Distribution of Salem numbers and their connection to the eigenvalues of Jacobi β-ensemble

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joint work with F. Götze

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Algebraic numbers: basic definitions and notations

- A complex number α is called an **algebraic number** if there exists a polynomial $P \in \mathbb{Z}[t]$ such that $P(\alpha) = 0$.
- For every algebraic number α there exists a unique polynomial $P_\alpha \in \mathbb{Z}[t]$, which is
  - $P_\alpha(\alpha) = 0$;
  - P_α has a minimal degree among all polynomial $P \in \mathbb{Z}[t]$ such that $P(\alpha) = 0$;
  - has co-prime coefficients;
  - has positive leading coefficient.

  This polynomial is called the **minimal polynomial** of algebraic number α.
- Algebraic numbers α_1 and α_2 are called **algebraic conjugates** if they have the same minimal polynomial (note, that α and $\bar{\alpha}$ are always algebraic conjugates).
- Algebraic number α is called an **algebraic integer** if the leading coefficient of its minimal polynomial P_α is equal to 1.

Example

- Take the set of polynomials $P(t) = at - b$, where $a, b \in \mathbb{Z}$. Their roots form the set of rational numbers $\frac{a}{b}$;
- Take the set of polynomials $P(t) = t - b$, where $b \in \mathbb{Z}$. Their roots form the set $\mathbb{Z}$.

Salem numbers

A **Salem number** is a real algebraic integer $\alpha > 1$ such that all its algebraic conjugates have absolute value less or equal to 1 and at least one of them has absolute value equal to 1.

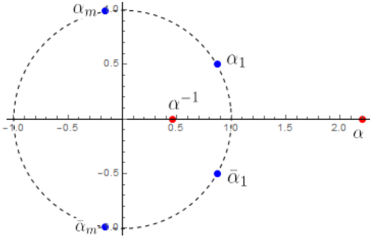
Properties:

- The minimal polynomial P_α of a Salem number α is self-reciprocal, meaning that

$$P_\alpha(t) = t^n + a_1 t^{n-1} + \ldots + a_1 t + 1;$$

- The degree of Salem number α is even;
- α and α^{-1} are algebraic conjugates;
- All algebraic conjugates of Salem number α, except for α^{-1}, have absolute value equal to 1 and we denote them by $\alpha_1, \bar{\alpha}_1, \ldots, \alpha_m, \bar{\alpha}_m$;
- Denote by Sal_m the set of all Salem numbers of degree $2(m+1)$.

Distribution of Salem numbers: what do we understand by this?



- Let us fix integer numbers $1 \leq k \leq m$ and consider the $(k+1)$-tuple

$$\boldsymbol{\alpha} := (\alpha, \arg \alpha_1, \ldots, \arg \alpha_k) \in \mathbb{R} \times [0; \pi]^k,$$

  where $\alpha \in Sal_m$, $\alpha_1, \ldots, \alpha_k$ are distinct algebraic conjugates of α;
- let $I_1, \ldots, I_k \subset [0; \pi]$ be disjoint intervals;
- let $Q > 1$ be some real number;
- denote by $Sal_{m,k}(Q, I_1, \ldots, I_k)$ the number of ordered $(k+1)$-tuples $\boldsymbol{\alpha}$ lying in $(1; Q] \times I_1 \times \ldots \times I_k$.

Our problem

Find the asymptotic behaviour of the value $Sal_{m,k}(Q, I_1, \ldots, I_k)$ as $Q \to \infty$.

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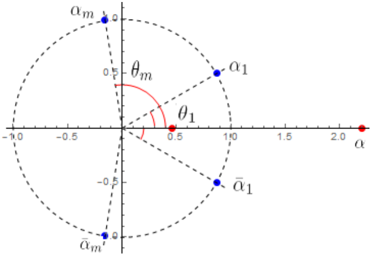

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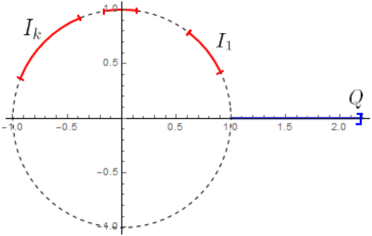

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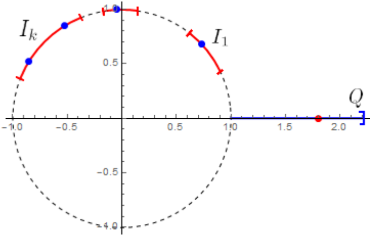

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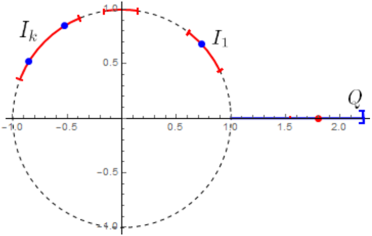

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Find the asymptotic behaviour of the value $Sal_{m,k}(Q, I_1, \ldots, I_k)$ as $Q \to \infty$.

Distribution of Salem numbers: what do we expect?

Theorem (Götze, G., 2019+)

For any integer m we have

$$Sal_{m,0}(Q) = \omega_m \, Q^{m+1} + O\left(Q^m\right), \quad Q \to \infty,$$

where

$$\omega_m := \frac{2^{m(m+1)}}{m+1} \prod_{k=0}^{m-1} \frac{k!^2}{(2k+1)!}.$$

For $I_1, \ldots, I_k \subset [0; \pi]$ we expect (!!)

$$\boxed{Sal_{m,k}(Q, I_1, \ldots, I_k) = \omega_m \, Q^{m+1} \int\limits_{I_1} \ldots \int\limits_{I_k} \rho_{m,k}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} + O\left(Q^m\right)}$$

We call $\rho_{m,k}(\boldsymbol{\theta})$ a k-point correlation function of conjugates of Salem numbers.

Random matrix theory

Define a random matrix ensemble by specifying **the joint probability density function (JPD)** for its eigenvalues as follows

$$p_N^\beta(x_1,\dots,x_N) = \prod_{i=1}^N w(x_i) \prod_{1\le i<j\le N} |x_i - x_j|^\beta,$$

where β is an in general complex parameter and the so called weight function $w(x)$ can be chosen to suit the needs.

- The k-point correlation functions of eigenvalues is defined by

$$R_{N,k}^\beta(x_1,\dots,x_k) := Z_{N,\beta}^{-1} \frac{N!}{(N-k)!} \int p_N^\beta(x_1,\dots,x_N)\mathrm{d}x_{k+1}\dots\mathrm{d}x_N,$$

  where $Z_{N,\beta}$ is normalization constant.

- Consider the following counting measure $\mu := \sum_{i=1}^N \delta_{X_i}$. Then for any family of mutually disjoint intervals $I_1,\dots,I_k \subset \mathbb{R}$ we have

$$\mathbb{E}\left[\prod_{i=1}^k \mu(I_i)\right] = \int_{I_1}\dots\int_{I_k} R_{N,k}^\beta(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Jacobi β-ensemble

The Jacobi β-ensemble with $\beta = 1$ is the random matrix ensemble with the following JPD function for eigenvalues

$$p_N(x_1, \dots, x_N) = \prod_{1 \le i < j \le N} |x_i - x_j|, \quad x_i \in [-1; 1].$$

The weight function in this case $w(x) = \mathbb{1}_{[-1;1]}(x)$ is a special case of more general class of weight functions

$$w(x) := w(a, b; x) = \begin{cases} (1-x)^a (1+x)^b, & -1 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad a, b > -1,$$

which gives an ensemble corresponding to the Jacobi orthogonal polynomials $J_j^{(a,b)}(x)$. The Jacobi β-ensemble is an example of random matrix ensemble with eigenvalues forming a Pfaffian point process, which means that all k-point correlation functions have the form

$$R_{N,k}(x_1, \dots, x_k) = \mathrm{Pf}\left[K_N(x_i, x_j)\right]_{i,j=1,\dots,k},$$

where $K_N(x, y)$ is a so called **Kernel function.**

Main result

Theorem (Götze, G., 2019+)

For any integer m and any disjoint intervals $I_1, \ldots, I_k \subset [0;\pi]$, $1 \le k \le m$ *we have*

$$\boxed{Sal_{m,k}(Q, I_1, \ldots, I_k) = \omega_m \, Q^{m+1} \int\limits_{I_1} \ldots \int\limits_{I_k} \rho_{m,k}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} + O\left(Q^m\right), \qquad Q \to \infty.}$$

Moreover, the function $\rho_{N,k}(\boldsymbol{\theta})$ *can be written in the following form*

$$\begin{aligned} \rho_{N,k}(\theta_1, \ldots, \theta_k) &= \mathrm{Pf}\left[K_N(\cos\theta_i, \cos\theta_j)\right]_{i,j=1,\ldots,k} \prod_{l=1}^{k} \sin\theta_l \\ &= R_{N,k}(\cos\theta_1, \ldots, \cos\theta_k) \prod_{l=1}^{k} \sin\theta_l, \end{aligned}$$

where $K_N(x,y)$ *and* $R_{N,k}(x_1, \ldots, x_k)$ *is a Kernel function a k-point correlation function (respectively) of Jacobi* β*-ensemble with* $\beta = 1$*.*

Idea of the proof

1. Use the locally Lipschitz mapping between the roots and the coefficients of a polynomial: given a Salem number α with conjugates $\alpha^{-1}, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_m}, e^{-i\theta_m}$ we can express the coefficients of P_α via $\alpha, \theta_1, \ldots, \theta_m$.
2. Consider the vector of coefficients of a polynomial P_α as a point in $\mathbb{R}^{m+1}$.
3. The amount of integer points inside some set $B \in \mathbb{R}^{m+1}$ is approximately equal to its volume.
4. The number of reducible polynomials with integer coefficients is relatively small in compare to the number of irreducible coefficient with integer coefficients.
5. The Jacobian of the transformation from the step 1 is

$$2^{\frac{m(m+1)}{2}} \left(1 - \frac{1}{\alpha}\right) \prod_{l=1}^{m} \left(\alpha + \frac{1}{\alpha} - 2\cos\theta_l\right) \prod_{l=1}^{m} \sin\theta_l \prod_{1 \le i < j \le m} |\cos\theta_i - \cos\theta_j| .$$

6. Integrating over $\alpha \in (1, Q]$ we conclude

$$Sal_{m,k}(Q, I_1, \ldots, I_k) = \omega_m \, Q^{m+1}$$

$$\times Z_m^{-1} \frac{m!}{(m-k)!} \int_{I_1} \cdots \int_{I_k} \int_0^{\pi} \cdots \int_0^{\pi} \prod_{1 \le i < j \le m} |\theta_i - \theta_j| \prod_{l=1}^{m} \sin\theta_l \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_m + O(Q^m).$$

Thank you for your attention!