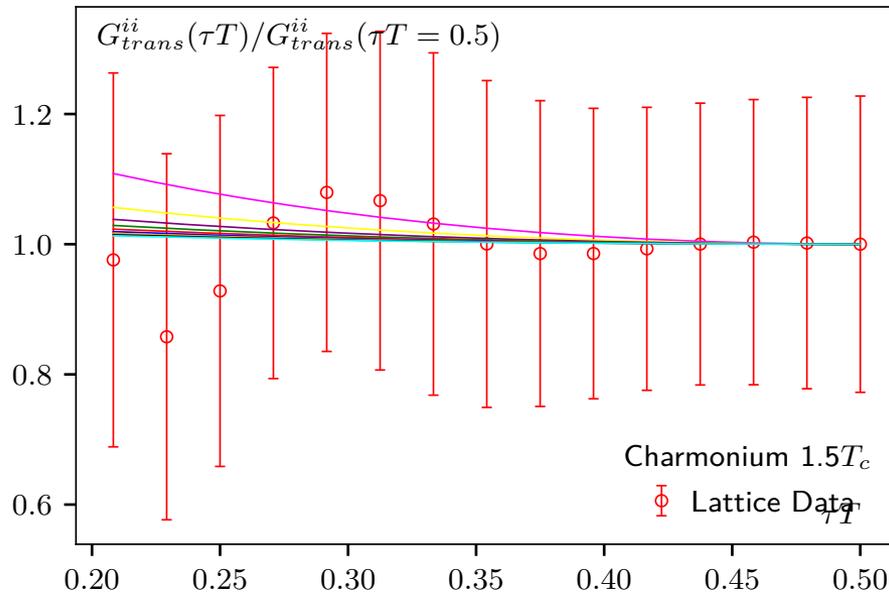
bottomonium:

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transport contribution in the vector channel: $\rho(\omega \ll T) \simeq 2\chi_{00} \frac{T}{M} \frac{\omega\eta}{\omega^2 + \eta^2}$, $\eta = \frac{T}{MD}$

charmonium:

bottomonium:



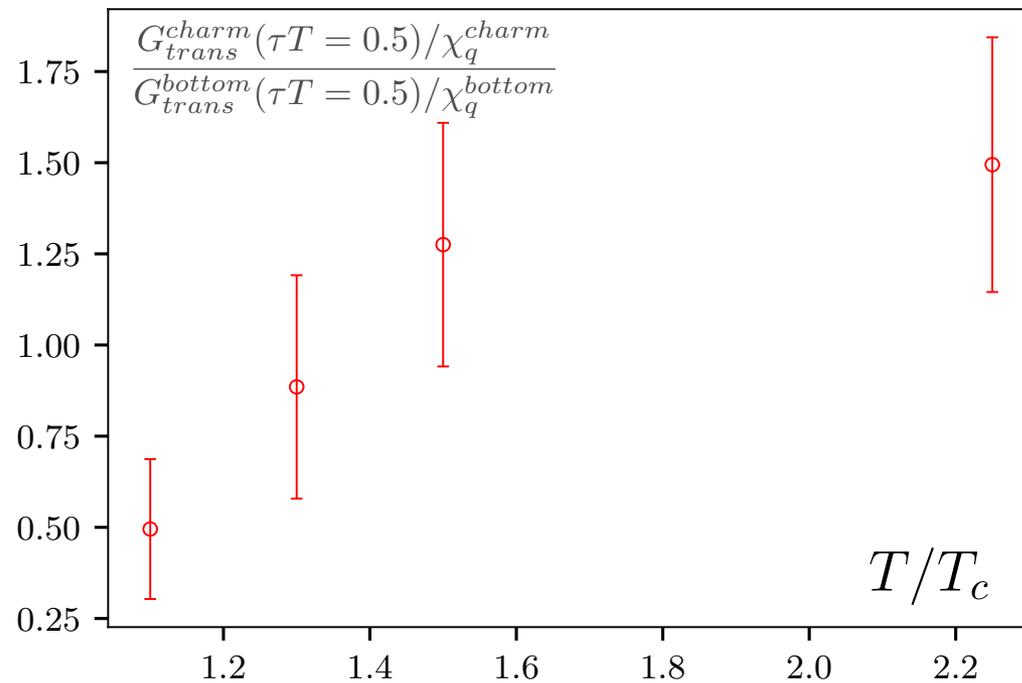
varying $2\pi DT$ between 1 and 9

- only small curvature due to transport contribution within our errors
- hard to determine transport coefficients with present data
- easier in the heavy quark mass limit

transport contribution in the vector channel: $\rho(\omega \ll T) \simeq 2\chi_{00} \frac{T}{M} \frac{\omega\eta}{\omega^2 + \eta^2}, \quad \eta = \frac{T}{MD}$

ratio of transport contributions to the correlator for charm and bottom:

$$\frac{G_{trans}^{charm} / \chi_q^{charm} / T}{G_{trans}^{bottom} / \chi_q^{bottom} / T} \approx \frac{M_{bottom}}{M_{charm}} \frac{\tan^{-1}(T/\eta^{charm})}{\tan^{-1}(T/\eta^{bottom})}$$



using $M_{bottom}/M_{charm} \approx 3$:

$$\frac{\tan^{-1}(T/\eta^{charm})}{\tan^{-1}(T/\eta^{bottom})} < 1$$

$$\rightarrow \eta^{charm} > \eta^{bottom}$$

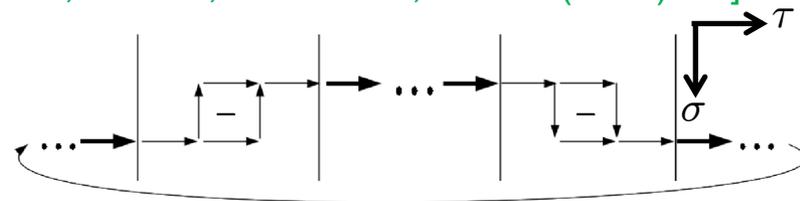
Heavy Quark Effective Theory (HQET) in the large quark mass limit

for a single quark in medium

leads to a (pure gluonic) “color-electric correlator”

[J.Casalderrey-Solana, D.Teaney, PRD74(2006)085012,
S.Caron-Huot,M.Laine,G.D. Moore,JHEP04(2009)053]

$$G_E(\tau) \equiv -\frac{1}{3} \sum_{i=1}^3 \frac{\left\langle \text{Re Tr} \left[U\left(\frac{1}{T}; \tau\right) g E_i(\tau, \mathbf{0}) U(\tau; 0) g E_i(0, \mathbf{0}) \right] \right\rangle}{\left\langle \text{Re Tr} \left[U\left(\frac{1}{T}; 0\right) \right] \right\rangle}$$



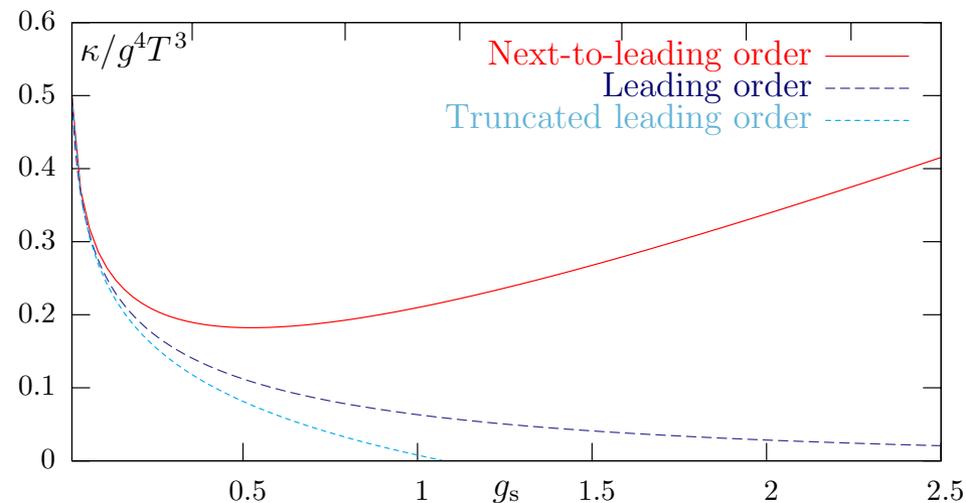
Smooth limit expected from NLO PT

[Caron-Huot, M.Laine, G.Moore, JHEP 0904 (2009) 053]

$$\kappa = \lim_{\omega \rightarrow 0} \frac{2T \rho_E(\omega)}{\omega}$$

NLO perturbative calculation:

[Caron-Huot, G. Moore, JHEP 0802 (2008) 081]



→ large correction towards strong interactions

→ non-perturbative lattice methods required

[A.Francis, OK, M.Laine, T.Neuhaus, H.Ohno, PRD92(2015)116003]

Quenched Lattice QCD on large and fine isotropic lattices at $T \simeq 1.5 T_c$

- standard Wilson gauge action
- algorithmic improvements to enhance signal/noise ratio
- fixed aspect ratio $N_s/N_t = 4$, i.e. fixed physical volume $(2\text{fm})^3$
- perform the continuum limit, $a \rightarrow 0 \leftrightarrow N_t \rightarrow \infty$
- determine κ in the continuum using an Ansatz for the spectral fct. $\rho(\omega)$
- scale setting using r_0 and t_0 scale [A.Francis,OK,M.Laine, T.Neuhaus, H.Ohno, PRD91(2015)096002]
- multilevel combined with link-integration techniques to improve the signal

β_0	$N_s^3 \times N_\tau$	confs	$T\sqrt{t_0}^{(\text{imp})}$	$T/T_c _{t_0}^{(\text{imp})}$	$T\sqrt{t_0}^{(\text{clov})}$	$T/T_c _{t_0}^{(\text{clov})}$	Tr_0	$T/T_c _{r_0}$
6.872	$64^3 \times 16$	172	0.3770	1.52	0.3805	1.53	1.116	1.50
7.035	$80^3 \times 20$	180	0.3693	1.48	0.3739	1.50	1.086	1.46
7.192	$96^3 \times 24$	160	0.3728	1.50	0.3790	1.52	1.089	1.46
7.544	$144^3 \times 36$	693	0.3791	1.52	0.3896	1.57	1.089	1.46
7.793	$192^3 \times 48$	223	0.3816	1.53	0.3955	1.59	1.084	1.45

similar studies by [Banerjee,Datta,Gavai,Majumdar, PRD85(2012)014510]
and [H.B.Meyer, New J.Phys.13(2011)035008]

[A.Francis, OK et al., PRD92(2015)116003]

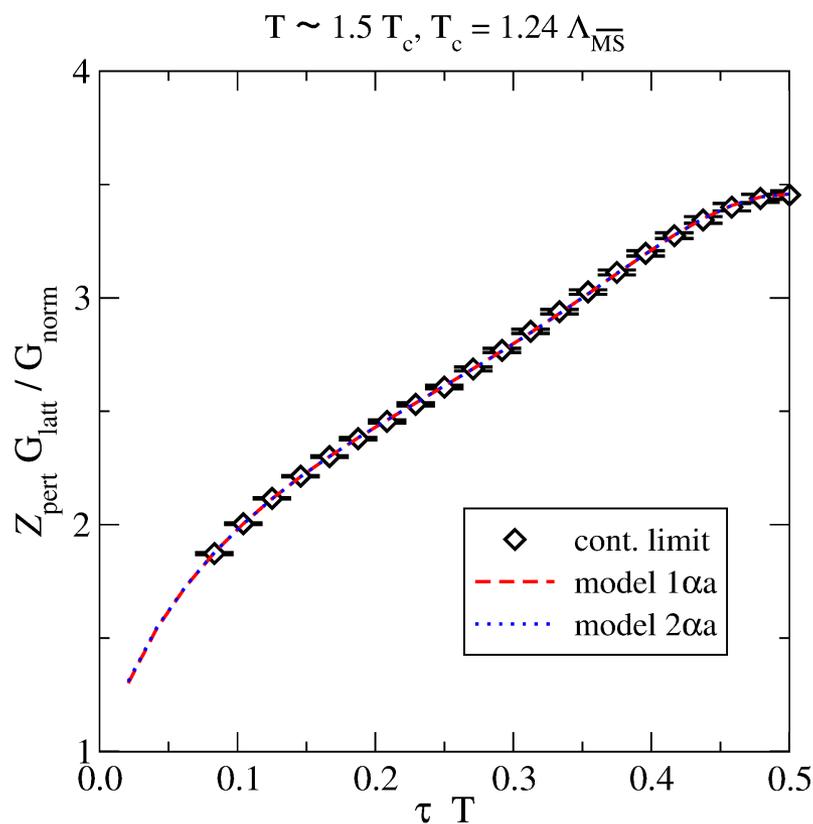
Using continuum extrapolated lattice correlators

Spectral function models with correct asymptotic behavior

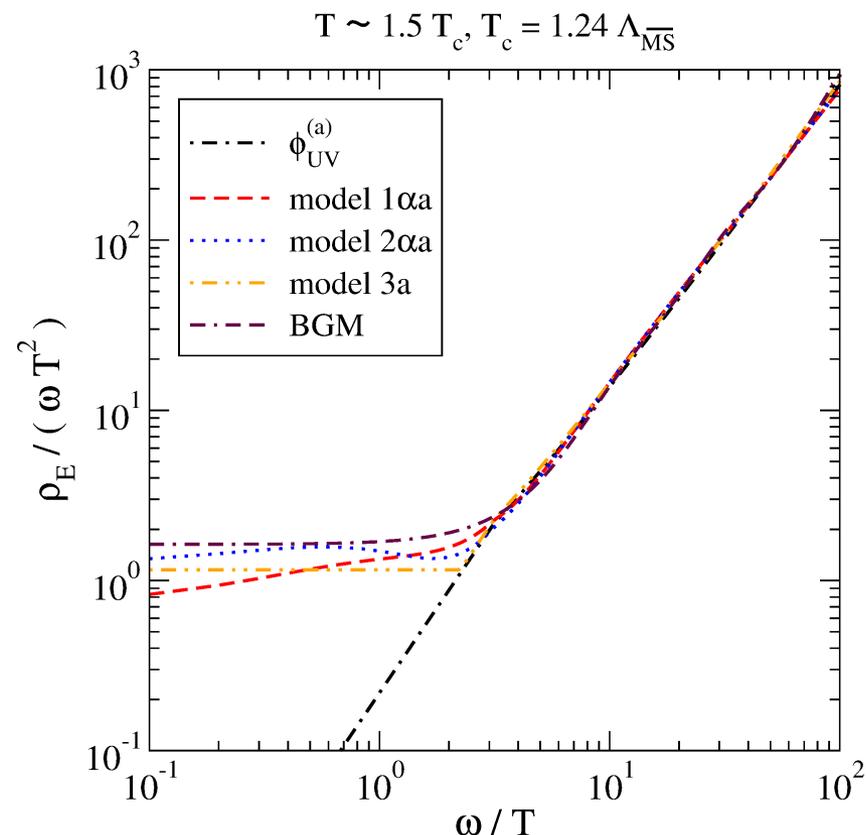
modeling corrections to ρ_{IR} by a power series in ω

$$\rho_{UV}(\omega) = \frac{g^2(\bar{\mu}_\omega) C_F \omega^3}{6\pi}$$

$$\rho_{IR}(\omega) = \frac{\kappa\omega}{2T}$$

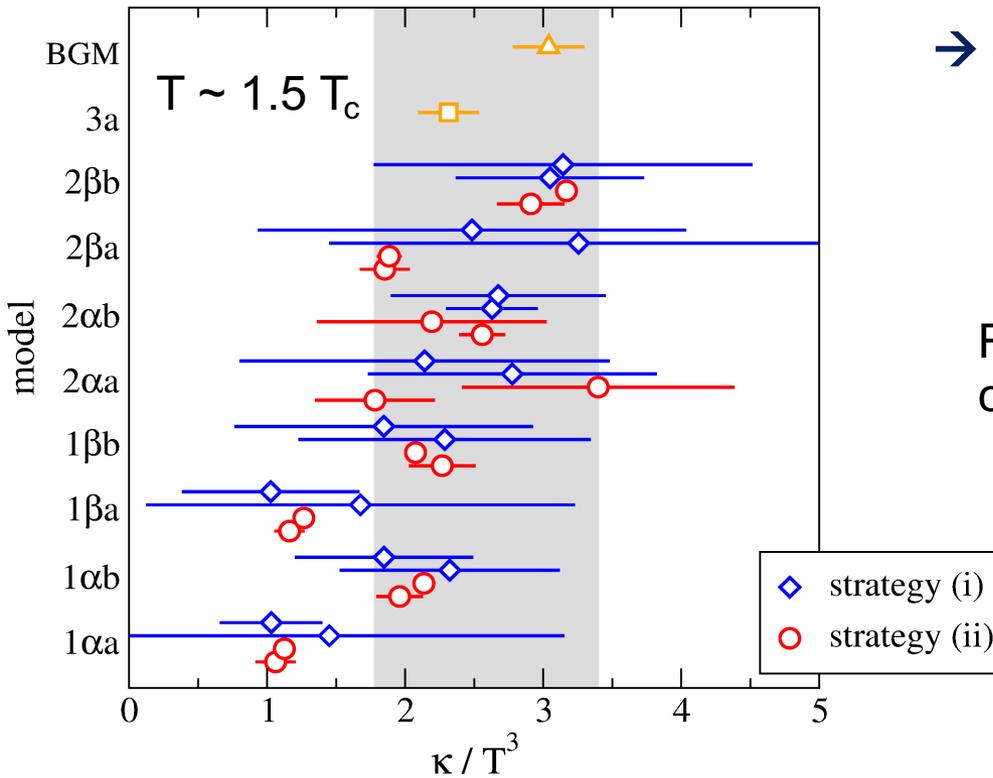


$$G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh\left(\frac{1}{2} - \tau T\right) \frac{\omega}{T}}{\sinh \frac{\omega}{2T}}$$



$$\kappa / T^3 = \lim_{\omega \rightarrow 0} \frac{2T \rho_E(\omega)}{\omega}$$

[A.Francis, OK et al., PRD92(2015)116003]



Detailed analysis of systematic uncertainties

→ **continuum estimate of κ** :

$$\kappa/T^3 = \lim_{\omega \rightarrow 0} \frac{2T \rho_E(\omega)}{\omega} = 1.8 \dots 3.4$$

Related to diffusion coefficient D and drag coefficient η_D (in the non-relativistic limit)

$$2\pi T D = 4\pi \frac{T^3}{\kappa} = 3.7 \dots 7.0$$

$$\eta_D = \frac{\kappa}{2M_{kin}T} \left(1 + O\left(\frac{\alpha_s^{3/2}T}{M_{kin}}\right) \right)$$

time scale associated with the kinetic equilibration of heavy quarks:

$$\tau_{kin} = \frac{1}{\eta_D} = (1.8 \dots 3.4) \left(\frac{T_c}{T}\right)^2 \left(\frac{M}{1.5 \text{ GeV}}\right) \text{ fm/c}$$

→ **close to T_c , $\tau_{kin} \simeq 1 \text{ fm/c}$ and therefore charm quark kinetic equilibration appears to be almost as fast as that of light partons.**

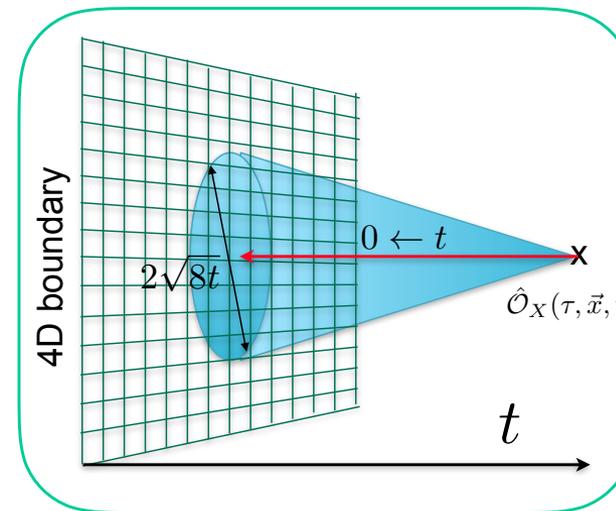
Noise reduction methods used in the quenched approximation not applicable in full QCD

Gradient flow - *diffusion* equation for the gauge fields along extra dimension, *flow-time* t

[M. Lüscher, 2010]

$$\frac{\partial}{\partial t} A_\mu(t, x) = -\frac{\partial S_{\text{YM}}}{\partial A_\mu}$$

$$A_\mu(t = 0, x) = A_\mu(x)$$



- continuous smearing of the gauge fields, effective smearing radius: $r_{\text{smear}} \sim \sqrt{8t}$
- gauge fields become smooth and renormalized
- no UV divergences at finite flow-time $t \rightarrow$ operators of flowed fields are renormalized
- UV fluctuations effectively reduces \rightarrow noise reduction technique
- Applicable in quenched and full QCD \rightarrow first case study in quenched

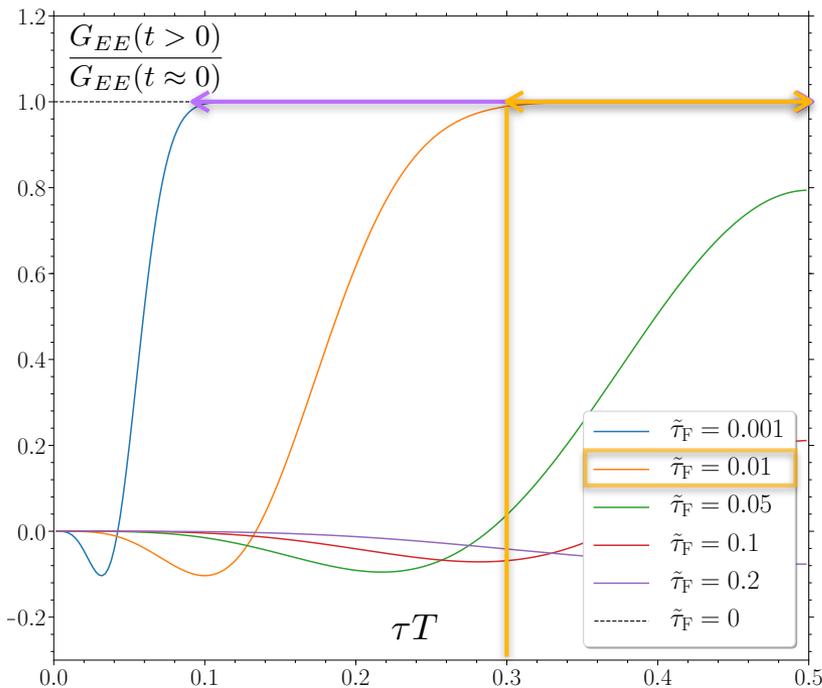
What is the flow time dependence of correlation functions?

How to perform the continuum and $t \rightarrow 0$ limit correctly?

LO perturbative limits

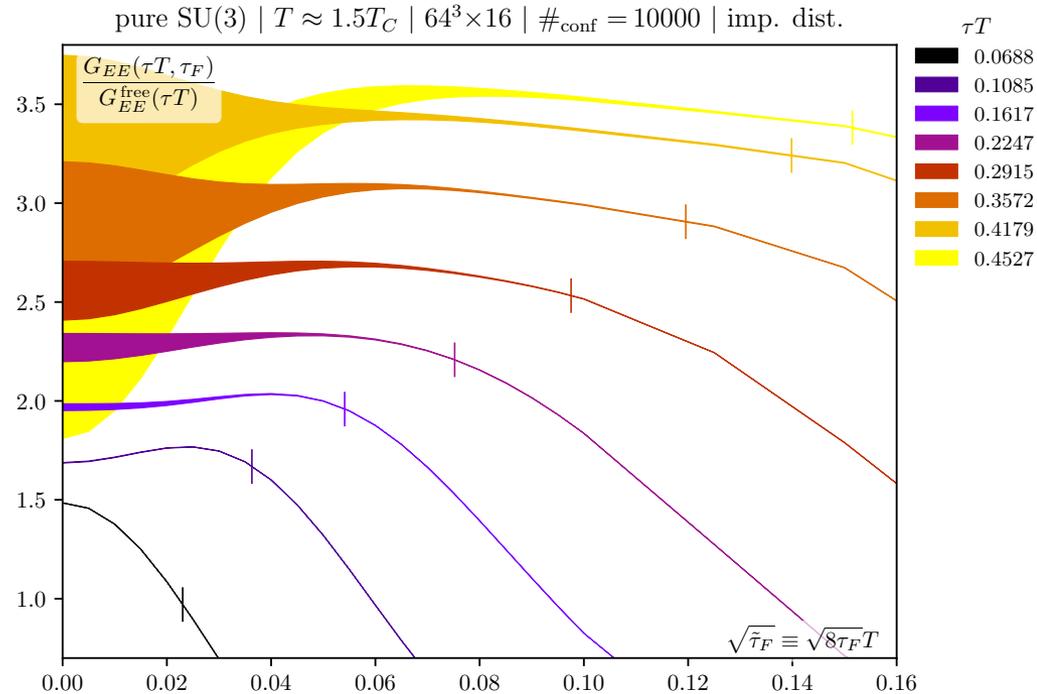
for the flow-time dependence:

$$\tilde{\tau}_f < 0.1136(\tau T)^2$$



First lattice QCD results on the flow

dependence of the color-electric correlator:



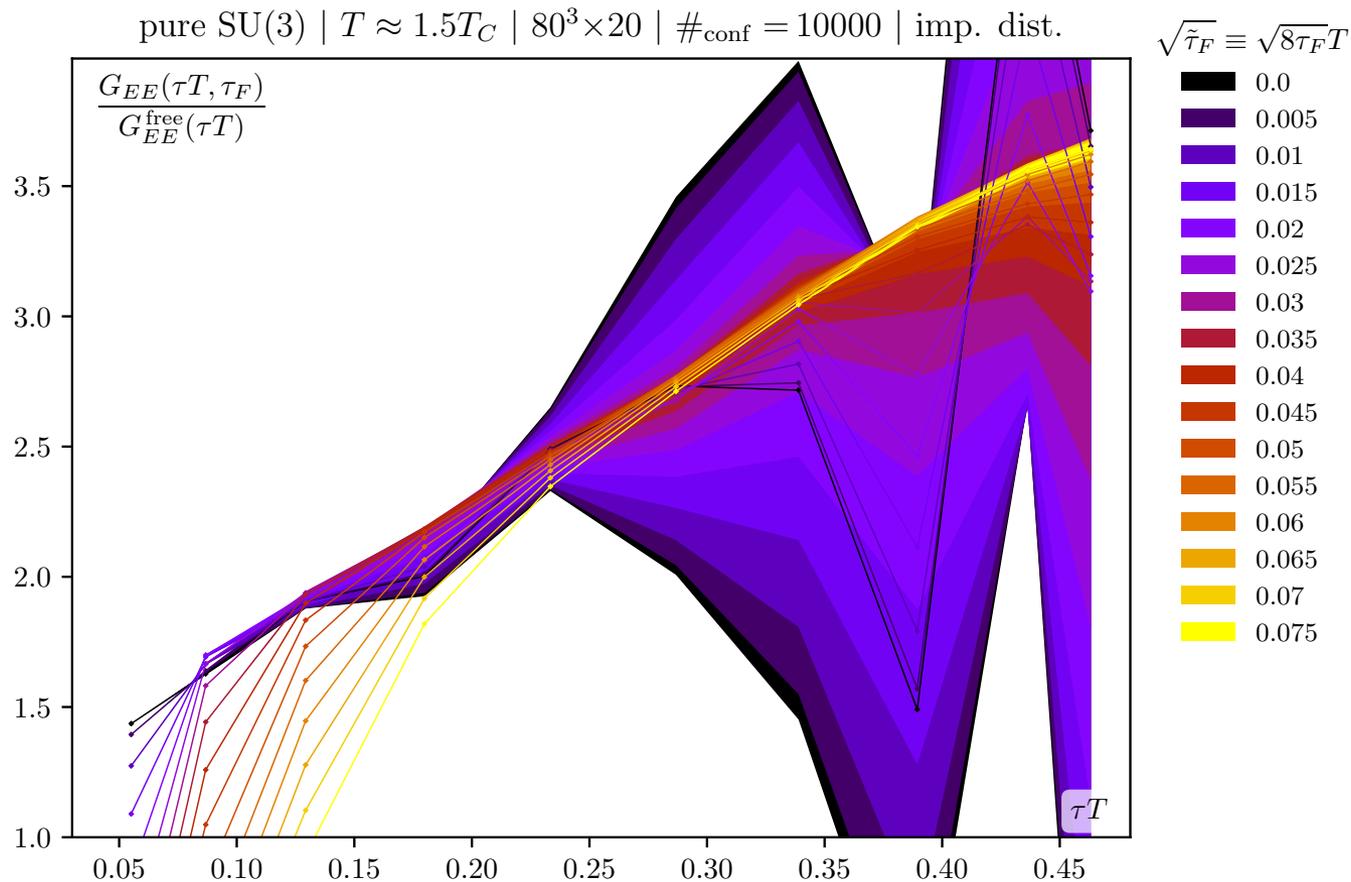
[A.M Eller, G.D. Moore, PRD97 (2018) 114507]

[L. Altenkort, H.T. Shu, OK et al., work in progress]

Effective reduction of UV fluctuations → good noise reduction technique

Signal gets destroyed at flow times above the perturbative estimate

Linear behavior at intermediate flow times

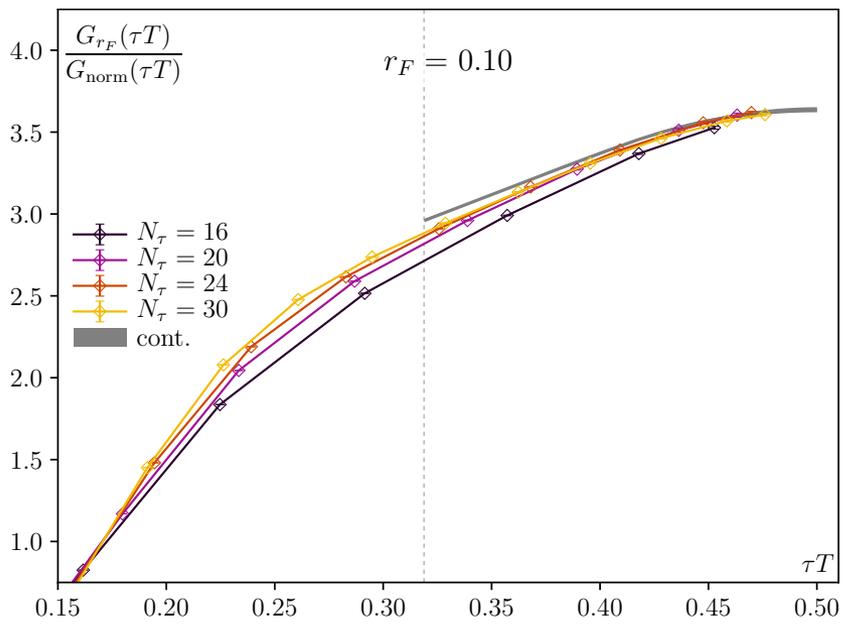


Effective reduction of UV fluctuations → good noise reduction technique

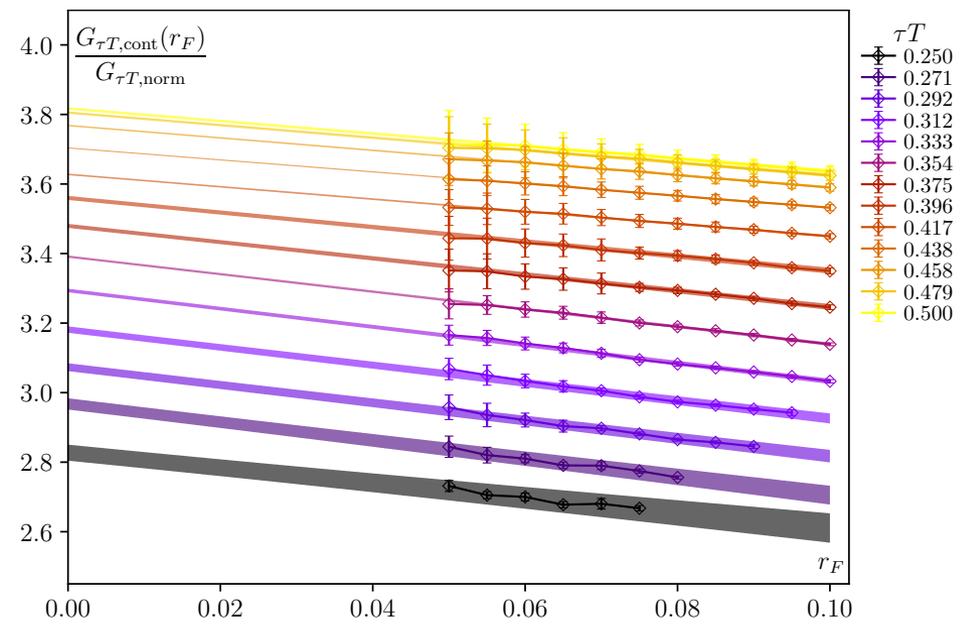
Signal gets destroyed at small distances → large- ω part of the spectral function modified

Final goal: Continuum limit at fixed physical flow time + $t \rightarrow 0$ limit

Continuum limit, $a \rightarrow 0$ ($N_t \rightarrow \infty$),
at fixed physical flow time:



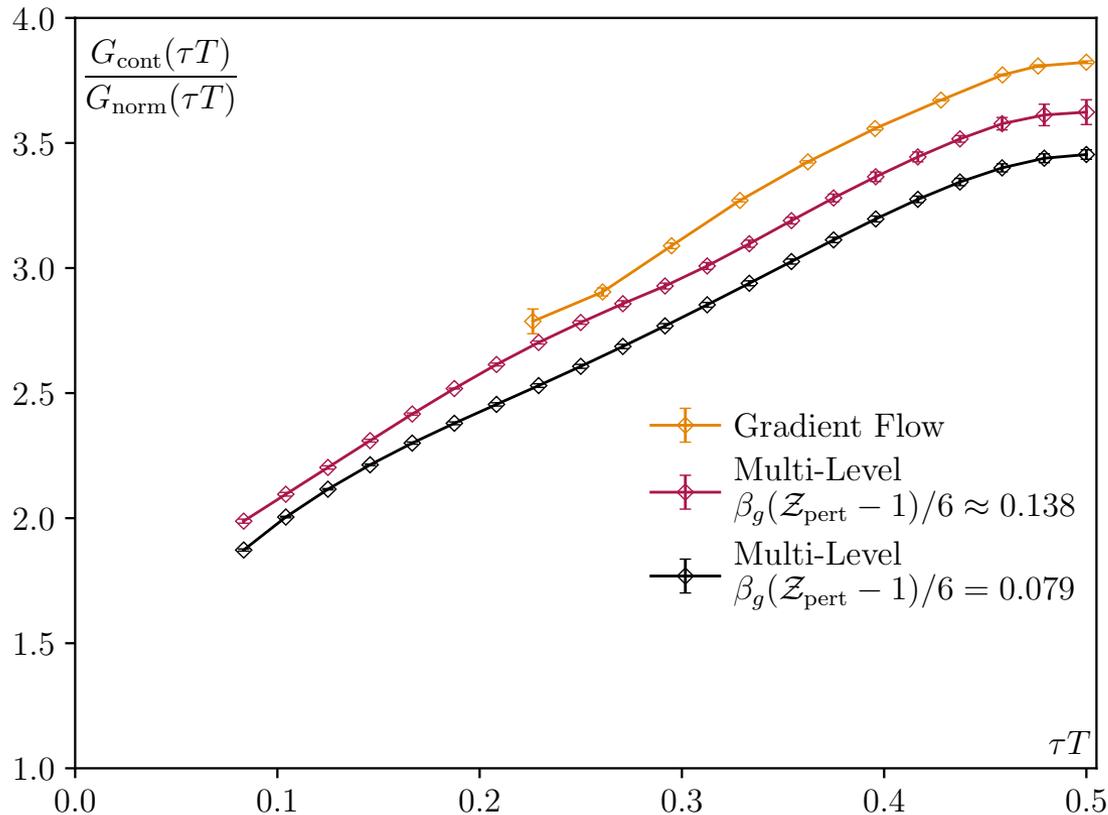
Flow time limit, $t \rightarrow 0$,
for each distance:



→ well defined continuum and flow time extrapolation

→ well defined renormalized correlation function

Comparison of **gradient flow** and **multi-level** method:



gradient flow: $N_t=16,20,24,30$

(more in progress to reach smaller τT)

[L. Altenkort, H.T. Shu, OK et al.,
work in progress]

multi-level: $N_t=20,24,36,48$

[A.Francis, OK et al., PRD92(2015)116003]

Comparable τT dependence at large distances

Uncertainty in the renormalization resolved by gradient flow

Heavy quark momentum diffusion coefficient expected to be slightly larger

Extension to full QCD possible using gradient flow method

Well defined methodology to extract spectral and transport properties from lattice QCD

Continuum extrapolated correlators from quenched lattice QCD are well described by perturbative model spectral functions down to $T \approx T_c$ for observable with an external scale (mass, momentum) $\gtrsim \pi T$

All results in this talk were obtained in the quenched approximation

What may change when going to full QCD?

$$\Lambda_{\overline{\text{MS}}}|_{N_f=0} \approx 255\text{MeV}$$

$$\Lambda_{\overline{\text{MS}}}|_{N_f=3} \approx 340\text{MeV}$$

$$T_c|_{N_f=0} \approx 1.24\Lambda_{\overline{\text{MS}}}|_{N_f=0}$$

$$T_c|_{N_f=3} \approx 0.45\Lambda_{\overline{\text{MS}}}|_{N_f=3}$$

$$\alpha_s^{EQCD}|_{T \simeq T_c} \simeq 0.2$$

$$\alpha_s^{EQCD}|_{T \simeq T_c} > 0.3$$

1st order deconfinement transition

chiral crossover transition

Physics may become more non-perturbative, more interesting, more complicated...

Quenched theory is a nice playground but **full QCD studies crucial!**

Stay tuned in the next years for the first results in full QCD...

Nuclear Science Computing Center at CCNU



Installation in 2018:
18 GPU-servers with
144 Nvidia V100
2.3 PFlops (single precision)

Extended in 2019 to:
38 GPU-servers with
304 Nvidia V100
4.8 Pflops (single precision)



Bielefeld IQCD GPU Cluster



Installation in 2019:
28 GPU-servers with
224 Nvidia V100
3.5 PFlops (single precision)

**Provides the resources to continue the fruitful collaboration
between the lattice QCD groups at CCNU and Bielefeld**