Compact Stars as a Laboratory for Nuclear Matter

Bachelor thesis

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1. Introduction

At the end of its evolution an ordinary star forms a compact stellar object. It may be a white dwarf, a neutron star or a black hole. In this thesis we concentrate on neutron stars, which typically are assumed to be almost entirely composed of neutrons. To emphasize that the matter inside them can be in the deconfined state, we call them compact stars. Their masses are at least of the order of the solar mass $M_\odot = 1.989 \cdot 10^{30} kg$ and their radii of the order of 10 km [1]. It follows that the average density in compact stars is at least of the order of that of the baryon number saturation density $n_0 = 0.15 - 0.16 fm^{-3}$. Compact stars are of special interest because they are the only places in nature where cold nuclear matter can reach densities above that of a single nucleus. To understand this and to introduce the underlying physical aspects, we will at first talk about them in their own right.

1.1. Quarks and Gluons

Quarks are elementary particles which compose hadrons. Each quark is charged with one of three colors. They can’t be observed isolated, due to color confinement which means that only colorless composed particles are observed in nature.

Quarks were first proposed by M. Gell-Mann and G. Zweig independently in 1964. Gell-Mann’s proposal can be traced back to his Eightfold Way, a classification of particles according to SU(3) flavor symmetry. In 1968 they were discovered indirectly at the Stanford Linear Accelerator Center: Experiments on deep inelastic scattering of protons have shown that protons are composite objects of point-like-particles, quarks. Along with the observation of the tau-neutrino at Fermilab in 2000, the standard model of particle physics (see Figure 1, for a classification of the elementary particles) is experimentally verified as far as known today.

The strong interaction of quarks is mediated by eight colored gluons and is described by QCD in the context of the standard model. QCD is a QFT and its non-Abelian gauge invariant Lagrangian is

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1 The matter is formed of the three lightest, up, down and strange quarks, abbreviated strange quark matter. Additionally, electrons and muons may be present. Anyway, we are interested in cold and dense matter described by quantum chromodynamics (QCD), abbreviated cold and dense matter.

2 The saturation density is the density at which nuclear matter is bound at zero pressure. The value corresponds to a mass density of $7 \cdot 10^{14} g \cdot cm^{-3}$. At the surface the mass density is that of iron, $7.85 g \cdot cm^{-3}$.

3 Their temperatures are approximately zero. Nuclear matter is matter of nucleons, that is neutrons and protons are the degrees of freedom.

4 Nuclear matter is formed mostly of neutrons and protons, in turn composed of up and down quarks.

5 Quantum electrodynamics (QED), which describes the electromagnetic interaction, and the weak interaction are invariant under SU(2) and U(1), respectively.

6 The Higgs boson is hoped to be observed at the Large Hadron Collider (LHC).

7 Additionally, quarks are subjected to electromagnetic and weak interactions.

8 Non-abelian means not commutative from which self-interaction of the gauge-bosons follows.
Figure 1: Elementary particles of the Standard Model of particle physics, taken from [2]. They are divided into six quarks, six leptons and five force carriers (two W bosons). The Higgs boson is not included in this figure, as it is not verified, yet. Quarks and leptons, spin 1/2 particles, are the building blocks of matter. They are allocated to three generations, grouped by a similar coupling to the weak interaction. Ordinary matter is built up of the first generation. The fundamental forces are carried by the gauge bosons, spin 1 particles. Photons, gluons and both the charged W and the neutral Z bosons are the mediators of the electromagnetic, the strong and the weak interactions, respectively. If the energy is of the order of 100GeV, QED and the weak interaction are unified to a single electroweak interaction. The gravitational force, which is described by the theory of general relativity (GR), is not part of the standard model: So far, it is not possible to formulate GR as a quantum field theory (QFT). The properties of the elementary particles are being collected by the Particle Data Group (see [3]). They are characterized by quantum numbers. Corresponding to each massive particle, an associated antiparticle with opposite quantum numbers exists. According to the standard model, it is assumed that neutrinos are massless, which, however, seems not to be the case, because of neutrino oscillation.
Figure 2: The coupling constant $\alpha_s$ as a function of the energy scale $Q$, taken from [4]. The curves are the QCD predictions for the value of $\alpha_s(M_Z)$ and the symbols are measured values (see [4], for further explanatory notes on them).

\[ L_{QCD} = \bar{\Psi} \left( i \hbar c \gamma^\mu \partial_\mu - m c^2 \right) \Psi - \frac{1}{16\pi} F_{\mu\nu}^j F_{\mu\nu}^j - (g_s \bar{\Psi} \gamma^\mu \lambda_j \Psi) A_j^\mu, \] (1.1)

\[ F_{\mu\nu}^j = \partial_\mu A_\nu^j - \partial_\nu A_\mu^j - \frac{2g_s}{\hbar c} f_{jkl} A_k^\mu A_l^\nu, \] (1.2)

\[ [\lambda^j, \lambda^k] = 2i f^{jkl} \lambda^l. \] (1.3)

Here, $\Psi$ are the Dirac fields and $\bar{\Psi} = \Psi^* \gamma^0$ the Dirac-adjoint ones, $\gamma^\mu$ are the four Dirac matrices, $\lambda^j$ are the eight Gell-Mann matrices, $A^\mu$ are the eight gauge fields and $g_s = \sqrt{\alpha_s/4\pi}$ is the coupling constant of the strong interaction $^9$.

In equation (1.1) we identify the kinematic terms of the quark and the gluon fields, and the quark-gluon, three gluon and four gluon interaction vertices. The interaction terms depend

$^9$ $F_{\mu\nu}^j$ is the gluonic field strength tensor and $f^{jkl}$ are the structure constants connected to the generators of the SU(3) gauge group.
on the coupling constant $\alpha_s$. The coupling strength depends on the energy scale (see Figure 2).

Taking a look at Figure 2, we see that the strength of the coupling constant decreases with increasing energy $^{10}$. At small energy scales it is strong which implies that quarks are confined, that is they form hadrons $^{11}$. On the other side at sufficiently high energy scales the coupling constant becomes arbitrarily small. It follows that the interaction terms in equation (1.1) become of small magnitude and that the theory becomes non-interacting. This phenomenon is called asymptotic freedom, discovered by D. Gross, F. Wilczek and D. Politzer in 1973.

If the coupling constant is small enough we can in practical calculations expand the quantity of interest in powers of the coupling constant up to an adequate order and neglect the terms of higher order. This is called perturbation theory, however, it does not work at energy densities of relevance to most systems in nature.

1.2. Compact Stars

In 1934, W. Baade and F. Zwicky first proposed the existence of neutron stars and suggested that they are formed in a supernova explosion resulting from the gravitational collapse of an ordinary star $^{[5]}$ $^{12}$.

The evolution of a typical star is described in $^{[7]}$. We are going to outline the important aspects in the context of neutron stars. Thermonuclear fusion is the source of the energy for the evolution of a star across the various stages of combustion $^{13}$. It is exothermic up to $^{56}Fe$, the nucleus of the lowest energy. Hence, given enough time the star consists of iron. This is the case for stars of masses of more than $8M_\odot$ which is the lower limit for a neutron star to be formed.

The gravitational pull in the core is counterbalanced by degenerated non-relativistic electrons, a Fermi gas $^{14}$. Burning in the outer shells adds more heavy elements to the core and the electrons start becoming relativistic. At a mass density of about $10^7g \cdot cm^{-3}$, before reaching the nuclear matter saturation density neutrons are an energetically more favorable state. They are converted from electrons and protons by the inverse beta decay ensuring the stability of the star in the same way as electrons do to a white dwarf $^{15}$.

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$^{10}$This behavior is the same for the weak interaction and opposed to that of QED.

$^{11}$It is strong compared with the coupling constants of the electromagnetic, weak and gravitational interaction.

$^{12}$The supernova is of type II and more rare of type Ib and Ic. Additionally, Baade and Zwicky identified supernovas as a new class of astronomical objects. The neutron was discovered in 1932 by J. Chadwick $^{[6]}$.

$^{13}$Thermonuclear fusion refers to nuclear fusion induced by tunneling through the Coulomb barrier between nuclei. Thermal motion provides the energy for tunneling.

$^{14}$It provides a pressure, as a consequence of the Pauli exclusion principle. Non-compact stars are stable due to thermal pressure. If this is the end of the stellar evolution, the star will be a white dwarf.

$^{15}$At this point the chemical potential of the neutrons is similar to their mass. If the neutrons become relativistic their number is 8 times that of protons.

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Adding even more heavy elements to the core causes the gravitational collapse of a star which is followed by a stellar explosion. It depends on the residual mass whether a compact star will be formed. If it is below $1.38M_\odot$, the Chandrasekhat limit, the star will be a white dwarf [8]. Supernova simulations suggest that the minimum mass of a neutron star is about $1.1M_\odot$ [9]. The upper limit, the Tolman-Oppenheimer-Volkoff (TOV) limit, is not known for sure, because of the uncertainties in the Equation of State (EoS) of dense matter. According to [10] it is $2.2M_\odot$. Above this limit GR predicts the gravitational collapse to a black hole, the densest object known today.

We are going to collect some more properties of compact stars based on [1]. As a compact star is the remnant of a supernova, it moves through space with a velocity of up to $0.03c$, where $c$ denotes the velocity of light in vacuum. Additionally it rotates with a frequency up to about $1ms^{-1}$. The magnitude of its magnetic field at the surface is of the order of $10^{12}G$ up to $10^{15}G$, in the CGS-system of units.

The temperature in a supernova is of the order of $10^{MeV}$. Due to primarily neutrino emission, the temperature of a compact star decreases down to temperatures in the keV range, during its evolution (see section 2.2 for further explanations). Anyway, compact stars are said to be cold, because temperatures in the keV range are small compared to the scale set by QCD. The temperature of the deconfinement transition at zero chemical potential is about $170MeV$, for example. Moreover, their temperatures are small compared to the chemical potential. This implies that we can approximate the temperature being zero in practical calculations.

Once formed, an isolated compact star will cool down and its magnetic field may decay. However, the internal constitution of a compact star is frozen within a few seconds of birth. There is no reason, other than an interaction with another star that they should not live for an infinite time as a compact star [7].

In 1967 J. B. Burnell and A. Hewish first discovered a highly magnetized rotating neutron star [11] and [12], for a possible realization which allows stable hybrid stars of masses up to $2.1M_\odot$ and strange stars of masses up to $2.75M_\odot$, respectively. Both, the Chandrasekhat and the TOV limits were originally calculated to be smaller. S. Chandrasekhat’s suggested in 1931 the limiting mass to be $0.91M_\odot$ [13] and J. R. Oppenheimer and G. Volkoff, based on the work of R. C. Tolman [14], supposed in 1939 the limiting mass to be approximately $0.7M_\odot$ [15].

Black holes may also be formed during the cooling process, due to accretion of the core and deleptonization. Taking a typical radius of $10km$, this implies that a point on the equator moves with a velocity of $0.2c$. See [11], for further notes on its importance. However, we will consider static stars.

The magnitude of the magnetic field of the earth is about $0.6G$.

We are setting $k_B = 1$, where $k_B$ denotes the Boltzmann constant.

Photon emission contributes less to the cooling process, as they are mainly absorbed by the matter. In general, the mantle is hotter than the core, as a nuclear lattice has the potential to store more heat.

Furthermore, eventually, given enough time, all stars will have evolved into white dwarfs, compact stars or black holes.
star, called a pulsar [16]. A pulsar is observed in periodic pulses, because of the electromagnetic radiation along the magnetic axis and a different alignment of the magnetic and the rotation axes.

In 1974, R. Hulse and J. Taylor discovered a binary system of a pulsar and another star [18]. A Pulsar may be identified by the Doppler shift, the effect of the frequency depending on the relative motion of observer and source of the waves [19]. As a compact star is a residue of a supernova, the detection of such an bright event is a hint for its existence. Additionally a compact star may be identified by its cooling process [21]. Some compact stars, their masses and rotation frequencies are listed in [11] and references therein.

1.3. Cold and Dense Matter in Compact Stars

To understand, what kind of matter compact stars may be built up from, we now review the phases and phase transitions of dense matter. Let us consider the schematic phase diagram of QCD shown in Figure 3.

A compact star is build up of cold and dense matter, classified in the following. If it is composed of nuclear matter or deconfined quark matter it is called a neutron star or quark star, respectively. Furthermore it may simultaneously be composed of both confined and deconfined quarks, in which case it is a hybrid star. As we expect the density in the interior to be larger than at the surface, a hybrid star consists of a quark core and a nuclear mantle. The exact identity of the observed compact stars is presently unknown.

If a pure quark star contains strange quarks, it is said to be a strange star. Their existence is made possible by the 'strange quark matter hypothesis' proposed by A. Bodmer in 1971 [22] and E. Witten in 1984 [23]. According to it, strange quark matter is the ground state of QCD at zero pressure, not nuclear matter. This stability is possible, as three-flavor quark matter consists of more quarks being in a lower energy state than two-flavor quark matter, because of the Pauli exclusion principle. If this hypothesis is true, strange stars are composed of an approximately equal number of up, down and strange quarks. In this context, nuclear...
Figure 3: Our present understanding of the phase diagram of strongly interacting matter in the plane of quark chemical potential $\mu_q$ and temperature $T$, taken from [1]. We can find a neutron star (or a compact star, more generally speaking) in the range of $\mu_q = 100 \text{MeV}$ to $1\text{GeV}$. It is composed of cold and dense matter, which may consist of superfluid nuclear matter and/or deconfined quark matter (yellow colored). Deconfined quark matter may be in a superfluid and color-superconducting color-flavor locking (CFL) state or other color-superconducting non-CFL state. CFL is the ground state of three-flavor quark matter at $\mu_q \to \infty$, where quarks form Cooper pairs, whose color and flavor properties are correlated in a symmetric pattern. Superfluidity and color-superconductivity is the property of a fluid being of no viscosity and of no electrical resistance, respectively. It emerges due to breaking the underlying gauge symmetry, manifesting at a critical temperature. In the case of superconductivity, fermions pair to form bosons on the Fermi surface. CFL is the highest density phase of three-flavor quark matter. At high densities and temperatures above $170\text{MeV}$, quarks and gluons are almost free, termed quark gluon plasma. The dots label critical points at which phase boundaries cease to exist. The hadronic phase is located at low chemical potentials and temperatures, subdivided in a gas and a liquid phase (and a solid phase, not part of this figure), at a chemical potential of $310\text{MeV}$ for zero temperature. It is accessible due to nuclear physics methods, the hadron resonance gas models, for example, that is treating the system as a nearly ideal gas of hadrons (finite volumes and interactions of hadrons are taken into account at higher energies). The region of low chemical potentials and high temperatures is explored by heavy ion collisions. There is no single method to cover the entire plane.
matter is a long living metastable state, as a transformation of nuclear matter to strange quark matter due to the weak interaction is extremely slow. A quark star may be formed in an energy region above the mass threshold of the strange quark \(^{29}\). The strange quark matter hypothesis does not disagree with our present knowledge \(^{24}\). A possible experimental hint for the existence of strange stars would be a star with a rotation period above the limit for neutron stars \(^{7}\), or the detection of a star with mass much larger than \(2M_\odot\) \(^{12}\).

Even a star containing only hadronic matter may be exotic. At sufficiently high densities, hyperons, which are baryons with nonzero strangeness, may evolve inside compact stars \(^{30}\). If the density is high enough, pions and kaons may condense, as a condensate is the ground state of bosons. Even though kaons are heavier than pions, and therefore more unlikely to condense, their effective mass becomes sufficiently small to allow for a condensate \(^{25}\). Pions and kaons are moreover able to establish a Bose-Einstein-condensate in the CFL phase of quark matter \(^{1}\). The possible phases of nuclear and/or quark matter inside a star are summarized in Figure 4.

It is also possible that a star consist of multiple phases. A hybrid star, built up of a quark

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\(^{29}\)The mass of the strange quark is about \(101\text{MeV}\), setting \(c = 1\).

\(^{30}\)Hyperons are of zero charmness, bottomness and topness.
core and a nuclear mantle, is a feasible example of this \( ^{31} \). The question, what the interface between these two phases is like, arises. Is it sharp with a jump in the density profile or is there a shell, in which both phases coexist in a mixed phase? The conclusion is that if global (not local) charge neutrality is required, it is unlikely that a mixed phase of quarks and hadrons is realized \(^{1}\) \(^{32}\).

As noted in Figure 3, the region of low chemical potential/density and high temperature is experimentally explored in relativistic heavy ion collisions. This is for example done at the Relativistic Heavy Ion Collider (RHIC) \(^{26, 27, 28, 29}\) and the LHC \(^{30}\). Further experiments in the region of lower temperatures and higher densities are done at RHIC \(^{31}\) and the Facility for Antiproton and Ion Research \(^{32}\), for example. These experiments may give some insight in the EoS of cold nuclear matter. However, they are not able to probe the region of very low temperatures and high densities.

The region of high temperature and/or high density, far above the phase transitions, is theoretically explored by first principle calculations of QCD (see equation (1.1)) using perturbation theory, ensured by asymptotic freedom \(^{33, 34, 35, 36}\) \(^{33}\). Due to the non-perturbative and non-linear nature of QCD, it is difficult to perform first principle calculations at non-asymptotic temperatures/densities, in particular for cold and dense matter.

On the other side, lattice QCD has provided controlled numerical results in the region where the chemical potential is lower than the temperature, close to the deconfinement transition \(^{37, 38}\). At larger chemical potentials, particularly in the region of cold and dense matter, lattice QCD is inapplicable, because of the sign problem \(^{34}\).

One way to approach the region of cold and dense matter is to interpolate between the known regions of high density and low density. As an example, it can be assumed that color-superconductivity, being the stable ground state of deconfined quark matter at high densities, lasts in the region where compact stars exist \(^{39}\). In doing that, we have to be aware that such extrapolations are not numerically valid in nature \(^{40}\). Another way is to use one of the models that have been built to describe the relevant physics in this region \(^{41}\).

As a result, we summarize that compact stars are of special interest, as they are the only existing laboratory for cold and dense nuclear matter. The region of cold dense matter is hardly accessible today, or in a more positively speaking, interesting due to new physics. We can learn something about compact stars from their microscopic constitution or we can learn

\(^{31}\)In a compact star further mixed phases may exist, like a neutron superfluid coexisting with a lattice of irons in the inner crust.

\(^{32}\)See \(^{1}\), for an argumentation that the requirement of charge neutrality for a compact star is valid.

\(^{33}\)In high density systems small distances are related to large momenta, corresponding to high energies. This is traced back to the Fourier transform operating on the position space and the momentum space.

\(^{34}\)According to the Pauli exclusion principle, the wavefunction of a system of fermions changes sign by interchanging any two fermions. An integral over all these states is therefore highly oscillating and can’t be solved for useful accuracy, unless the numerical method is of high precision.
something about dense matter by ruling out theories which does not fit to the observational data.

This is what we will be concerned with in this thesis. To do this quantitatively, we will solve the mass-radius relation of a compact star, a system of two coupled differential equations [1]

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon(r),
\]

\[
\frac{dp}{dr} = -\frac{G[\epsilon(r) + p(r)][m(r) + 4\pi r^3 p(r)]}{r[r - 2Gm(r)]}.
\]

Here, \(m\) is the mass up to the radius \(r\) for the interior of the static, isotropic and relativistic star in hydrostatic equilibrium and the gravitational constant \(G = 6.674 \cdot 10^{-11} m^3 kg^{-1} s^{-2}\). Equation (1.1) is called the TOV equation [7]. A derivation of it can be found in [7].

To solve the relation between the mass and radius of the star a closed system of equations is required. This input can be obtained from the microscopic physics which yields an EoS of the form \(p(\epsilon)\), where \(p\) denotes the pressure and \(\epsilon\) the energy density [8]. It is an example of how microscopic physics affects quantities measured in astrophysical experiments, such as the mass and radius of a star.

1.4. Outline of the Thesis

After this introduction on elementary particles and compact stars and their connection, we will deal with the EoS of nuclear and quark matter. Therefore we will introduce some basic thermodynamics. We will give an overview about EoSs of nuclear matter by introducing phenomenological models and realistic potentials and comparing them. As an example for a phenomenological model we will outline the Walecka model. We will talk about quark matter EoSs and state-of-the-art three loops EoS calculations. Additionally we will show some results of model calculations.

Thereafter we will concentrate on the TOV equation. We will recall the TOV equation, comment on it and show some results of the mass-radius relation of compact stars for a given EoS. Next we will consider a constant energy density as the EoS and solve the TOV equation analytically and see what these results imply. As an example for a numerical solution of the

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35Equation (1.4) is derived by the differential mass of a star at a given radius, that is the mass of a thin spherical layer, \(dm = \epsilon(r)dV\), where \(\epsilon(r)\) is the energy density (\(c = 1\)) and \(dV = 4\pi r^2 dr\) is the volume of the thin spherical layer at the given radius \(r\). This definition of the energy density \(\epsilon\) is related to the density of the mass-energy in the rest frame of the fluid.

36Compact stars are rotating, not static. The corrections will be at most a scaling of the mass by a factor 1.1 and an increasing of the equatorial radius by typically 1.5 – 2km [7]. The hydrostatic equilibrium is preserved between the attractive gravitational force and the repulsive forces, which are the Fermi pressure and the pressure of the strong interactions of the nuclear and/or quark matter inside.

37Sometimes equation (1.1) is called the Oppenheimer-Volkoff (OV) equation (see footnote 16).

38This could be a polytrope of the form \(p = 1/3\epsilon\) for relativistic electrons and nucleons, for example.
TOV equation we will consider an EoS describing a hybrid star and calculate its mass-radius relation. After this we will draw conclusions and present an outlook.
2. Equation of State of Nuclear and Quark Matter

2.1. Basic Thermodynamics

The EoS is a constitutive equation describing a mathematical relationship between state variables. It is a thermodynamical equation describing the state of matter in equilibrium under a given set of physical conditions. The internal energy density $\epsilon$ for a fixed volume of a grand canonical ensemble is given by the fundamental Euler integral

$$\epsilon = -p + Ts + \sum_i \mu_i n_i,$$

(2.1)

where $p$ is the pressure, $T$ the temperature, $s$ the entropy density, $\mu_i$ the chemical potential and $n_i$ the number density of a type of particle and the label $i$ denotes the possibility of several kinds of contributing particles. According to the Gibbs-Duhem equation

$$\sum_i n_i d\mu_i + sdT - dp = 0,$$

(2.2)

where $d$ denotes the differential of the corresponding quantity, the first law of thermodynamics for a given volume

$$d\epsilon = Tds - dp + \sum_i \mu_i dn_i$$

(2.3)

follows, where $T$ and $\mu_i$ are given variables. The temperature, the pressure, the chemical potential and densities are intensive properties of matter, whereas the internal energy $U$, the entropy $S$, the volume and the particle number are extensive properties. With these quantities we can define the enthalpy $H$, the Helmholtz free energy $F$, the Gibbs free energy $G$ and the grand potential $\Omega$ as

$$H = U + pV,$$

(2.4)

$$F = U - TS,$$

(2.5)

$$G = U + pV - TS,$$

(2.6)

$$\Omega = U - TS - \mu N = -pV.$$  

(2.7)

Their differentials may be derived via Legendre transformations. A physical quantity of interest can be deduced from the differentiation of the corresponding relation with respect to an appropriate variable.

All these quantities have in common that they are macroscopic ones. To involve the microscopic constitution of matter we first introduce the statistical definition of the entropy

$$S = \ln Z,$$

(2.8)

39Keep in mind that we consider thermal equilibrium. It means that every microscopic reaction is counter-balanced by an inverse reaction.

40According to the Gibbs-Duhem equation, the extensive properties are not independent.
where \( Z \) is the partition function, which is the number of microstates corresponding to the observed thermodynamic macrostate. According to the canonical ensemble and the grand canonical ensemble

\[
F = -T \ln Z_c, \tag{2.9}
\]

\[
\Omega = -T \ln Z, \tag{2.10}
\]

where \( Z_c \) is the canonical partition function and \( Z \) the grand canonical partition function. The latter is a function of \( \beta = 1/T \) and the chemical potential \( \mu_i \)

\[
Z = \text{Tr} \left( e^{-\beta \left( H + \sum_i \mu_i N_i \right)} \right). \tag{2.11}
\]

Using the path integral formalism, the grand potential becomes

\[
\Omega = -T \ln \int_\text{periodic} \mathcal{D}\phi \, e^{\beta/\hbar} \int_0^\beta d\tau \int d^3x L. \tag{2.12}
\]

Here, the functional integral \( \int \mathcal{D}\phi \) is an abbreviation for all fields contributing to an appropriate Lagrangian \( L \) with ‘periodic’ boundary conditions \( \phi(\tau = 0) = \phi(\tau = \beta), \tau = it \) and the reduced Planck constant \( \hbar = 6.626 \cdot 10^{-34} \text{m}^2\text{kgs}^{-1} \). From equation (2.7), we then obtain different equilibrium thermodynamical quantities, such as

\[
n_i V = -\partial_{\mu_i} \Omega, \tag{2.13}
\]

\[
s V = \partial_T \Omega. \tag{2.14}
\]

### 2.2. EoS of Nuclear Matter

The most common way to compute the EoS of nuclear matter is to do microscopic many-body calculations using phenomenological relativistic mean-field theories (MFT) and realistic nucleon-nucleon potentials. The potentials have to provide an accurate description of the measured nucleon-nucleon scattering data.

The EoS may then be computed by variational or quantum Monte-Carlo (MC) techniques. Variational techniques rely on the parameterization of the nuclear wave-function combined with correlated interactions between pairs of nucleons, arising because of Fermi statistics. In contrast, MC techniques are not limited by the form of the variational wave-function and can potentially include all possible correlations in the many-body system. But this approach is computationally intensive as the number of required operations grows roughly as \( 2^A A!/(N! Z!) \), where \( A, N \) and \( Z \) are the baryon number, neutron number and proton number, respectively. This method suffers, in principle, from the sign problem (see [45], for a calculation of the ground-state energy of 14 neutrons in a periodic box).

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41 This section is based on [42].
One method that uses realistic nucleon-nucleon potentials is the Brueckner-Hartree-Fock (BHF) approach. Here, the bare nucleon-nucleon interaction is used to determine the interaction energy for pairs of nucleons. They are treated as independent particles and correlations are incorporated. This method was originally developed by Brueckner, Bethe and Goldstone. It is non-perturbative in the coupling but utilizes a perturbative expansion in the number of independent hole lines [46], the convergence of which is improved if the single-particle dispersion relation is obtained using the Hartree-Fock approximation. The lowest order BHF results are in agreement with those obtained using variational methods [47].

The realistic nucleon-nucleon potentials are constrained by a collection of scattering data [48]. But these data probe the scattering for energies below the pion threshold of about \(350\, MeV\). Thus, various models for the nucleon-nucleon interactions which are equivalent up to this limit and behave differently from there on can be constructed. Anyway these models can differ in their predictions for many-body systems, because additional many-body forces, higher than two, can be relevant, as the strength of these forces depends on the two-body force at high energies. However, it is sufficient to add three-body forces to the applied realistic two-body potentials to describe light nuclei [49].

Another approach is the development of nucleon-nucleon interactions based on mean-field theory [50]. It allows a systematic treatment of physics of sufficiently high energies by organizing the calculation in powers of momenta, where a small expansion parameter is identified at small momenta. It is shown that a momentum space potential with a cutoff of \(2\, fm^{-1}\) may describe low-energy data and that all realistic nucleon-nucleon potentials evolve to this form, in the sense of the renormalization group [51].

In nuclear mean-field models the nucleon-nucleon scattering is replaced by a phenomenological interaction. The parameters are determined by fitting the model predictions in the mean field approximation to measured properties of bulk nuclear matter at saturation density. We will state these properties of bulk nuclear matter at saturation nucleon number density \(n_0 = 0.15 - 0.16 \, fm^{-3}\): the binding energy per nucleon is \(16\, MeV\); the nuclear compression modulus is in the range of \(200 - 300\, MeV\); the nucleon effective mass is \(0.7 - 0.8 m_N\); the nuclear symmetry energy defined through the relation \(S = 1/2 \partial^2 \chi_{(1-2x_p)} (\epsilon/n)\), where \(x_p\) is the proton fraction, is in the range of \(30 - 35\, MeV\).  

We will now review a variant of the Walecka model, as an example of a mean-field model describing nuclear matter. Originally, Walecka proposed in 1974 a field-theoretical model for interacting nuclear matter based on Yukawa couplings of the nucleons with scalar and vector

\[K = k_T^2 \partial_{k_T^2} \chi_{(1-2x_p)} (\epsilon/n)\]  

A small and a large value correspond to so-called soft and stiff matter, respectively. See [1], for a derivation of the relation \(\chi^{-1} = n_0 K/9\) between \(K\) and the thermodynamic definition of compressibility \(\chi\). An equivalent definition of soft and stiff matter follows from the comparison between two different EoS. The one with the larger pressure over a given energy range is termed stiff and the one with the smaller pressure is termed soft. Soft EoS can sustain less gravitational pull, which leads to stars with lower maximum masses.

\[15\]
mesons [52]. We will concentrate on a refined model introduced by Boguta and Bodmer in 1977 describing a short-range interaction between nucleons via scalar $\sigma$, vector $\omega$ and vector $\rho$ mesons [53]. Its Lagrangian is

\[ \mathcal{L}_N = \bar{\Psi}_N \left( i \gamma^\mu \partial_\mu - m_N^* - g_{\omega N} \gamma^\mu V_\mu - g_{\rho N} \gamma^\mu \tau_N \cdot R_\mu \right) \Psi_N \]

\[ + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{4} V_\mu V^\mu \]

\[ + \frac{1}{2} \frac{m_\omega^2 \sigma V_\mu V^\mu - \frac{1}{4} R_\mu \cdot R_\mu + \frac{1}{2} m_\rho^2 R_\mu \cdot R_\mu}{m_\rho}, \]

where $m_N^* = m_N - g_{\sigma N} \sigma$ is the nucleon effective mass and $m_N = 939 \text{MeV}$, $m_\sigma = 600 \text{MeV}$, $m_\omega = 782 \text{MeV}$ and $m_\rho = 770 \text{MeV}$ are the masses of the nucleon, the scalar field $\sigma$, the vector field $V_\mu$ corresponding to the isoscalar $\omega$ meson and the vector field $R_\mu$ corresponding to the isovector $\rho$ meson, respectively. $\tau_N$ is the nucleon isospin operator. Equation (2.18) describes the self interaction between scalar $\sigma$ mesons. The dimensionless coupling constants $c_1$ and $c_2$ are introduced to obtain accurate agreement with the empirical value of the nuclear compressibility (see [53]). The five coupling constants $g_{\sigma N}$, $g_{\omega N}$, $g_{\rho N}$, $c_1$ and $c_2$ are chosen to reproduce the five empirical properties of nuclear matter at saturation density listed above [7].

Solving the model in the mean-field approximation, according to [7], we obtain the minimized grand potential

\[ \Omega_{\text{nuclear}} (\mu_n, \mu_p, \mu_e) = \frac{1}{\pi^2} \left( \int_{0}^{k_{F_n}} dk \ k^2 [\epsilon_n(k) - \mu_n] + \int_{0}^{k_{F_p}} dk \ k^2 [\epsilon_p(k) - \mu_p] \right) \]

\[ + \frac{1}{2} \left( m_\sigma^2 \langle \sigma \rangle^2 - m_\omega^2 \langle \omega \rangle^2 - m_\rho^2 \langle \rho \rangle^2 \right) + U(\sigma) - \frac{\mu_e^4}{12 \pi^2}, \]

\[ \epsilon_n(k) = \sqrt{k^2 + m_N^*} + g_{\omega N} \langle \omega \rangle - \frac{1}{2} g_{\rho N} \langle \rho \rangle, \]

\[ \epsilon_p(k) = \sqrt{k^2 + m_N^*} + g_{\omega N} \langle \omega \rangle + \frac{1}{2} g_{\rho N} \langle \rho \rangle, \]

where $\epsilon_n(k)$ and $\epsilon_p(k)$ are the neutron and proton single-particle energies. The single-particle energy at the Fermi surface defines the respective chemical potential. These chemical potentials are related by the condition of weak interaction equilibrium given by $\mu_n = \mu_p + \mu_e$ (see footnote 28). If we assume local electrical charge neutrality we require the charge density
Figure 5: Thermodynamic properties of charge neutral nuclear matter, taken from [42]. The left figure shows the baryon chemical potential, the electron chemical potential and the energy per baryon as a function of the baryon number density. And the right figure shows the EoS, a relation between the energy density and the pressure.

\[ \rho_Q = \partial \mu_e \Omega_{\text{nuclear}} \] (see equation (2.13)) to be zero. Consequently only one independent quantity \( \mu_n \), corresponding to baryon number conservation, is left and the EoS of dense nuclear matter is specified as a function of \( \mu_n \). The results are summarized in Figure 5.

Like the nucleon-nucleon potentials, different nuclear EoSs constructed to satisfy the empirical constraints at nuclear density can differ at lower and higher densities. Additionally, the difference in symmetric nuclear matter EoS and neutron-rich matter EoS could be important. This difference arises mainly because of the dependence of the nuclear symmetry energy on the proton fraction. In the mean-field model consider here, the nuclear symmetry energy arises due to the isovector force from the exchange of \( \rho \) mesons and its density dependence being linear \(^{44}\).

If we compare the mean-field EoSs with those of a variational treatment, such as those reported in the work of Akmal, Pandharipande and Ravenshall (APR) \(^{43}\), we will identify mainly two differences. First, the mean-field EoS for matter in beta equilibrium according to weak interactions is stiffer than the APR EoS at low densities and softer at high densities, leading to different predictions of maximum masses of stars. Second, the typical proton fraction in the mean-field EoS is higher than in the APR EoS at high densities. As the differences can in part be attributed to differences in the density dependence of the nuclear symmetry energy, precise measurements of the neutron skin of heavy nuclei \(^{54, 55}\) and heavy-ion collisions \(^{56, 57}\) may rule out one of these theories.

The lower proton fraction in the context of APR suppresses neutrino cooling of the star. The

\(^{43}\)See [1], for a discussion on the importance of the scalar interactions.

\(^{44}\)Considering realistic nucleon-nucleon interactions and correlations beyond the mean-field theory, the nuclear symmetry energy has non-trivial density dependence.
most efficient process for neutrino emission is given by $n \rightarrow p + e + \bar{\nu}_e$ and variants thereof, called direct Urca process \(^{45}\). These reactions, taking place at the Fermi surface, have to conserve momentum, termed $k$. This is true, only if $|k_{Fn}| \leq |k_{Fp} + k_{Fe}|$, termed the triangle inequality. Requiring electric charge neutrality or equivalently $|k_{Fn}| \approx |k_{Fp}|$, the triangle inequality is difficult to satisfy, if the proton fraction is small. It is shown that the proton fraction has to be at least $1/9$ to yield in the Urca process significantly contributing to the cooling mechanism of stars \(^{58}\). If the value of the proton fraction is smaller, the modified Urca process will be the primary cooling mechanism. It includes charged-current processes, such as $nn \rightarrow npe\bar{\nu}_e$ and $np \rightarrow ppe\bar{\nu}_e$ (and their inverses). Additionally neutrino bremsstrahlung $nn \rightarrow nnu\bar{\nu}_e\bar{\nu}_e$ will be a significant process. For typical temperatures of interest in neutron star cooling, these reactions are five orders of magnitude slower than the direct Urca process.

As one increases the energy density and approaches the deconfinement transition region the quantitative accuracy of these models begins to suffer. The primary uncertainties in the EoS are coming from the unknown composition of the matter, that is the conjectured presence of hyperons or kaon condensation, for example. Secondary uncertainties are details of the calculations such as the exact form of the respective variational ansatz or the effect of neglecting the simultaneous interactions of more than two nuclei \(^{43, 60, 61}\).

### 2.3. EoS of Quark Matter

Next, we look at the EoS of high-density quark matter. The phase structure of this region is badly known. Only at asymptotically large baryon densities the physical phase is known, it is the CFL superconductor. The question of how far down the baryon number density axis this phase extends to and by which it will be ultimately replaced, are still open. However, not all equilibrium thermodynamic quantities, in particular the EoS, are sensitive to these details. A CFL superconductor can be described by a perturbatively exponentially small energy gap $\Delta \sim e^{-C/g_s}$, where $C$ denotes a positive constant \(^{46}\). This is taken into account in state-of-the-art EoS calculations at three loops (see \([12]\)) \(^{47}\).

In Figure 6 we see that the '2+1 flavor' result matches the '3 flavor' result at large chemical potential $\mu$ and approaches the '2 flavor' result at small $\mu$. It has been shown that a comparison with the two-loop results exhibits convergence for $\mu > 1GeV$ \([12]\).

When one interpolates between the EoSs of nuclear and dense quark matter requiring thermodynamic consistent one can find that the range of the allowed EoSs is reduced, see Figure 7. Thermodynamically consistent means that the pressure of the hadronic phase is equal to that of the quark matter phase at the matching point.

\(^{45}\)This process is as efficient in sucking the energy out of the star as the Casino de Urca in Rio de Janeiro is in sucking money out of the pockets of the gamblers and, hence, the name.

\(^{46}\)In a weak coupling expansion of the EoS this can formally be neglected.

\(^{47}\)For these calculations crossings of quark mass thresholds are taken into account.
Figure 6: The total quark number density $n(\mu, \Lambda)$ evaluated to $O(\alpha_s^2)$ as a function of the quark chemical potential $\mu$ ($\mu = \mu_d = \mu_s$, see footnote 28), taken from [12]. '2 flavor' and '3 flavor' label 2 and 3 massless quark flavors, respectively, and '2 + 1 flavor' denotes a 2 light and 1 massive flavor system. The calculations are done for the case of beta equilibrium and locally charge neutrality (which requires $\mu_e = 0$ for $N_f = 3$ and $\mu_e \neq 0$ for $N_f = 2$) and normalized to the density of three massless flavors $n^{(0)}(\mu)$. See [12], for further technical notes.

Figure 7: A comparison between several nuclear matter EoSs with a hybrid EoS obtained using the quark matter EoS of Figure 6, taken from [11].
3. From QCD to Compact Stars: The TOV Equation

In the previous section we have seen how we may get the EoS of cold and dense matter. The next step will be to solve the TOV equation by using an appropriate EoS. To get a better understanding of the TOV equation let us recall equation (1.4) and use it to rewrite equation (1.5) as

$$\frac{dm}{dr} = 4\pi r^2 \epsilon(r),$$  \hspace{1cm} (3.1)

$$4\pi r^2 dp(r) = -\frac{Gm(r)dm(r)}{r^2} \left[ 1 + \frac{p(r)}{\epsilon(r)} \right] \left[ 1 + \frac{4\pi r^3 p(r)}{m(r)} \right] \left[ 1 - \frac{2Gm(r)}{r} \right]^{-1}. \hspace{1cm} (3.2)$$

We may think of a shell of matter in the star of radius \(r\) and thickness \(dr\). The first equation gives the mass energy in this shell. \(p(r)\) is the pressure of matter interior to this shell and \(p(r) + dp(r)\) exterior to it \(^{48}\). Hence, the left hand side of the TOV equation (3.2) is the net force acting outward on the surface of the shell by the pressure difference \(dp(r)\). The first factor on the right hand side is the attractive Newtonian force of gravity acting on the shell by the mass interior to it. The remaining three factors are corrections due to GR. Thus, these equations are the equations of hydrostatic equilibrium in GR, as each fluid element is at rest in the star.

Newtonian stellar structure is obtained as a special case of the TOV equation \(^{49}\). It follows from the second and the third term in equation (3.2) being approximately equal to one, because \(p \ll \epsilon\), as \(\epsilon\) is dominated by the rest mass of the baryons, which does not contribute to the pressure. Furthermore the fourth term is also approximately equal to one for stars such as our sun, \(2GM/\mathcal{R} = 4.3 \cdot 10^{-6}\).

To solve the TOV equation and equation (3.1) for a given EoS we have to integrate from the origin with the initial conditions \(m(0) = 0\) and an arbitrary central density \(\epsilon(0)\) until the pressure \(p(r)\) becomes zero \(^{50}\). The point \(R\) where the pressure vanishes defines the radius of the star and \(m(R)\) its gravitational mass \(^{51}\).

For a given EoS there is a unique relationship between the stellar mass and the central energy density \(\epsilon(0)\). Thus, for each possible EoS there is a unique sequence of stars parametrized by the central energy density which is shown in Figure 8.

Furthermore, recent advances in observations of compact stars have ruled out regions of their mass-radius plane. From Figure 9 we can see that the hybrid EoSs that we obtain by matching the hadron and quark phases result in mass-radius curves are within the standard deviation of the observed neutron stars.

\(^{48}\)According to equation (3.2) \(dp\) is negative.

\(^{49}\)The Newtonian limit follows from equation (3.1) and the requirement that pressure balances gravity, that is \(4\pi r^2 dp(r) = -Gm(r)dm(r)/r^2\). The last term in equation (3.2) follows from the metric where \(R_s = 2GM\) is the Schwarzschild radius. For neutron stars this value is usually in the range of 3km to 6km.

\(^{50}\)In a vicinity of \(r = 0\) we may write \(m(r) = 4/3\pi r^3 \epsilon(0)\).

\(^{51}\)The energy density is finite at this point.
Figure 8: A rough summary of the present understanding of the solution of the mass $M$ of the TOV equation over a broad range of central energy densities $\epsilon$, taken from [7]. $\epsilon_0$ is the saturation energy density. The solid lines are stable configurations. A necessary condition for stability is $\partial M(\epsilon)/\partial \epsilon > 0$ [7]. Be aware of the dependence of a solution on the applied EoS.
Figure 9: Mass-radius relation of observed neutron stars and pure and mixed phases hybrid EoSs, taken from [11]. We assume negligible spin frequencies for both. 'Case I' and 'Case II' label the matching of the quark matter and nucleonic EoSs at low ($n_B \lesssim 0.39\text{ fm}$) and high ($n_B \gtrsim 0.64\text{ fm}$) baryon densities, respectively [12].

### 3.1. Solving the TOV Equation for a Constant Energy Density

The TOV equation with equation (3.1) and an appropriate EoS is usually solved numerically, as it is in general not analytically solvable. For a constant energy density $\epsilon_0$ as the EoS, describing a self-bound star, it is however possible to solve it by integration. A self-bound star is a star consisting of matter which is bound by strong interactions, even without gravity. Gravitationally bound means that the matter is bound by gravity.

A constant density is not natural, as it implies an incompressible fluid which contradicts the principle of causality in GR.

From equation (3.1) the mass

$$m(r) = \begin{cases} 
\frac{4}{3}\pi \epsilon_0 r^3, & r < R \\
M = \frac{4}{3}\pi \epsilon_0 R^3, & r \geq R 
\end{cases}$$

\[ (3.3) \]
follows directly. With equation (3.3) we can rewrite the TOV equation as

$$ \frac{4\epsilon_0}{(\epsilon_0 + p)(\epsilon_0 + 3p)} \, dp = -\frac{16}{3} G \pi \epsilon_0 r \, dr. $$  \hspace{1cm} (3.4)

We can restate the left hand site of equation (3.4) as

$$ \frac{4\epsilon_0}{(\epsilon_0 + p)(\epsilon_0 + 3p)} \, dp = 2 \left( \frac{3}{\epsilon_0 + 3p} - \frac{1}{\epsilon_0 + p} \right) \, dp. $$ \hspace{1cm} (3.5)

If we combine equation (3.4) and equation (3.5) and integrate the result from \( p_0 = p(r = 0) \) to \( p = p(r') \) and \( r = 0 \) to \( r = r' \), respectively, we will get

$$ 2 \left( \ln \left( \frac{\epsilon_0 + 3p}{\epsilon_0 + p} \right) + \ln \left( \frac{\epsilon_0 + p_0}{\epsilon_0 + 3p_0} \right) \right) = \ln \left( 1 - \frac{8}{3} G \pi \epsilon_0 r'^2 \right) $$ \hspace{1cm} (3.6)

or equivalently

$$ \frac{\epsilon_0 + p}{\epsilon_0 + p_0} = \frac{\epsilon_0 + 3p_0}{\epsilon_0 + p_0} \sqrt{1 - \frac{8}{3} G \pi \epsilon_0 r'^2} $$ \hspace{1cm} (3.7)

With equation (3.3), equation (3.7) results in

$$ p = \epsilon_0 \sqrt{1 - \frac{2GMr'^2}{R^3}} - \frac{\epsilon_0 + p_0}{\epsilon_0 + 3p_0} \sqrt{1 - \frac{2GMr'^2}{R^3}}. $$ \hspace{1cm} (3.8)

Finally, we apply the condition of vanishing pressure for \( r' = R \), yielding

$$ \frac{\epsilon_0 + p_0}{\epsilon_0 + 3p_0} = \sqrt{1 - \frac{2GM}{R}}, $$ \hspace{1cm} (3.9)

or equivalently the (linear) mass-radius relation \( ^{55} \)

$$ M = \frac{1 - \left( \frac{\epsilon_0 + p_0}{\epsilon_0 + 3p_0} \right)^2}{2G} \, R $$ \hspace{1cm} (3.10)

and the solution (renaming \( r' \) \( r \))

$$ p(r) = \epsilon_0 \sqrt{1 - \frac{2GMr^2}{R^3}} - \sqrt{1 - \frac{2GM}{R}} \, R \sqrt{1 - \frac{2GMr^2}{R^3}}. $$ \hspace{1cm} (3.11)

To study the behavior of this solution we consider the central pressure

$$ p_0 = \epsilon_0 \frac{1 - \sqrt{1 - \frac{2GM}{R}}}{3 \sqrt{1 - \frac{2GM}{R}} - 1}. $$ \hspace{1cm} (3.12)

\(^{54}\)Remember that the pressure outside a star is zero.

\(^{55}\)The polytrope \( p = 1/3 \epsilon \) would yield the same relationship.
For a fixed energy density $\epsilon_0$, the central pressure $p_0$, the mass $M$ (see equation (3.3)) and, hence, the ratio $M/R$ increases monotonically, as the radius $R$ increases. This is what we anticipate, since, as more matter is added to a star a greater pressure is required to counterbalance it. The central pressure becomes infinite if

$$R_{\text{max}} = \left(\frac{9}{4}\right)GM_{\text{max}}. \quad (3.13)$$

When we consider $\epsilon_0 = 10^{17} \text{kg} \cdot \text{m}^{-3}$, as an example for a density of the order of the saturation density, we will get the results $R_{\text{max}} = 21 \text{km}$ and $M_{\text{max}} = 4 \cdot 10^{31} \text{kg}$ from equation (3.13), ensuring a finite central pressure, and equation (3.3). These values match the typical values for a compact star introduced at the beginning of this paper at hand. Further properties of equation (3.11) are discussed in [62].

### 3.2. Solving the TOV Equation for a Hybrid EoS

In this section we will solve the TOV equation numerically, as one has to do in general. We will apply a nuclear matter/quark matter hybrid EoS, taken from [11]. The data are computed by thermodynamic concistence matching of the hadronic (APR, see section 2.2) and quark matter phases (see section 2.3). We will do the computation using ’Mathematica’ (see [64]). The code and comments on it are appended. Below we will describe how the computation is done.

Initially the data have to be described by a continuous function. However, as the data shows a jump in the energy density we will describe it by piecewise continuous functions. The jump originates from a first order phase transition, as we consider a hybrid star. We choose polynomials of roughly the order of the square root of the number of the data points to match them, as they behave smooth. Additionally, we have to be aware of the initial condition of the mass. If we will define the mass to be zero at zero radius this will cause a singularity in the TOV equation. Hence, we will define the mass to be zero at some small radius close to zero. This should not affect the result and thus, we choose it to be in the order of the accuracy of the computer and small compared to the radius of a star. Thereafter we solve the TOV equation numerically for various central pressures and calculate the radius and the mass thereby. The result implies masses which are related to two different radii, which is unphysical. If the star has the lower radius, it rotates faster which means that its energy

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56 As this is a qualitative statement, we assume that the central pressure is well defined.
57 This limit is a purely relativistic phenomena and does not occur in Newtonian theory.
58 Be aware that we have to multiply the right hand site of equation (3.13) by $c$.
59 See [63] and [12], for further explanatory notes on the computation.
60 A non-injective radius is physically allowed, though. As a star is described by its mass this means that they are different stars.
Figure 10: The mass-radius relation of a nuclear matter/quark matter hybrid star. Its calculation is appended. We have done a cutoff on radii higher than 14 km, as all stars with a mass below a limit of approximately 1 $M_\odot$ are white dwarfs. We could have done the cutoff at this limit, however, it is not exactly known and the mass-radius curve describes physically allowed small masses states. Starting from the right end of the shown curve the central pressure increases with going along the curve. These stars contain a small quark core at its centers. Be aware that no uncertainties are included.

is higher [7]. Hence, we will do a cutoff on the lower radii 61. The result is shown in Figure 10 62.

Figure 10 is in good agreement with the pure phases 'Case I' in Figure 9, as it should be. Corresponding to Figure 10, the maximum mass of a non-rotating compact star is about 1.9 $M_\odot$ and their radii are in the range of 11.8 km to 12.6 km 63. These values are in agreement with observational data (see [11] and references therein, for example).

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61 Additionally the region $R \leq 2GM = R_s$ is forbidden. Sometimes the region $R \leq 3GM = R_s$ is forbidden by GR causality of the EoS, too.
62 A change in the slope changes the stability of a mode.
63 The maximum mass of a compact star decreases with an increasing quark matter core [11].
4. Conclusions and Outlook

In this thesis we have investigated nuclear matter at low temperatures and densities slightly above that of the nuclear saturation density. Such kind of matter can’t be produced or probed in laboratories, so the only window to it is a compact star. In fact, this region of nuclear matter is not accessible from the theory, either. Neither perturbative QCD calculations at such low densities can be performed, lattice QCD simulations for cold and dense matter are possible, nor extend existing nuclear matter models for low densities to this region can be extended. Usually, one combines microscopic calculations and macroscopic observations by solving the TOV equation for an proper EoS yielding the mass-radius relation of a star. The EoS may be given by phenomenological models (see section 2.2) or an interpolation between EoSs of the known low density nuclear matter and high density quark matter regions (see section 2.3).

In this thesis, we have applied an EoS describing a star consisting of a strange quark matter core and a nuclear mantle. We have determined the mass-radius relation numerically, finding that hybrid stars are realizable up to masses of $1.9 M_\odot$, which matches recent observations on compact stars. This value is not the ultimate limit for the maximum mass of compact stars as one has to take into account rotational effects.

Additionally, one may apply an EoS describing mixed phases, condensations and hyperons yielding further solutions to the TOV equation (see [11]). As described in this thesis, the TOV equation yields two families of stable solutions - white dwarfs and neutron stars. In fact, a third family of solutions exists. It is shown that any first order phase transition in the matter may result in that solution [65]. Further observations on compact stars, their masses, radii, frequencies and further properties will constrain the theoretical considerations and may rule out EoSs.
A. Solving the TOV Equation Numerically

The computation is done using 'Mathematica 8', see http://reference.wolfram.com/mathematica/guide/Mathematica.html.
Msol = 1.98892 \times 10^{30}; (*solar mass [kg]*)

k = 10^{3};

G = 6.67384 \times 10^{(-11)} \times Msol/k^3; (*gravitational constant [km^3 * Msol^(-1) * s^(-2)]*)

c = 2.99792 \times 10^{8}/k; (*velocity of light in vacuum [km * s^(-1)]*)

datUnit = 1.0218 \times 10^{(-10)}/c^2/(10^(-15))^3/Msol * k; (*GeV/fm^3 \rightarrow Msol/km^3*)

file = Import[“(*file path*)\EoS_H0L0.378X3.5ms0.0922D0.1M0.955379mix0.dat”, “Table”]

(*http://theory.physics.helsinki.fi/~aekurkel/neutron/*)

(*{pressure [GeV/fm^3], energy density [GeV/fm^3]}*)

Show[ListPlot[Take[file, {100, 140}]], AxesLabel \rightarrow {“p [GeV/fm]”, “\varepsilon [GeV/fm]”}]

(*jump in \rho at p = 0.002 (hybrid star]*)

(*matching the data by functions*)

data1 = Join[Table[{file[[i, 1]] * datUnit, file[[i, 2]] * datUnit}, {i, 1, 120}],

{{file[212, 1]] * datUnit, file[212, 2] * datUnit}}];

data2 = Table[{file[[i, 1]] * datUnit, file[[i, 2]] * datUnit}, {i, 121, 211}];

series1 = Flatten[Table[{p[r]^((i/10))}, {i, 0, 10}]]; 

series2 = Flatten[Table[{p[r]^((i))}, {i, 0, 10}]];
\( \varepsilon_1[r] = \text{Fit}[\text{data1}, \text{series1}, p[r]]; \)

\( \varepsilon_2[r] = \text{Fit}[\text{data2}, \text{series2}, p[r]]; \)

Show[Plot[\( \varepsilon_1[r] \), \{p[r], 0.002 * \text{datUnit}, 85\text{datUnit} \}], \text{ListPlot}[\text{data1}],
AxesLabel \rightarrow \{"p [\text{Msol/km}]", "\varepsilon [\text{Msol/km}]"\}]

Show[Plot[\( \varepsilon_2[r] \), \{p[r], 0, 0.002 * \text{datUnit} \}], \text{ListPlot}[\text{data2}],
AxesLabel \rightarrow \{"p [\text{Msol/km}]", "\varepsilon [\text{Msol/km}]"\}]

\( \varepsilon[r_] := \text{Piecewise}[\{\{\varepsilon_1[r], p[r] > 0.002 * \text{datUnit}\}, \{\varepsilon_2[r], p[r] < 0.002 * \text{datUnit}\}\}] \)

(*EoS*)

(*energy density \( \varepsilon \) as a function of the radius \( r \) (and as an implicit function of the pressure \( p[r] \))*
\[
\text{eps} = 10^{-8};
\]
(*definition of the radius as some small number, that is eps, close to zero
for the initial condition of the mass, because zero radius causes a singularity*)

\[
\text{sol[pc] := NDSolve}\left\{\{p'[x] == -G (\varepsilon [x] + p[x]) * (m[x]/c^2 + 4 \pi \varepsilon x^3 \cdot p[x]/c^2),
\right.
\]
\[
(x \cdot (x - 2 \cdot G \cdot m[x]/c^2), m'[x]/c^2 == 4 \pi x^2 \varepsilon [x]/c^2, p[\text{eps}] == \text{pc,}
\right.
\]
\[
m[\text{eps}] == 0\}, \{p, m\}, \{x, \varepsilon, 100\}\}
\]

(*TOV equation in units of the solar mass and km with initial conditions that the mass \(m\)
at zero radius \(x\) is zero (\(\equiv\varepsilon\)) and the central pressure \(p(\text{at radius zero})\) is some constant \(\text{pc}\*)

\[
\text{M[r_, pc_] := First}[m[r]/\text{sol[pc]}]
\]
(*mass \(M\) of the star as a function of the radius \(r\) and the central pressure \(\text{pc}\*)

\[
\text{P[r_, pc_] := First}[p[r]/\text{sol[pc]}]
\]
(*pressure \(P\) of the star as a function of radius \(r\) and the central pressure \(\text{pc}\*)

\[
\text{R[pc_] := Last}[\text{Last}[\text{FindRoot}[P[r, \text{pc}] == 0, \{r, 10\}]]]
\]
(*radius \(R\) of the star as a function of central density \(\text{pc}\*)

\[
\text{tabMR} = \text{Join}[\text{Table}\{\{\text{R}[i/10^7], \text{M}[\text{R}[i/10^7], i/10^7]\}, \{i, 1, 10\}],
\]
\[
\text{Table}\{\{\text{R}[i/(2 \cdot 10^6)], \text{M}[\text{R}[i/(2 \cdot 10^6)], i/(2 \cdot 10^6)]\}, \{i, 2, 20\}],
\]
\[
\text{Table}\{\{\text{R}[i/(2 \cdot 10^5)], \text{M}[\text{R}[i/(2 \cdot 10^5)], i/(2 \cdot 10^5)]\}, \{i, 2, 20\}],
\]
\[
\text{Table}\{\{\text{R}[i/(2 \cdot 10^4)], \text{M}[\text{R}[i/(2 \cdot 10^4)], i/(2 \cdot 10^4)]\}, \{i, 2, 20\}];
\]
(*table of the radius and the mass of the star for various central pressures*)

\[
\text{ListPlot}[\text{tabMR, AxesLabel} \rightarrow \{"R \text{ [km]}", "M / Msol"\}, \text{PlotRange} \rightarrow \{\{0, 50\}, \{0, 2.5\}\}]
\]
(*mass-radius relation of the star*)
ListPlot[Take[tabMR, 51], AxesLabel -> {"R [km]", "M /"Subscript[M, solar]"},
PlotRange -> {{11, 14}, {0, 2}}, Joined -> True, PlotStyle -> Thick,
PlotLabel -> "strange quark matter hybrid star"]

(*mass has to be injective*)
References


[3] [http://pdg.lbl.gov/].


[20] [http://www.esa.int/SPECIALS/Integral/SEMACK0VRHE_0.html].


[63] [http://theory.physics.helsinki.fi/~aekurkel/neutron/].

[64] [http://www.wolframuniv.com/mathematica/].

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Declaration

I hereby affirm that this bachelor thesis represents my own work and has not been previously submitted to any examination office. All resources used have been referenced.

October 18, 2011