pQCD at High Temperature

Diplomarbeit

an der
Fakultät für Physik
Universität Bielefeld

vorgelegt von
Ervin Bejdakic

31. März 2006
# Contents

1 Introduction 5

2 QCD at high temperature 9
   2.1 Finite Temperature (scalar) Field Theory ............... 9
   2.2 QCD Partition Function .................................. 10

3 Nested Sums 15
   3.1 Introduction ............................................. 15
   3.2 Harmonic sums, Euler-Zagier sums ......................... 19
   3.3 Harmonic Polylogarithms .................................. 23

4 Methods of Computation 27
   4.1 Difference Equations .................................... 27
   4.2 Differential Equations .................................... 30
   4.3 x-Space Evaluation ....................................... 32
   4.4 Mellin-Barnes Method .................................... 35

5 Applications 39
   5.1 1 and 2 Loop ............................................. 39
   5.2 3 Loop .................................................. 39
   5.3 4 Loop .................................................. 43

6 Discussion and Outlook 53

A Example Codes 55

B Additional Results 61

C Acknowledgments 63
Chapter 1

Introduction

There are four forces in nature: gravity, electromagnetism and the strong and weak nuclear forces. The strong interaction has up to now been successfully described by Quantum Chromodynamics (QCD), a non-Abelian gauge theory. Its theoretical foundations was established in the early 1970’s [1],[2],[3]. QCD allows one to calculate the interactions of quarks and gluons, which carry color charge. Because of the strongly interacting nature, the quarks and gluons are confined into hadrons, so that no free quarks can be seen. The non-Abelian structure of QCD leads to non-linear equations of motion for gluons, which unlike photons in QED, interact with each other. Yet, exactly this property of gluons leads to an astonishing feature of QCD, asymptotic freedom [2],[3], which leads to a deconfinement at small distances, or high energies, of hadrons into a weakly interacting quark-gluon plasma. Numerical computations suggest that at a critical temperature $T_c = 170$ MeV this phase transition takes place [5].

In the energy region above $T_c$, one can, since the running coupling $g$ is small, apply the perturbation theory and compute amplitudes through Feynman diagrams (diagram and graph will be used as synonyms). Some of these graphs lead to infinities, and need to be regularized and/or renormalized, something which can be done by so-called dimensional reduction [6]. One computes the integrals in $d = 4 - 2\epsilon$ (or in $d = 3 - 2\epsilon$ in three dimensional theory) dimensions and expands in $\epsilon$. The infinities now become poles in $\epsilon$, which cancel each other when one sums all diagrams that contribute to a physical quantity.

This thesis is about the analytic calculation of typical scalar integrals that appear in Feynman diagrams. We will take a closer look at several techniques that are used in computation of Feynman graphs and apply these methods
on the set of the so-called master integrals (MI’s) of the SU($N_c$) + Adjoint Higgs Theory, which is an effective theory for QCD at high temperatures. The thesis is organized as follows. In the second chapter we introduce perturbative quantum field theory (pQFT) and perturbative QCD (pQCD) at high temperature and derive the generic Feynman rules. There we argue that even at temperatures above $T_c$ one cannot apply the simple perturbative expansion in the gauge coupling $g$ since at high temperature the free energy or pressure gets contributions from different momentum scales $T$, $gT$ and $g^2T$, so that in reality we have a multi-scale system. The perturbative expansion breaks also at $g^6$ order for other reasons [7]. What one has to do is to apply the approach of constructing effective three-dimensional theories [8], [9], that can be computed by analytical and numerical methods. The resulting electrostatic QCD (EQCD) can be used to compute the pressure or free energy up to $g^5$ order [11], [10], at $g^6$ order one has to take into account the contribution of $g^2T$ scale, which enters the stage. The contribution of $g^2T$ scale can be described by the magnetostatic QCD (MQCD), which is entirely non-perturbative. Still one has to compute the EQCD contribution up to four loops for the $g^6$ order. The set of all graphs of EQCD up to four loops can be reduced by general integration-by-parts (IBP) method to a set of master graphs which were given in [39]. The $\epsilon$-expansion of these Feynman graphs is expressible in terms of so called S/Z-sums.

In chapter 3 we introduce the concept of nested sums and their algebra, which will be used as a natural language for the results of the expansion of Feynman graphs. Two special cases of nested sums, the harmonic sums and harmonic polylogarithms or polylogs for short, are treated in more detail. The following chapter deals with four methods of computing Feynman diagrams, namely via difference equations, differential equations, the x-space method and the Mellin-Barnes method. Some simple examples will be given to illustrate the methods. There will appear diagrams which can be computed with several methods, like $\square$, where the solid lines stand for scalar massive propagators and dotted lines for massless propagators. Unfortunately, there will be diagrams which cannot (yet) be solved with any method presented here. These are $\bigotimes$ and $\bigotimes$.

In the fifth chapter we apply these methods to the MI’s of SU($N_c$) + Adjoint Higgs Theory. Since some of the master integrals of SU($N_c$) + Adjoint Higgs Theory also appear in computations in various models we try to solve all graphs in $d$ dimensions if possible, if not then in 3 dimensions for EQCD or in 4 dimensions for Standard Model (SM). At one loop level there is only
one graph , and there are no two loop integrals. At three loop level there are two master integrals, and . The first is computed solving the corresponding difference equation, the second by using a relation to a master integral which can be computed using differential equation method from [37]. In [37] also three four loop integrals , and could have been computed using the same method. The master integrals , and can be solved using difference equations, but can only be expanded in . The two master integrals and couldn’t have been solved, because essentially the result of for all is necessary and we were not been able to compute it.

In the last chapter we discuss the methods used and take an outlook on possible future developments. In the appendix we collect some sample programs.
Chapter 2

QCD at high temperature

2.1 Finite Temperature (scalar) Field Theory

Let us first start with the statistical Quantum Mechanics, and generalize it then to Field Theory (we follow in this chapter basically [18]). The most important entity in statistical physics is the (grand-canonical) partition function $Z(T)$ which is defined as:

$$Z(T) = \text{Tr} \left[ e^{-\beta \hat{H}} \right]$$  

(2.1)

where $\beta = \frac{1}{T}$, $T$ being the temperature of the system and $\hat{H}$ is the Hamiltonian. From $Z(T)$ one can compute all thermodynamic quantities through fundamental thermodynamic relations, e.g. free energy $F = -T \ln(Z(T))$ or entropy $S = \frac{F}{T} + \frac{E}{T}$. It will be useful to write the partition function as a functional integral over the eigenstates of the system:

$$Z(T) = \lim_{N \to \infty} \int_{-\infty}^{+\infty} dx \langle x | e^{-\beta \hat{H}} | x \rangle$$

(2.2)

Inserting $\int \frac{dp}{2\pi\hbar}|p\rangle\langle p| = \mathbb{1}$ and using the relations $\langle p|x \rangle = e^{ipx}$ and $\langle p|e^{-\frac{\beta}{2} \hat{H}}|x \rangle = e^{ipx-\frac{\beta}{2} \hat{H}+O(\epsilon^2)}$ we get:

$$Z(T) = \lim_{N \to \infty} \int \prod_{i=1}^{N} dx_i \frac{dp_i}{2\pi\hbar} e^{\left\{ -\frac{\hbar}{2} \sum_{j=1}^{N} \left( \frac{p_j^2}{2m} + ip_j x_{j+1} - x_j + V(x_j) \right) \right\}}$$  

(2.3)

where $x_{N+1} - x_1$ and $\epsilon = \frac{\hbar}{N}$. Performing the limes and integrating over $p_i$, since it is Gaussian integral, we can write formally the path integral:

$$Z(T) = \int_{x(0)=x(\beta \hbar)} \mathcal{D}x e^{-\frac{\beta}{2} \int_0^{\beta \hbar} d\tau \mathcal{L}_E(\frac{dx}{d\tau}, x(\tau))}$$

(2.4)
where $\mathcal{L}_E = \mathcal{H} - px$ is the Euclidean Lagrangian which is connected with the one in Minkowski space by $\tau = -it$.

Now let us interpret the argument $x$ for a given $\tau$ in the above path integral as an internal degree of freedom of a general function, called $\phi(\tau, \bar{x}) = \phi(x)$. We get then the path integral representation of a partition function for a scalar field theory:

$$Z_{SFT}(T) = \int_{\phi(0, \bar{x}) = \phi(\tau, \bar{x})} \mathcal{D}\phi(x)e^{-\frac{1}{\tau} \int_0^{\beta \tau} d\tau \int d^3 x \mathcal{L}_E}$$  \hspace{1cm} (2.5)

For the fermions analog way leads to fermionic partition function for a scalar field theory:

$$Z_{\text{ferm.}}(T) = \int \psi(0, \bar{x}) = -\psi(\tau, \bar{x}) \mathcal{D}\psi(x)\mathcal{D}\bar{\psi}(x)e^{-\frac{1}{\tau} \int_0^{\beta \tau} d\tau \int d^3 x \bar{\psi}[\gamma^E_\mu \partial_\mu + m] \psi}$$

$$= \int \psi(0, \bar{x}) = -\psi(\tau, \bar{x}) \mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x)e^{-\frac{1}{\tau} \int_0^{\beta \tau} d\tau \int d^3 x \mathcal{L}_E}$$  \hspace{1cm} (2.6)

with Euclidean $\gamma$-functions with properties $\{\gamma^E_\mu, \gamma^E_\mu\} = 2\delta_{\mu\nu}$, and $(\gamma^E_\mu)^\dagger = \gamma^E_\mu$, $\gamma^E_0 = \gamma^0$, $\gamma^E_i = -i\gamma^i$.

### 2.2 QCD Partition Function

The Euclidean Lagrangian is given for QCD by:

$$\mathcal{L}_{QCD} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi} \gamma_\mu D_\mu \psi$$  \hspace{1cm} (2.7)

where $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_b^\mu A_c^\nu$ is the field strength tensor and $D_\mu \equiv \partial_\mu - igA_\mu \equiv \partial_\mu - igA_\mu^a T^a$ the covariant derivative, the $T^a$, $a = 1, \ldots, N^2 - 1$ are generators of the fundamental representation of SU(N) and $f^{abc}$ are the structure coefficients of SU(N) given by $[T^a, T^b] = if^{abc}T^c$. The QCD Lagrangian is invariant under:

$$A_\mu \equiv A_\mu^a T^a \rightarrow G^{-1} A_\mu G + \frac{i}{g} (\partial_\mu G^{-1}) G \psi \rightarrow G^{-1} \psi$$

$$G \equiv \exp[i g T^a \alpha^a]$$  \hspace{1cm} (2.8)
2.2. QCD PARTITION FUNCTION

with \( \alpha^a \) being a smooth function. The partition function for QCD is:

\[
Z_{QCD} = \int_{\text{periodic}} DA_\mu^a \int_{\text{periodic}} D\tilde{\eta} D\eta \int_{\text{antiperiodic}} D\bar{\psi} D\psi e^{\left\{ -\int_0^\beta d\tau \int d^3x \left[ -\frac{1}{2} A_\mu^a \left( \delta_{\mu\nu} \partial^2 - (1 - \frac{1}{\xi}) \partial_\nu \partial_\mu \right) A_\nu^a + \bar{\psi} \left( \gamma_\mu \partial_\mu - m \right) \psi + \bar{\eta} \partial^2 \eta \right] \right\}}
\]

\[
= \int_{\text{periodic}} DA_\mu^a \int_{\text{periodic}} D\tilde{\eta} D\eta \int_{\text{antiperiodic}} D\bar{\psi} D\psi e^{S_0 + S_{INT}}
\]

(2.9)

where \( \tilde{\eta}, \eta \) are the Faddeev-Popov ghosts, which have the same boundary conditions as the gauge fields.

The QCD Lagrangian, split in free and interaction part, gives us the Feynman rules. From the free (quadratic) part of the action,

\[
S_0 = \int_0^\beta d\tau \int d^3x \left\{ -\frac{1}{2} A_\mu^a \left( \delta_{\mu\nu} \partial^2 - (1 - \frac{1}{\xi}) \partial_\nu \partial_\mu \right) A_\nu^a + \bar{\psi} \left( \gamma_\mu \partial_\mu - m \right) \psi + \bar{\eta} \partial^2 \eta \right\}
\]

(2.10)

one gets the propagators in momentum space by a Fourier transformation, which are:

\[
\begin{align*}
\rightarrow & = \frac{[i\gamma_\mu p_\mu + m]}{p^2 + m^2} & (2.11) \\
\overrightarrow{\quad} & = \frac{1}{p^2} & (2.12) \\
\overleftrightarrow{\quad} & = \frac{1}{p^2} \left[ \delta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right] & (2.13)
\end{align*}
\]

The vertex functions can be obtained from the interaction part,

\[
S_{INT} = g \int_0^\beta d\tau \int d^3x \left\{ \frac{g}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e + f^{abc} (\partial_\mu A_\nu^a) A_\rho^b A_\sigma^c - i\bar{\psi} \hat{A} \psi + f^{abc} \bar{\eta}^a A_\rho^c \partial_\rho \eta^b \right\}
\]

(2.14)

and are given in momentum space by:

\[
\begin{align*}
\rightarrow & = -g^2 \left\{ f^{ebd} f^{fcc} (\delta_{\alpha\epsilon} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\epsilon}) + \begin{pmatrix} b & c \leftrightarrow d \end{pmatrix} \right\} \\
= & ig \{ \delta_{\alpha\beta} (p - q)_\gamma + \delta_{\beta\gamma} (q - r)_\alpha + \delta_{\alpha\gamma} (r - p)_\beta \} \\
= & ig \alpha_\gamma (T^a)_{ij} \\
= & ig f^{abc} p_\alpha
\end{align*}
\]

(2.15)
However, although at high temperature the coupling constant is small, the straightforward perturbation theory in powers of $g^2$ fails. This is due to the fact that at high temperature we are dealing with a multi-scale system. The reason for this is that the non-static modes gain effective masses that grow linearly with increasing temperature and then decouple, leaving the zero-modes of the gauge fields

$$A^a_\mu(x) = T \sum_{-\infty}^{\infty} \exp[i\omega_n^b \tau] A^a_{\mu,n}(x)$$

where $\omega_n^b = 2n\pi T$ are the Matsubara frequencies , (2.16) as true degrees of freedom contributing. These modes can be described by an electrostatic scalar field $A^a_0(x)$ and magnetostatic gauge field $A^a_i(x)$ of a three dimensional effective theory, called electrostatic QCD (EQCD), with the Lagrangian:

$$L_{EQCD} = \frac{1}{2} Tr F_{ij}^a + Tr [D_i, A_0]^2 + m_E^2 Tr A_0^a + \frac{ig^3}{3\pi^2} \sum_f \mu_f Tr A_0^3 +$$

$$+ \lambda_E^{(1)} (Tr A_0^2)^2 + \lambda_E^{(2)} Tr A_0^4 + \text{higher order operators}$$

(2.17)

with

$$F_{ij}^a = \partial_i A^a_j - \partial_j A^a_i + g_E f^{abc} A^b_i A^c_j$$

(2.18)

$$D_i = \partial_i - ig_E A_i$$

(2.19)

Using this Lagrangian the partition function, or to use a physical quantity, minus the grand canonical energy density (that is the pressure), can be written (valid for sufficiently high energy) as:

$$p_{QCD}(T) = p_E(T) + \frac{T}{V} \ln \int \mathcal{D} A^a_0 \mathcal{D} A^a_0 \exp\{-S_E\}$$

(2.20)

where $p_E = p_{EQCD}$ is a parameter of the effective theory computable in perturbative full QCD [10]. With this theory one is able to compute the pressure of the full theory to the order $g^5$ [10]. To be able to compute higher orders the $g^5$, one must make further separations since there are still two dynamical scales $gT$ and $g^2 T$ [8]. The non-perturbative scale $g^2 T$ which enters in the computation at order $g^6$, originates from the magnetostatic sector, that is from the fields $A^a_0$, so that we can write:

$$p_{QCD} = p_{EQCD} + p_{MQCD} + \frac{T}{V} \ln \int \mathcal{D} A^a_0 \exp\{-S_{MQCD}\}$$

(2.21)
where

\[
\mathcal{L}_{MQCD} = \frac{1}{2} Tr F^a_{ij} \\
F^a_{ij} = \partial_i A_j^a - \partial_j A_i^a + g_M f^{abc} A_i^b A_j^c
\]  

(2.22)

g_M = g_{MQCD} is, analogous to \( p_E \), computable through perturbative expansion of EQCD. The next goal in computation of \( p_{QCD} \) is \( g^6 \) order (for a review see [44]) and that means in particular that one needs to go for four loop Feynman graphs in EQCD. The set of all four loop graphs has been given in [39] and after applying the general partial integration identities [17] this set is further reduced to the set of so called “master” integrals [23], which have to be computed in \( d = 3 - 2\epsilon \). Again, since most of the master integrals are also needed in other theories, we will try to solve them all in \( d = 4 - 2\epsilon \) as well.
Chapter 3

Nested Sums

3.1 Introduction

The Z-sums are defined recursively by

\[ Z(n) = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases} \]

\[ Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{i=1}^{n} \frac{x_i^j}{i^w} Z(i-1; m_2, \ldots, m_k; x_2, \ldots, x_k) \quad (3.1) \]

where \( k \) is called the depth and \( w = m_1 + m_2 + \ldots + m_k \) the weight of the Z-sum. Equivalent definition can be given by

\[ Z(n) = \sum_{n \geq i_1 \geq i_2 \geq \ldots \geq i_k > 0} \frac{x_{i_1}^{i_1}}{i_1^{m_1}} \ldots \frac{x_{i_k}^{i_k}}{i_k^{m_k}} \quad (3.2) \]

Analogous definition can be given for the S-sums

\[ S(n) = \begin{cases} 1 : n > 0 \\ 0 : n \leq 0 \end{cases} \]

\[ S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{i=1}^{n} \frac{x_i^j}{i^w} S(i; m_2, \ldots, m_k; x_2, \ldots, x_k) \quad (3.3) \]

or

\[ S(n) = \sum_{n \geq i_1 \geq i_2 \geq \ldots \geq i_k > 1} \frac{x_{i_1}^{i_1}}{i_1^{m_1}} \ldots \frac{x_{i_k}^{i_k}}{i_k^{m_k}} \quad (3.4) \]
Notice that the difference between the S- and Z-sums is the upper summation boundary, (i-1) for Z- and (i) for S-sums. With the help of the following formula, one can easily convert Z-sums into S-sums and vice versa

\[
S(n; m_1, \ldots; x_1, \ldots) = \sum_{i=1}^{n} \sum_{m_1}^{i-1} \sum_{m_2}^{i-1} S(i; m_3, \ldots; x_3, \ldots) + S(n; m_1 + m_2, \ldots; x_1 x_2, x_3, \ldots)
\]

\[
Z(n; m_1, \ldots; x_1, \ldots) = \sum_{i=1}^{n} \sum_{m_1}^{i} \sum_{m_2}^{i} \sum_{m_3}^{i} Z(i; m - 1; m_3, \ldots; x_3, \ldots) - Z(n; m_1 + m_2, \ldots; x_1 x_2, x_3, \ldots)
\]  (3.5)

Since, as we will see, Z-sums form an algebra, the product of two Z-sums with the same upper summation, that is the same argument, can be written in terms of single Z-sums

\[
Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) \times Z(n; m_1, \ldots, m_l; x_1, \ldots, x_l) = \sum_{i=1}^{n} \sum_{m_1}^{i} \sum_{m_2}^{i} \sum_{m_3}^{i} Z(i - 1; m - 1; m_2, \ldots, m_k; x_1, \ldots, x_k) \times Z(i - 1; m_1 - 1; m_3, \ldots; x_1, \ldots, x_l)
\]

\[
+ \sum_{i=1}^{n} \sum_{m_1}^{i} \sum_{m_2}^{i} \sum_{m_3}^{i} Z(i - 1; m_1 - 1, m_3; x_1, \ldots, x_k) \times Z(i - 1; m_2 - 1, m_3; x_2, \ldots, x_l)
\]

\[
+ \sum_{i=1}^{n} \sum_{m_1}^{i} \sum_{m_2}^{i} \sum_{m_3}^{i} Z(i - 1; m_1 - 1, m_3; x_1, \ldots, x_k) \times Z(i - 1; m_2 - 1, m_3; x_2, \ldots, x_l)
\]  (3.6)

This formula can be applied recursively on the RHS and since per definition it has an ending, it is a well defined recursion which at the end leaves us with single Z-sums. For example:

\[
Z(n; m_1, m_2; x_1, x_2) \times Z(n; m_3; x_3) =
\]

\[
Z(n; m_1, m_2, m_3; x_1, x_2, x_3) + Z(n; m_1, m_3, m_2; x_1, x_3, x_2) + Z(n; m_1, m_2, m_3; x_1, x_2, x_3) + Z(n; m_1 + m_2, m_3; x_1 x_3, x_2)
\]  (3.7)
3.1. INTRODUCTION

Similarly the product of two $S$-sums simplifies to sum of single sums:

\[
S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) \times S(n_1; m_1', \ldots, m_{i'}; x_1', \ldots, x_{i'}) = \\
\sum_{i=1}^{n} \frac{x_i^{i_1}}{i_1} S(i_1; m_2, \ldots, m_k; x_2, \ldots, x_k) S(i_1, m_1', \ldots, m_{i'}; x_1', \ldots, x_{i'}) \\
+ \sum_{i=1}^{n} \frac{x_i^{i_2}}{i_2} S(i_2; m_1, \ldots, m_k; x_1, \ldots, x_k) S(i_2, m_2', \ldots, m_{i'}; x_2', \ldots, x_{i'}) \\
- \sum_{i=1}^{n} \frac{(x_i x_i')^i}{m_1 + m_{i'}} S(i; m_2, \ldots, m_k; x_2, \ldots, x_k) S(i, m_2', \ldots, m_{i'}; x_2', \ldots, x_{i'})
\]  
(3.8)

The proof for the equation (3.6) uses the triangle relation (see Fig. 3.1):

\[
\sum_{i=1}^{n} \sum_{j=1}^{i} a_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_{ij} + \sum_{j=1}^{n} \sum_{i=1}^{j-1} a_{ij} + \sum_{i=1}^{n} a_{ii}
\]  
(3.9)

The equation (3.6) actually states that the $Z$-sums form a so called Hopf Algebra. To see this we first introduce some notation:

\[
\cdot : A \times A \to A \\
(m_1, x_1) \cdot (m_2, x_2) = (m_1 + m_2, x_1 x_2)
\]  
(3.10)

where $\cdot$ is a multiplication of the set of letters, called the alphabet $A$, a pair $(m_1, x_1)$ is called the letter of the alphabet and $m_i$ is called the degree of the letter. A word is then a concatenation, that is an ordered sequence, of letters, e.g.

\[
W = (m_1, x_1), (m_2, x_2), \ldots, (m_k, x_k)
\]
A word of length zero is called $e$. We define the quasi-shuffle algebra $\mathcal{A}$ as a vector space of words with:

$$
e \circ W = W \circ e = W$$

$$(m_1, x_1), W_1 \circ ((m_2, x_2), W_2) = (m_1, x_1), (W_1 \circ (m_2, x_2), W_2)$$

$$+ (m_2, x_2), (((m_1, x_1), W_1), W_2) \cdot ( (m_1, x_1) \cdot (m_2, x_2)), (W_1 \circ W_2)$$

(3.11)

where $\cdot$ is multiplication of letters and $\circ$ is product of algebra $\mathcal{A}$. One can see that the equation (3.6) is actually defining a quasi-shuffle algebra so Z-sums form a quasi-shuffle algebra. Now from [24] one knows that it is sufficient to show that some objects form a quasi-shuffle algebra and a general theorem proves that they also form a Hopf algebra. (The proof that a quasi-shuffle algebra forms a Hopf algebra will not be given here, for details see [24].)

The $Z/S$-sums are a fairly general object, in a lots of cases it wont be necessary to consider these general objects, but instead some simpler ones (see fig. 3.2). If one for example takes the index $n$ in Z-sums to be infinity, one ends with the so called multiple polylogarithms of Goncharov [25]:

$$Z(\infty; m_1, \ldots, m_k; x_1, \ldots, x_k) = \text{Li}_{m_1, \ldots, m_k}(x_1, \ldots, x_k).$$

(3.12)

If, in addition to $n = \infty$ one also sets $x_1 = \cdots = x_k = 1$ then one gets
3.2. HARMONIC SUMS, EULER-ZAGIER SUMS

By taking only \( x_1 = \cdots = x_k = 1 \) and leaving \( n \) general, we get Euler-Zagier sums ([27] [28]):

\[
Z(n; m_1, \ldots, m_k; 1, \ldots, 1) = Z_{m_1, \ldots, m_k}(n)
\]

On the other hand, the S-sums for values \( x_1 = \cdots = x_k = 1 \) and \( m_i > 0 \) reduce to harmonic sums [22]:

\[
S(n; m_1, \ldots, m_k; 1, \ldots, 1) = S_{m_1, \ldots, m_k}(n)
\]

Multiple polylogs, in turn contain as a subset the classical polylogs \( \text{Li}_n(x) \), Nielsen’s generalized polylogs [29]:

\[
S_{n;p}(x) = \text{Li}_{1,1,n+1}(1, \ldots, 1, x)
\]

and harmonic polylogs introduced by Vermaseren and Remiddi [21]

\[
H_{m_1,\ldots,m_k}(x) = \text{Li}_{m_k,\ldots,m_1}(1, \ldots, 1, x)
\]

In this work we will specially use harmonic sums and harmonic polylogs, therefore we will take a closer look of these two subclasses in the following section.

3.2 Harmonic sums, Euler-Zagier sums

As we have seen, the harmonic sums (or Euler-Zagier sums) are a special case of general S(/Z)-sums (or speaking in historical terms, the S(/Z)-sums are a generalization of harmonic sums (Euler-Zagier sums), namely when in the S(/Z)-sums the arguments are \( x_1 = \ldots = x_k = 1 \). So starting from S-sums and setting all \( x_i = 1 \), we can define harmonic sums as (we follow the convention of [22]):

\[
S_m(n) = \sum_{i=1}^{n} \frac{1}{i^m}
\]

\[
S_{-m}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^m}, \quad m > 0
\]
Recursively, one can write higher harmonic sums, that is sums with bigger weight, by:

\[ S_{m_1,\ldots,m_k}(n) = \sum_{i=1}^{n} \frac{1}{i^m} S_{j_1,\ldots,j_p}(i) \]  

(3.19)

Similarly, the so called Euler-Zagier sums are defined the same way, only starting from Z-sums instead of S-sums. Or in other words, the relation between harmonic sums and S-sums is analogous to the relation between Euler-Zagier sums and Z-sums, only difference being the upper summation limit. Here we will therefore only consider harmonic sums. One example would be

\[ S_{1,3,2}(n) = \sum_{i=1}^{n} \frac{1}{i} \sum_{j=1}^{i} \frac{1}{j^3} \sum_{k=1}^{j} \frac{(-1)^k}{k^2} \]  

(3.20)

The above formula is a well defined recursion, which can be easily implemented in computer algebra system, like FORM [30], where harmonic sums are implemented as

\[ S_{1,\ldots,i_m} = S(R(i_1,i_2,\ldots,i_m),n) \]

What we will be interested in, is to sum over a product of harmonic sums and possible denominators, which will occur as formal solutions of difference equations in next chapter. Since harmonic sums inherit the algebra from S-sums it is possible to write product of two harmonic sums as sum of single, higher harmonic sums. Example similar to equation (3.8) would be:

\[ S_{m_1,\ldots,m_k}(n) \times S_{m_1',\ldots,m_l'}(n) = \sum_{j_1=1}^{n} \frac{1}{j_1^{m_1}} S_{m_2,\ldots,m_k}(j_1) S_{m_1',\ldots,m_l'}(j_1) \]

\[ + \sum_{j_2=1}^{n} \frac{1}{j_2^{m_1}} S_{m_1,\ldots,m_k}(j_2) S_{m_2',\ldots,m_l'}(j_2) - \sum_{j=1}^{n} \frac{1}{j^{m_1+m_1}} S_{m_2,\ldots,m_k}(j) S_{m_2',\ldots,m_l'}(j) \]  

(3.21)

In the case that the arguments are not the same, one has to synchronize the harmonic sums first, that means to bring the arguments, possible denominators and/or factorials to the equal form, for example :

\[ \sum_{i=1}^{n} \frac{S_1(i+1)}{i} = \sum_{i=1}^{n} \frac{S_1(i)}{i} + \sum_{i=1}^{n} \frac{1}{(i+1)i} \]

\[ = \sum_{i=1}^{n} \frac{S_1(i)}{i} + \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i+1} \]  

(3.22)
This of course doesn’t work if the argument is a difference of symbolic values. In that case one can use recursively one of the formulas:

\[
\frac{S_{j,r_1,\ldots,r_s}(i + m)}{i} = \frac{S_{j,r_1,\ldots,r_s}(i + m - 1)}{i} + \frac{S_{r_1,\ldots,r_s}(i + m)}{(i + m)}
\]

On the right hand side one can use the general partial fractioning formula

\[
\frac{1}{i + a} \frac{1}{i + b} = \delta_{a,b} \frac{1}{(i + a)^2} + (\Theta'(a - b) + \Theta'(b - a)) \frac{1}{b - a} \left( \frac{1}{i + a} - \frac{1}{i + b} \right)
\]

where \(\Theta'(n)\) is zero if \(n \leq 0\) and one if \(n > 0\). Applying the above formulas recursively, (since it has a defined ending, it is a well defined recursion), one gets “synchronized” harmonic sums over which one can perform summation. This synchronization and summation is in FORM implemented in the package called summer. An example code for the typical expression:

\[
\sum_{j=1}^{n} \frac{S_{1}(j_1)S_{2,1}(j_1 - 1)}{j_1 + 2}
\]

is given in the appendix, together with the result. The harmonic/Euler-Zagier sums can be used to expand Gamma functions in \(\epsilon\) around integer valued numbers, through the formula:

\[
\Gamma(a + be) = \Gamma(1 + be)\Gamma(a)\left(1 + \sum_{j=1}^{\infty} (be)^j Z_{1,\ldots,1}(a - 1)\right)
\]

and

\[
\Gamma(a + be)^{-1} = \frac{1}{\Gamma(1 + be)\Gamma(a)}\left(1 + \sum_{j=1}^{\infty} (be)^j Z_{1,\ldots,1}(a - 1)\right)^{-1}
\]

\[
= \frac{1}{\Gamma(1 + be)\Gamma(a)}\left(1 + \sum_{j=1}^{\infty} (-be)^j S_{1,\ldots,1}(a - 1)\right)
\]

for \(a \in \mathbb{N}\) and \(a > 0\). For negative \(a\) one uses the analytic continuation formula

\[a\Gamma(a) = \Gamma(a + 1)\]
Using this property and the fact that harmonic/Euler-Zagier sums form an algebra, one can expand in $\varepsilon$ the so called hypergeometric functions $jF_{j-1}$, which are defined by:

$$jF_{j-1}(\{A_1, \ldots, A_k\}, \{B_1, \ldots, B_k\}, x) = \sum_{i=0}^{\infty} \frac{(A_1)_i \cdots (A_k)_i x^i}{(B_1)_i \cdots (B_k)_i i!} = 1 + \sum_{i=1}^{\infty} \frac{(A_1)_i \cdots (A_k)_i x^i}{(B_1)_i \cdots (B_k)_i i!} = 1 + \frac{\Gamma(B_1) \cdots \Gamma(B_{j-1})}{\Gamma(A_1) \cdots \Gamma(A_j)} \sum_{i=1}^{\infty} \frac{\Gamma(A_1 + i) \cdots \Gamma(A_j + i)}{\Gamma(B_1 + i) \cdots \Gamma(B_{j-1} + i)} \frac{x^i}{\Gamma(i + 1)}$$

(3.28)

where $(A)_i = A(A + 1) \cdots (A + i - 1) = \frac{\Gamma(A + i)}{\Gamma(A)}$ is the Pochhammer symbol.

Here we will briefly sketch the algorithm from [12] which allows to compute expressions like the one above (There it is called Algorithm A). For algorithms for other, similar expressions see [12]. Let us write down the most general expression which one can solve with this algorithm:

$$\sum_{i=1}^{n} \frac{x^i}{(i + c)^m} \frac{\Gamma(i + a_1 + b_1 \varepsilon) \cdots \Gamma(i + a_k + b_k \varepsilon)}{\Gamma(i + c_1 + d_1 \varepsilon) \cdots \Gamma(i + c_k + d_k \varepsilon)} \times Z(i + o - 1; m_1, \ldots, m_l; x_1, \ldots, x_l)$$

(3.29)

First one has to expand the $\Gamma$ functions to Z-sums (in our case it suffices to expand in Euler-Zagier sums, but we keep the algorithm description general) according to eq. (3.26)(3.27), and synchronize the remaining Z-sum $Z(i + o - 1; m_1, \ldots, m_l; x_1, \ldots, x_l)$. Then we have terms like:

$$\sum_{i=1}^{n} \frac{x^i}{(i + c)^m} Z(i - 1, m_1 \ldots)$$

$c \geq 0$

(3.30)

The only thing one has to do now is to reduce the offset $c$ to zero, since then one would per definition have another Z-sum. To do that, use recursively the formula:

$$\sum_{i=1}^{n} \frac{x^i}{(i + c)^m} = \frac{1}{x} \sum_{i=1}^{n} \frac{x^i}{(i + c - 1)^m} - \frac{1}{c^m} + \frac{x^n}{(n + c)^m}$$

(3.31)

in case that the depth of the Z-sum in (3.30) is zero, or:

$$\sum_{i=1}^{n} \frac{x^i}{(i + c)^m} Z(i - 1; m_1 \ldots) = \frac{1}{x} \sum_{i=1}^{n} \frac{x^i}{(i + c - 1)^m} Z(i - 1; m_1 \ldots)$$


3.3. HARMONIC POLYLOGARITHMS

\[-\sum_{i=1}^{n} \frac{x^i}{(i+c)^m} \frac{x_1^i}{i^{m_1}} Z(i-1;m_2\ldots;x_2\ldots) + \frac{x^n}{(n+c)^m} Z(n-1;m_1\ldots)\]  
(3.32)

if the depth of the Z-sum is not zero. At the end one has only terms of the form:

\[\sum_{i=1}^{n} \frac{x^i}{i^m} Z(i-1;m_1\ldots)\]  
(3.33)

which is of course again a Z-sum. Notice that in the formula (3.28), in the definition of the hypergeometric function, the upper limit of the sum is infinity. In that case, as we already mentioned in eq. (3.12), we get from the equation (3.30-33) generalized polylogarithms. If further the Z-sum is an Euler-Zagier sum, then we get multiple zeta values (eq. (3.13)). Of course the last term in the equations (3.31) and (3.32) respectively do not contribute. The above algorithm, and others, are available in FORM package XSUMMER [13], or in GINAC [14] nestedsums library [15].

One important remark should be made here, namely, from the algorithm described above, especially from reduction formula of the offset c to zero (3.31),(3.32) one can immediately see, that one can expand only hypergeometric functions with integer valued coefficients. If our c, or equivalently the offset o in Z-sum in equation (3.29), would be a rational number, then the algorithm would not work, since the offset is recursively reduced by one until it reaches zero and similarly synchronization of the subsum also works the same way, so that in case of a rational number the algorithm would never reach zero. This means that hypergeometric functions, like \(2F_1(\frac{1}{2} + \epsilon, 2\epsilon; 1 + \epsilon; x)\) cannot be expanded with the algorithm described. There is however a generalization of the above algorithm which includes rational numbers [16], but these rational number up to now have to have certain form. The general algorithm for no matter what distribution of rational numbers has not been found.

In the appendix we give an example of a FORM program that expands hypergeometric function \(2F_1(2\epsilon - 1, \epsilon; 2\epsilon; x)\).

3.3 Harmonic Polylogarithms

As we have seen in the first section, harmonic polylogarithms (HPL) are a special case of multiple polylogarithms in eq. (3.17). On their own, one
can define HPL’s recursively as following:

\[
H(0, \ldots, 0; x) = \frac{1}{n} \log^n x
\]

\[
H(a, a_1, \ldots, a_k; x) = \int_0^x f_a(t) H(a_1, \ldots, a_k; x) dt
\]  
(3.34)

for general vector of length or weight \(n\), where \(a_i = 1, 0, -1\) and functions \(f_a(x)\) are

\[
f_1(x) = \frac{1}{1 + x}, \quad f_0(x) = \frac{1}{x}, \quad f_{-1}(x) = \frac{1}{1 - x}
\]  
(3.35)

The beginning of the recursion also has to be given, in this case that would be the lowest weight:

\[
H(1; x) = \int_0^x \frac{1}{1 - t} dt = -\ln(1 - x)
\]

\[
H(0; x) = \int_0^x \frac{1}{t} dt = \ln(x)
\]

\[
H(-1; x) = \int_0^x \frac{1}{1 + t} dt = \ln(1 + x)
\]  
(3.36)

An alternative definition would be:

\[
\frac{d}{dx} H(a, a_1, \ldots, a_k; x) = f_a(x) H(a_1, \ldots, a_k; x)
\]  
(3.37)

From the equation (3.28), it is easy to see that HPL’s are a generalization of Nielsen polylogarithms [29]. Historically, that was the reason for their introduction [21].

HPL’s also form an algebra, so one can write, just like in case of S/Z-sums or harmonic/Euler-Zagier-sums, the product of two HPL’s (with the same argument of course) as a sum of single HPL’s of higher weight. For example:

\[
H(a_1, a_2; x) H(b_1, b_2; x) = H(a_1, a_2, b_1, b_2; x) + H(a_1, b_1, a_2, b_2; x) + H(a_1, b_1, b_2, a_2; x) + H(b_1, a_1, a_2, b_2; x) + H(b_1, a_1, b_2, a_2; x) + H(b_1, b_2, a_1, a_2; x)
\]  
(3.38)

Notice, that in the above formula the relative order of the elements of a vector \(\vec{a} = (a_1, a_2)\) and \(\vec{b} = (b_1, b_2)\) respectively, is preserved. This is due to shuffle algebra. The general formula is then:

\[
H(a_1, \ldots, a_k; x) H(b_1, \ldots, b_k; x) = \sum_{c_i \in a_i, \bigcup b_i} H(c_1, \ldots, c_{k_1+k_2}; x)
\]  
(3.39)
3.3. **HARMONIC POLYLOGARITHMS**

where the symbol $\bigcup^>$ stands for the fact mentioned earlier, namely that the internal order of the elements $a_i$ and $b_i$ resp. is preserved.

The HPL’s can be Mellin transformed and Taylor expanded. Since we wont need Mellin transforms and the Taylor expansion of HPL’s, we refer the interested reader to original literature [21].

What we will need are the HPL’s with argument $x=1$. These are actually nothing else then harmonic/Euler-Zagier sums at infinity, which are nothing else then multiple zeta values (MZV) for positive $a$’s, or colored MZV for arbitrary $a$’s.

$$
H(a;1) = \zeta(a), \quad a > 0
$$

$$
H(-a;1) = (1 - 2^{1-a})\zeta(a), \quad a > 0
$$

$$
H(a_1, \ldots, a_k;1) = (-1)^{\#(a_i<0)}\zeta(a_1, \ldots, a_k), \quad k > 1
$$

where $\zeta$’s are:

$$
\zeta(a_1, \ldots, a_k) = \sum_{i_1}^{\infty} \sum_{i_2}^{i_1-1} \cdots \sum_{i_k}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{\text{sgn}(a_j)^{i_j}}{i_j^{[a_j]}} \quad (3.41)
$$

and vector $\bar{a} = (a_1, \text{sgn}(a_1)a_2, \ldots, \text{sgn}(a_{i-1})a_i, \ldots, \text{sgn}(a_{k-1})a_k)$.

The MZV’s themselves possess an algebra, which means that they can be expressed in terms of a few mathematical constants, like powers of $\pi$, $\zeta$-functions and certain polylogarithms. For the relations see for example [31], [32].

The HPL’s are implemented in FORM package called HARMPOL [21], and also in MATHEMATICA package HPL from [33]. We will use HPL’s to calculate, or integrate, differential equations, therefore we will need to evaluate expressions like:

$$
\int_0^x \frac{1}{1+t} H(1,0,1;t)H(1,1;t) \quad (3.42)
$$

The solution of the above formula using FORM package HARMPOL is given in the appendix.
Chapter 4

Methods of Computation

4.1 Difference Equations

Difference equations or recurrences occur from the so called integration-by-parts (IBP) method [34], [35]. Feynman integrals

\[ F(q_1, \ldots, q_n; d) = \int d^{d}k_1 \cdots \int d^{d}k_n \prod_{l=1}^{L} D_{F,l}(p_l) \]  \hspace{1cm} (4.1)

where \( k_i \) are internal momenta, \( q_i \) external momenta, \( L \) the number of lines, or loop momenta \( p \) which are certain linear combinations of external momenta \( q_i \) and \( n = L - V + 1 \), where \( V \) is number of vertices, and \( n \) is the number of loops, and

\[ D_{F,l}(p_l) = \frac{iZ_l(p)}{(p_l^2 - m_l^2 + i0)}, \]  \hspace{1cm} (4.2)

where \( Z_l(p) \) is a polynomial in \( p \), have the property that in dimensional regularization a derivative of an integral of the form (4.1), with respect to mass or a momentum equals the corresponding integral of the derivative. As a direct consequence of this fact one deduces:

\[ \int d^{d}k_i \frac{\partial}{\partial k_i^v} \prod_{l=1}^{L} D_{F,l}(p_l) = \underbrace{0}_{\text{surfaceterm}}, \quad i = 1 \ldots n \]  \hspace{1cm} (4.3)

Application of this formula gives many relations of the given integral as a linear combination of some other integrals with one "pinched", that is contracted, line, and/or denominators risen to a different power then the
original integral. Since IBP relations give an under-determined, homogeneous, linear system of equations whose unknowns are the integrals, there exist some integrals, called "master" integrals, whose values cannot be determined by IBP relations. They have to be calculated by other means. The masters form so to speak the basis in which the other integrals can be expressed. Since one can by a linear transformation always transform a given basis into another one, it is not unique. So instead of solving all Feynman integrals, one "only" needs to solve the master integrals. In the literature there exists an approach which automatizes the procedure of IBP of a particular integral. We illustrate the method on the most simple example possible, the massive tadpole:

\[ J(x) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 + m^2)^x} \]  \hspace{1cm} (4.4)

Here we will denote the denominator to the power \(x\) by a dot on a line in \(\circ\). Now we take the derivative of mass times mass \(\partial_m m\) (could also take impulse):

\[
\partial_m m J(x) = \int \frac{d^dp}{(2\pi)^d} \partial_m m \frac{1}{(p^2 + m^2)^x} = \\
= \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 + m^2)^x} - 2m^2x \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 + m^2)^{x+1}} = \\
= J(x) - 2m^2xJ(x + 1) \hspace{1cm} (4.5)
\]

On the other hand by rescaling \(p \rightarrow \frac{p}{m}\) and pulling \(m\) out of the integral and again taking derivative with respect to \(m\) times \(m\), we get:

\[
\partial_m m \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 + 1)^x} = (1 + d - 2x)m^{d-2x} \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 + 1)^x} = (1 + d - 2x)J(x) \hspace{1cm} (4.6)
\]

Setting the two equations (4.5),(4.6) equal, one gets:

\[-2m^2xJ(x + 1) = (d - 2x)J(x) \hspace{1cm} \text{or} \hspace{1cm} J(x + 1) = \frac{d - 2x}{-2m^2x} J(x) \hspace{1cm} (4.7)\]

Now this is a homogeneous, linear difference equation (or recurrence) first order. In [17] there is general method, which describes how to solve difference
4.1. DIFFERENCE EQUATIONS

Equations numerically with high precision. We will however concentrate only on analytic calculations. Let us look at a general, inhomogeneous, difference equation of first order (see for example [19],[20]):

\[ \begin{align*}
    y(x+1) &= a(x) \cdot y(x) + g(x), \\
    y(x) &= y_0 , \quad x \geq x_0 \geq 0.
\end{align*} \]

(4.8)

The initial values can be found in our case easily, namely the initial value at \( x_0 = 0 \) means that we take our Feynman graph and “pinch” [see fig. 5.1] the dotted line which corresponds to a propagator to the power \( x \). To find a boundary value is also easy, in [17] it is said that for \( x \gg 1 \), the generic Feynman graph \( U(x) \) is given by:

\[ \lim_{x \to \infty} U(x) = \left( \frac{1}{(p^2 + 1)^x} \right) \times [g(0)] \sim (1)^x x^{-\frac{d}{2}} g(0) \]

(4.9)

where \( g(0) \) is the original graph with the dotted line cut away (not pinched). One can see a difference equation as a discrete differential equation, and just like one can formally solve, or ”integrate”, an ordinary differential equation of first order, one can formally solve, or ”sum”, the first order difference equation, with the result:

\[ y(x) = \left\{ \prod_{i=x_0}^{x-1} a(i) \right\} y_0 + \sum_{r=x_0}^{x-1} \left\{ \prod_{i=r+1}^{x-1} a(i) g(r) \right\} \]

(4.10)

This solution is valid for all \( x \in \mathbb{Z}^+ \).

Applied to our equation for the tadpole, we get with \( a(x) = \frac{d-2x}{2m^2 x} \):

\[ \begin{align*}
    J(x) &= \left( \prod_{i=1}^{x-1} \frac{d-2i}{-2m^2 i} \right) J(1) \quad \text{which is} \\
    J(x) &= \frac{(m^2)^{1-x}(1-\frac{d}{2})x^{-1}}{\Gamma(x)} J(1)
\end{align*} \]

(4.11)

where \( (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \) is the so called Pochhammer symbol. From the solution of the difference equation one can see that any \( J(x) \) for \( x > 1 \), can be expressed in terms of \( J(1) \), which is the only master integral. It can be solved analytically:

\[ J(1) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}} \equiv J \]

(4.12)
For a n-th order difference equation, like for n-th order differential equation, there is no formal solution. But since there is more knowledge in the field of differential equations, it might be useful to have a general way to transform difference into differential equations.

### 4.2 Differential Equations

One way to obtain differential equations is to use IBP relations and additional derivatives of the integrals [40]. Here we will use a general approach which uses the recurrences obtained by the IBP relations and transforms them by means of Laplace transform [20] into a differential equation. We will demonstrate this method on a second order difference equation, since no third or higher order equations will appear in this thesis. A homogeneous 2-nd order recurrence would be:

\[ a_0(x)y(x) + a_1(x)y(x + 1) + a_2(x)y(x + 2) = 0 \]  

(4.13)

where \( a_i \)'s are polynomials in \( x \) of maximum degree \( p \). To Laplace transform the equation we substitute:

\[ y(x) = \int_l dt \ t^{x-1} v(t) \]  

(4.14)

where \( l \) is a line of integration suitably chosen and \( v(t) \) is a solution of a certain differential equation which will be derived below. We rewrite the coefficients \( a_i(x) \) in the form:

\[
\begin{align*}
a_2(x) &= A_p(x + 2)(x + 3) \cdots (x + p + 1) + \ldots \\
&\quad + A_2(x + 2)(x + 3) + A_1(x + 2) + A_0 \\
a_1(x) &= B_p(x + 1)(x + 2) \cdots (x + p) + \ldots \\
&\quad + B_2(x + 1)(x + 2) + B_1(x + 1) + B_0 \\
a_0(x) &= C_p x(x + 1) \cdots (x + p - 1) + \ldots + C_2 x(x + 1) + C_1 x + C_0
\end{align*}
\]  

(4.15)

where \( A_p \neq 0 \) and \( C_p \neq 0 \).

Defining:

\[
\begin{align*}
\Phi_p(t) &= A_p t^2 + B_p t + C_p \\
\Phi_i(t) &= A_i t^2 + B_i t + C_i, \quad i = 0, 1 \ldots p - 1
\end{align*}
\]  

(4.16)
one can show [20] that (4.14) provides a solution of the difference equation

\[ t^p \Phi_p(t) \frac{d^p}{dt^p} v(t) - t^{p-1} \Phi_{p-1}(t) \frac{d^{p-1}}{dt^{p-1}} v(t) + \ldots (-1)^p \Phi_0(t) v(t) = 0 \]  

(4.17)

and a line of integration \( l \) is chosen so that

\[
I(x, t) = v(t) \sum_{k=0}^{p-1} \frac{d^k}{dt^k} [t^{x+k} \Phi_{k+1}(t)] - v'(t) \sum_{k=0}^{p-2} \frac{d^k}{dt^k} [t^{x+k+1} \Phi_{k+2}(t)] + \ldots 
+ (-1)^{p-1} v^{(p-1)}(t) [t^{x+p-1} \Phi_p(t)]
\]

(4.18)

has the same value at each endpoint.

One can immediately see that for a second order difference equation with coefficients of degree \( p \), one gets a \( p \)-th order differential equation with coefficients of degree \( 2+p \). For a non-homogeneous difference equation

\[
\sum_{i=0}^{N} a_i(x) y(x + i) = \sum_{i=0}^{N'} b_i(x) z(x + i)
\]

(4.19)

where \( b_i(x) \) are polynomials and \( z(x) \) is a solution of some difference equation that has already been solved before, one can analogously say that:

\[
y^{NH}(x) = \int_l dt \ t^{x-1} v^{NH}(t) \quad \text{and} \quad z(x) = \int_l dt \ t^{x-1} w(t)
\]

(4.20)

(where \( w(t) \) is a solution of a differential equation associated with \( z(x) \) by the method already described) are solutions of (4.19), if \( v^{NH}(x) \) and \( w(x) \) satisfy

\[
\sum_{i=0}^{p} \Phi_i(t)(-t)^i v^{(i)}_{NH}(t) = \sum_{i=0}^{p'} \Psi_i(t)(-t)^i w^{(i)}_{NH}(t).
\]

(4.21)

As an illustration we give an example from [38] of massive two loop graph \( S(x) \). Its difference equation is:

\[
S(x) \equiv \bigg( \begin{array}{c} 0 \\
\end{array} \bigg)_{x^2} 
\]

\[
0 = \frac{-3(x+1)S(x+2) + (2x + 3 - d)S(x+1) + (x+2 - d)S(x)}{\Gamma(x+1-d/2)(d-2) / \Gamma(x+1)(1-d/2)}
\]

(4.22)
with boundary conditions:

\[
S(0) = 1 \quad (4.23) \\
S(x \gg 1) = \frac{\Gamma(x - d/2)}{\Gamma(x)} \cdot \frac{1 - d/2}{\Gamma(1 - d/2)} \sim x^{-d/2} \cdot \frac{1 - d/2}{\Gamma(1 - d/2)} \quad (4.24)
\]

Simple manipulations (eq. (4.15-16-17)) lead to the differential equation:

\[
t(2 - d + (1 - d)t + 3t^2)v(t) - t(1 - t)(1 + 3t)v'(t) = \frac{(2 - d)}{\Gamma(1 - \frac{d}{2})\Gamma(\frac{d}{2})}t^{1-d/2}(1 - t)^{d/2-1} \quad (4.25)
\]

The homogeneous solution of the equation is

\[
v_H(t) = c_H t^{2-d}(1 - t)^{\frac{d-3}{2}} (1 + 3t)^{\frac{d-3}{2}} \quad (4.26)
\]

which does not satisfy the \( x \gg 1 \) boundary condition because it grows like \( x^{\frac{d}{2}-\frac{d}{4}} \), so \( c_H(t) = 0 \), and the inhomogeneous solution is formally obtained by varying the constant, leading to:

\[
S(x) = \frac{2 - d}{\Gamma(1 - \frac{d}{2})\Gamma(\frac{d}{2})} \int_0^1 dt t^{x+1-d}(1 - t)^{\frac{d-3}{2}} (1 + 3t)^{\frac{d-3}{2}} \int t^1 dz z^{\frac{d}{2}-2}(1 - z)^{-\frac{1}{2}}(1 + 3z)^{\frac{1-d}{2}} \quad (4.27)
\]

This integral is convergent for \( x > 2 \) in four dimensions. Using the relation

\[
S(3) = \frac{1}{36} (-3(d - 2)^2 + 2(d - 8)(d - 3)S(1))
\]

which follows from the difference equation by setting \( x = 0 \), one can compute at least numerically the \( S(1) \). In the appendix the analytical result up to second order in \( \epsilon \) is given.

### 4.3 x-Space Evaluation

In this section we present a calculation method for a class of so called sunset graphs, which is performed in coordinate space \([42]\). One writes in coordinate space the massive propagator according to:

\[
\int \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} = \frac{(m)^{\frac{d}{2}-1}}{(2\pi)^{\frac{d}{2}}} x^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(mx) \quad (4.28)
\]

where \( K_{\nu}(ax) \) is a modified Bessel function of third order or MacDonald function. The massless propagator to arbitrary power is:

\[
\int \frac{d^d p}{(2\pi)^d} e^{ip \cdot x} = \frac{\Gamma(\frac{d}{2} - \nu)}{\Gamma(\nu)} \frac{x^{2\nu-d}}{4^\nu \pi^{\frac{d}{2}}} \quad (4.29)
\]
4.3. X-SPACE EVALUATION

Analytically, at least in terms of hypergeometric functions, one can solve all sunrise topology n-loops diagrams with at most three massive lines. The reason for this constraint of number of massive lines is that in coordinate space one has to perform only one integration for sunrise-type graphs, and it is known that [41]:

\[
\int_0^\infty dx x^{\alpha-1} K_\lambda(ax) K_\mu(bx) K_\nu(cx) =
\]

\[
= \frac{2^{\alpha-4}}{\alpha^\alpha} \left\{ A(\alpha, \mu) + A(\alpha, -\mu) + A(-\alpha, \mu) + A(-\alpha, -\mu) \right\},
\]

for \( |\text{Re} \alpha| > |\text{Re} \lambda| + |\text{Re} \mu| + |\text{Re} \nu|; \text{Re}(a+b+c) > 0 \), where

\[
A(\alpha, \mu) = \left( \frac{a}{c} \right)^\alpha \left( \frac{b}{c} \right)^\mu \left[ \frac{\Gamma(-\lambda) + \Gamma(-\mu) + \Gamma\left( \frac{\alpha + \lambda + \mu - \nu}{2} \right)}{2} \right] 
\]

\[
\times F_1 \left( \frac{\alpha + \lambda + \mu - \nu}{2}, \frac{\alpha + \lambda + \mu + \nu}{2}; \lambda + 1, \mu + 1; \frac{a^2}{c^2}, \frac{b^2}{c^2} \right),
\]

(4.30)

where the \( F_1 \) is Appell’s hypergeometric function of two variables which has the form:

\[
F_1(a, b; c, d; z_1, z_2) = \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \frac{(a)_{j_1+j_2} (b)_{j_1+j_2}}{(c)_{j_1} (d)_{j_2}} \frac{z_1^{j_1} z_2^{j_2}}{j_1! j_2!};
\]

(4.31)

The above formula simplifies in case of only two massive propagators to:

\[
\int_0^\infty dx x^{\alpha-1} K_\mu(bx) K_\nu(cx) = A_{\mu, \nu}^\alpha, \text{ for}
\]

\[
\text{Re} (b+c) > 0; |\text{Re} \alpha| > |\text{Re} \mu| + |\text{Re} \nu|, \text{ where}
\]

\[
A_{\mu, \nu}^\alpha = 2^{\alpha-3} c^{-\alpha-\mu} \frac{b^\mu}{\Gamma(\alpha)} \left[ \frac{\Gamma\left( \frac{\alpha + \mu + \nu}{2} \right) + \Gamma\left( \frac{\alpha + \mu - \nu}{2} \right) + \Gamma\left( \frac{\alpha - \mu + \nu}{2} \right)}{2} \right] 
\]

\[
\times \left[ \frac{\Gamma\left( \frac{\alpha - \mu - \nu}{2} \right)}{2} \right] \times F_1 \left( \frac{\alpha + \mu + \nu}{2}, \frac{\alpha - \mu + \nu}{2}; \alpha - \frac{b^2}{c^2} \right)
\]

(4.32)

For only one massive propagator the formula is:

\[
\int_0^\infty dx x^{\alpha-1} K_\mu(bx) = \frac{2^{\alpha-3}}{b^\alpha} \Gamma\left( \frac{\alpha + \mu}{2} \right) \Gamma\left( \frac{\alpha - \mu}{2} \right)
\]

(4.33)
CHAPTER 4. METHODS OF COMPUTATION

A simple example for one massive (m=1) propagator would be [36]:

\[
\begin{align*}
&= \int_0^\infty \frac{x^{2-d}\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{\frac{d}{2}}} x^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(x) \\
&\quad \times \left[\frac{x^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(x)}{(2\pi)^{\frac{d}{2}}}\right] \\
&\quad \times \left[\frac{x^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(x)}{(2\pi)^{\frac{d}{2}}}\right]
\end{align*}
\]

(4.34)

Let us now apply the method to the two massive lines graph more explicitly, since it is a master integral. It has two massive lines, with the same mass, which we set to one, and three massless line. In x-space, using eq.(4.28) and (4.29) we get:

\[
\begin{align*}
&= \int d^d x \left(\frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{\frac{d}{2}} x^{d-2}}\right)^3 \left(\frac{K_{\frac{d}{2}-1}(x)}{(2\pi)^{\frac{d}{2}} x^{\frac{d}{2}-1}}\right)^2
\end{align*}
\]

(4.35)

Performing the trivial angular integration this simplifies to:

\[
\begin{align*}
&= \frac{\Gamma\left(\frac{d}{2} - 1\right)^3 2\pi^{\frac{d}{2}}}{4\Gamma\left(\frac{d}{2}\right)^2 (2\pi)^{d}} \int d^d x x^{7-3d} K_{\frac{d}{2}-1}(x)
\end{align*}
\]

(4.36)

Now we can use the equation (4.32), which in this case simplifies to

\[
\int_0^\infty dx x^{\alpha-1} K_\mu^2(x) = \frac{2^{\alpha-3}}{\Gamma(\alpha)} \Gamma\left(\frac{\alpha + 2\mu}{2}\right) \Gamma\left(\frac{\alpha - 2\mu}{2}\right)
\]

(4.37)

and setting \(\alpha = 8 - 3d\) and \(\mu = \frac{d}{2} - 1\) we get:

\[
\begin{align*}
&= \frac{\Gamma(5 - 2d)\Gamma^2(4 - 3d)\Gamma(3 - d)\Gamma^3\left(\frac{d}{2} - 1\right)}{(4\pi)^{2d} \Gamma(8 - 3d)\Gamma\left(\frac{d}{2}\right)}
\end{align*}
\]

(4.38)

Notice that this result looks different then the one in for example appendix of [23]. To match it with their result one has to multiply our result by \((4\pi)^4 \left(\frac{\pi}{\epsilon}\right)^{4\epsilon}\), which is due to a different normalization of the integral measure.

An example for a three massive lines graph would be:

\[
\begin{align*}
&= \int_0^\infty \frac{x^{2-d}\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{\frac{d}{2}}} x^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(x) \left(\frac{x^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(x)}{(2\pi)^{\frac{d}{2}}}\right)^3
\end{align*}
\]

(4.39)
4.4. MELLIN-BARNES METHOD

Since we set all three masses to be $m_1 = m_2 = m_3 = 1$, the $F_4$-function from equation (4.30) simplifies to $4F_3$- and $3F_2$-hypergeometric functions. The (somewhat long) result in terms of these functions is given in the appendix.

It is worth noticing, that not only the sunrise-type graphs can be computed with this method, also other topologies, whose massless lines can be combined in a way which gives effectively sunrise topology, can be computed. Take, for example, the graph $T2 = \bigcirc \bigcirc \bigcirc$. The massless lines can be written as:

$$\bigcirc \bigcirc \bigcirc = [G(1, 1)]^2$$

(4.40)

where G is the massless scalar one loop integral:

$$G(\alpha, \beta) \equiv \frac{1}{(k^2)^{\alpha + \beta - \frac{d}{2}} \int \frac{d^d p}{\pi^d (p^2)^\alpha ((p - k)^2)^\beta}}$$

$$= \frac{\Gamma(\alpha + \beta - \frac{d}{2}) \Gamma(\frac{d}{2} - \alpha) \Gamma(\frac{d}{2} - \beta)}{\pi^\frac{d}{2} \Gamma(d - \alpha - \beta) \Gamma(\alpha) \Gamma(\beta)}$$

(4.41)

Then one proceeds like in the example above, since then one has the same topology, with the massless propagator being of course different. The result is:

$$\frac{\bigcirc \bigcirc \bigcirc}{J^4} \equiv T2 = \frac{8^{d-3} \Gamma^3(\frac{d}{2}) \Gamma(6 - 2d) \Gamma^3(\frac{d}{2})}{\sin(\frac{3\pi d}{2}) \Gamma(\frac{11 - 3d}{2}) \Gamma^2(2 - \frac{d}{2}) \Gamma^2(d - 2)}$$

(4.42)

which coincides with the result given in [23] and [36].

4.4 Mellin-Barnes Method

In this section we present the so called Mellin-Barnes (MB) method for solving massive Feynman diagrams as was done in [46]. The method consists of the substitution for a massive denominator:

$$\frac{1}{(k^2 - m^2)^\beta} = \frac{1}{(k^2)^\beta \Gamma(\beta) 2\pi i \int_{-i\infty}^{i\infty} ds \left( \frac{m^2}{k^2} \right)^s \Gamma(-s) \Gamma(\beta + s)}$$

(4.43)

In other words one is reducing (or rewriting) massive lines in a Feynman diagram to massless ones at a price of an additional integration of Mellin-Barnes type. This integration can sometimes be performed using first Barnes
CHAPTER 4. METHODS OF COMPUTATION

lemma,

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \Gamma(a + s) \Gamma(b + s) \Gamma(c - s) \Gamma(d - s) = \frac{\Gamma(a + c) \Gamma(a + d) \Gamma(b + c) \Gamma(b + d)}{\Gamma(a + b + c + d)}
\]

(4.44)

the second Barnes lemma,

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(c + s) \Gamma(d - s) \Gamma(e - s)}{\Gamma(f + s)} = \frac{\Gamma(a + e) \Gamma(a + d) \Gamma(d + c) \Gamma(b + d)}{\Gamma(a + b + d + e) \Gamma(a + c + d + e)} \times \frac{\Gamma(b + e) \Gamma(c + e)}{\Gamma(b + c + d + e)}
\]

(4.45)

where \( f = a + b + c + d + e \) and/or by closing the integration contours in the complex plane and summing up the corresponding series.

Simple one-loop example would be:

\[
I(a, b; m) = \int \frac{d^d p}{(p^2)^a ((k - p)^2 - m^2)^b}
\]

(4.46)

Using equation (4.43) we get:

\[
\frac{1}{\Gamma(b)} \frac{1}{2\pi i} \int_{-\infty}^{\infty} ds (-m^2)^s \Gamma(\beta + s) \int \frac{d^d p}{(p^2)^a ((k - p)^2)^b}
\]

(4.47)

Now, the last integral is a massless one-loop integral which is known analytically in terms of Gamma functions to be

\[
\int \frac{d^d p}{(p^2)^a ((k - p)^2)^b} = \pi^{\frac{d}{2}} (k^2)^{\frac{d}{2} - a - b} \frac{\Gamma(\frac{d}{2} - a) \Gamma(\frac{d}{2} - b) \Gamma(a + b - \frac{d}{2})}{\Gamma(a) \Gamma(b) \Gamma(d + a - b)}
\]

(4.48)

so using this result and linearly shifting the variable of integration \( s = \frac{d}{2} - a - b - s \) we get:

\[
I(a, b; m) = \pi^{\frac{d}{2}} (-m^2)^{\frac{d}{2} - a - b} \frac{\Gamma(\frac{d}{2} - a)}{\Gamma(a) \Gamma(b)} \times
\]

\[
\times \frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \left( \frac{m^2}{k^2} \right)^s \frac{\Gamma(-s) \Gamma(a + s) \Gamma(a + b - \frac{d}{2} + s)}{\Gamma(\frac{d}{2} + s)}
\]

(4.49)
Closing the contour on the right, we obtain:

\[ I(a, b; m) = \pi^\frac{d}{2} (-m^2)^{\frac{d}{2} - a - b} \frac{\Gamma(\frac{d}{2} - a)}{\Gamma(a) \Gamma(b)} \times \]

\[ \times \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{k^2}{m^2} \right)^j \frac{\Gamma(a + j) \Gamma(a + b - \frac{d}{2} + j)}{\Gamma(\frac{d}{2} + j)} \] (4.50)

where we used the basic formula for the residuum of Gamma functions
\[ \text{res} \left( \Gamma(-x + a), \ x = a + j \right) = -\frac{(a + 1)^{x - 1}}{x}. \]
If we look at the sum in the above equation and compare it with our definition of hypergeometric function in eq. (3.28) we see that we can write our result as:

\[ I(a, b; m) = \pi^\frac{d}{2} (-m^2)^{\frac{d}{2} - a - b} \frac{\Gamma(\frac{d}{2} - a)}{\Gamma(a) \Gamma(b)} \times \]

\[ \times 2F_1(a, a + b - \frac{d}{2}; \frac{d}{2}; k^2/m^2) \] (4.51)

These hypergeometric functions can then be expanded in \( \epsilon \) with the program in the appendix, as long as the coefficients are integer valued.

Another, more complex example is \( \bigotimes \), which we cannot solve with methods from previous sections. With calculation analog to the one we just presented, \( \bigotimes \) can be also written in terms of one hypergeometric function (we look at the case where all three masses are equal, for the general case of different masses see [45]):

\[ \bigotimes = -\frac{3(d - 2)}{4(d - 3)} \left\{ 2F_1 \left( \frac{d}{2}; -\frac{d}{2}; 1; -\frac{5 - d}{2}; \frac{3}{4} \right) - 3\frac{4 - d}{2} \frac{2\pi \Gamma(5 - d)}{\Gamma(2 - \frac{d}{2}) \Gamma(3 - \frac{d}{2})} \right\} \] (4.52)

In \( d = 4 - 2\epsilon \) dimensions this graph is needed in Standard Model calculations, however the hypergeometric function has rational number coefficients and cannot yet be expanded in \( \epsilon \) in four or three dimensions. We have found a relation in the literature which transforms this hypergeometric function appearing in \( \bigotimes \) in four dimensions into a hypergeometric function which has only integer valued coefficients, namely [41]:

\[ 2F_1(\epsilon, 1; \frac{1}{2} + \epsilon; z) = \frac{(\sqrt{1 - z} + \sqrt{-z})^{1-2\epsilon}}{\sqrt{1 - z}} \times \]

\[ \times 2F_1(2\epsilon - 1, \epsilon; 2\sqrt{z^2 - z + 2z}) \] (4.53)
where \( z = \frac{3}{4} \). Finally, we can write the result in \( d = 4 - 2\epsilon \) as:

\[
\frac{\partial^2}{\partial \epsilon^2} \left( \frac{1}{\bar{J}^2} \right) = -\frac{3(2 - 2\epsilon)}{4(1 - 2\epsilon)} \left\{ \begin{array}{c}
2F_1(2\epsilon - 1, \epsilon; 2\epsilon; 2\sqrt{z^2 - z + 2z}) \\
-3 - \frac{1}{2} - \epsilon - \frac{2\pi \Gamma(1 + 2\epsilon)}{\Gamma(\epsilon) \Gamma(1 + \epsilon)}
\end{array} \right\}
\] (4.54)

In the appendix we have a sample code from [13] which computes the expansion of this hypergeometric function to the second order and we also printed the result of \( \frac{\partial^2}{\partial \epsilon^2} \left( \frac{1}{\bar{J}^2} \right) \) to the same order there.
Chapter 5

Applications

We would like to apply the methods introduced in the previous chapter to the following set of vacuum integrals:

\[
\begin{align*}
\circ, & \quad \nabla, \quad \Big\triangleleft, \quad \bigtriangleup, \quad \bigtriangledown, \quad \bigtriangledown, \\
\triangle, & \quad \bigcirc, \quad \square, \quad \bigtriangleup, \quad \bigtriangledown, \quad \bigtriangleleft
\end{align*}
\] (5.1)

They are all normalized to massive tadpole \( J(1) \) to the power of the number of loops of the diagram, that is, a n-loop diagram will be normalized (divided by) \( J^n(1) \). Since all diagrams have only one scale, we will set \( m = 1 \). The mass can in the end be reconstructed by dimensional analysis.

5.1 1 and 2 Loop

The master integrals for the SU(\( N_c \)) + Adjoint Higgs Theory are given in equation (5.1). There are no graphs at two loop level and at the one loop level there is only one graph which we solved in section 4.1 equation (4.11).

5.2 3 Loop

At this level there are two integrals which have to be solved, V3 and V4. The V3 is most conveniently solved by using the difference equations method. All difference equations in this thesis are taken from [38]. The difference equation for V3 is:

\[ V3(x) \equiv \frac{1}{J^3} \]
CHAPTER 5. APPLICATIONS

Figure 5.1: “Pinching” one line

\[ 0 = -(x + 6 - 2d)V3(x + 1) + (x + 4 - \frac{3d}{2})V3(x) + \]
\[ + \frac{\Gamma(x + 2 - \frac{d}{2})(2 - d)}{(x + 3 - d)\Gamma(x + 1)\Gamma(1 - \frac{d}{2})} \]  \hspace{2cm} (5.2)

This is a first order difference equation, which has a formal solution (see equation (4.10)). What we still need is an initial condition, say at \( x = 0 \). Setting \( x \) to zero means that we ”pinch” the line to which the dot is attached, that is, the neighboring vertices are pulled to one vertex (see fig. 5.1). So we have to calculate:

\[ V3(0) = B2(1) = \frac{\beta^2}{d^2} \]  \hspace{2cm} (5.3)

Since this is a sunrise type graph and it has only two massive lines, it can be computed simply in coordinate space with the method described in section 4.3. Straightforward calculation leads to a result:

\[ B2(1) = \frac{\beta^2}{d^2} = \frac{2^{d-3}\Gamma(4 - \frac{3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{3}{2} - d)\Gamma(1 - \frac{d}{2})} \]  \hspace{2cm} (5.4)

which coincides with [23], [36]. Now setting \( x = 0 \) in equation (5.2), we already get what we want, that is \( V3(1) \):

\[ V3(1) = \frac{(d - 2)^2}{4(d - 3)^2} + \frac{3d - 8}{4(d - 3)}V3(0) \]

\[ \Rightarrow V3(1) = \frac{(d - 2)^2}{4(d - 3)^2} + \frac{2^{d-5}(3d - 8)\Gamma(4 - \frac{3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{(d - 3)\Gamma(\frac{3}{2} - d)\Gamma(1 - \frac{d}{2})} \]  \hspace{2cm} (5.5)

This gives in \( d = 4 - 2\epsilon \) and \( d = 3 - 3\epsilon \):

\[ V3(1) \overset{d=4-2\epsilon}{=} \frac{2}{3} + \frac{5}{3}\epsilon + 5\epsilon^2 + \left(\frac{44}{3} - \frac{8\zeta_3}{3}\right)\epsilon^3 + O(\epsilon^4) \]
5.2. 3 LOOP

\[ V3(1) \quad d=\frac{3}{2+2\epsilon} \quad -\frac{\pi^2}{12} + \left( \frac{\pi^2}{3} + \frac{5\zeta_3}{2} \right) \epsilon + \left( -\frac{\pi^2}{3} - \frac{9\pi^4}{40} - 10\zeta_3 \right) \epsilon^2 + \]

\[ + O(\epsilon^3) \]  

Actually, by choosing a different basis of three loop master integrals, one would get from IBP relations [23]:

\[ \begin{align*}
\left( \begin{array}{c}
\int \\
J^3
\end{array} \right) &= J^3 \left( \frac{(d-2)^2}{(d-3)(3d-8)} \right) + \left( \begin{array}{c}
\int \\
\int
\end{array} \right) \left( \frac{4(d-3)}{(3d-8)} \right) \quad (5.7) \\
\left( \begin{array}{c}
\int \\
\int
\end{array} \right) &= J^3 \left( \frac{2(d-2)^2}{(d-3)(3d-8)} \right) + \left( \begin{array}{c}
\int \\
\int
\end{array} \right) \left( \frac{4(d-4)}{(3d-8)} \right) \quad (5.8)
\end{align*} \]

So equivalently, one could at three loop level also try to solve B2 and B4. We will use the relation from equation (5.8) and solve \( \left( \begin{array}{c}
\int \\
\int
\end{array} \right) \), and from that \( \left( \begin{array}{c}
\int \\
\int
\end{array} \right) \). The difference equation doesn’t bring much help, since it is a second order inhomogeneous difference equations, which we couldn’t manage to solve [38]:

\[ B4(x) = \frac{\left( \begin{array}{c}
\int \\
\int
\end{array} \right) J^3}{\left( \begin{array}{c}
\int \\
\int
\end{array} \right)} \]

\[ 0 = -16(x + 3 - d)(x + 1)B4(x + 2) + 2(7x^2 - 10xd + 27x + 3d^2 - 17d + 24)B4(x + 1) + (x^2 + 2 - d)(2x + 6 - 3d)B4(x) - \]

\[ \frac{\Gamma(x + 1 - \frac{d}{2})3(d-2)^2}{\Gamma(x + 1)\Gamma(1 - \frac{d}{2})} \]

with boundary conditions:

\[ B4(0) = 1 \quad \text{and} \]

\[ B4(x \gg 1) = \frac{\Gamma(x - \frac{d}{2})}{\Gamma(x)\Gamma(1 - \frac{d}{2})} \sim x^{-\frac{d}{2}} \quad \frac{\left( \begin{array}{c}
\int \\
\int
\end{array} \right) \Gamma(x)\Gamma(1 - \frac{d}{2})}{\Gamma(x - \frac{d}{2})} \quad (5.9) \]

Here one can see that we have only one initial condition, where the other one is a asymptotic condition. The Laplace transform of the homogeneous part of this difference equation leads to the following differential equation:

\[ 0 = \{ t^2[-16t^2 + 14t + 2]\partial_t^2 \]

\[ - t[16(1 - d)t^2 + (12 - 20d)t + 8 - 5d]\partial_t \]

\[ + \left( [(48d + 16)t^2 + (6d^2 - 14d + 8)t + 3d^2 - 6d + 12] \right) v(t) \quad (5.10) \]
where \( v(t) \) gives \( B_4(x) \) according to equation (4.14). This equation cannot be solved by means we have. Here we sketch a differential equation method and a result of \( B_4(1) \) from [37] which manages to solve \( B_4(1) \) with harmonic polylogarithms. First we define:

\[
\Phi(d, x) = \frac{M^2}{\int_\mu \frac{1}{p^2 + \Lambda^2}} \int_{P_1 \ldots P_3} \frac{1}{P_1^2 + x^2 M^2 P_2^2 + x^2 M^2 P_3^2 + M^2} \times \frac{1}{(P_1 + P_2 + P_3)^2 + M^2} \tag{5.12}
\]

which due to the symmetry of the the underlying Feynman diagram has to fulfill:

\[
\Phi(d, x) = x^{3d-8} \Phi(d, \frac{1}{x}) \tag{5.13}
\]

We can recover \( B_4 \) if set \( x = 1 \). By setting \( x = 0 \), we get the initial condition:

\[
\Phi(d, 0) = B_2(1) = \frac{2^{d-3} \Gamma(4 - \frac{3d}{2}) \Gamma(\frac{3d-4}{2}) \Gamma(d)}{\Gamma(\frac{7}{2} - d) \Gamma(1 - \frac{d}{2})} \tag{5.14}
\]

\( \Phi(d, x) \) satisfies following differential equation taken from [37]:

\[
(x(1 - x^2)\partial_x^2 - 2(1 - 2x^2)(d - 3)\partial_x - x(d - 3)(3d - 8)) \Phi(d, x) = (d - 2)^2 x^{d-3}(x^{d-2} - 1) \tag{5.15}
\]

Now, what one does is the Ansatz (in 4-dim):

\[
\Phi(d, x) = \sum_{n=0}^\infty \epsilon^n \Phi_n(x) \tag{5.16}
\]

and solving the differential equation order by order in \( \epsilon \), in terms of harmonic polylogarithms. Plugging this expression in equation (5.15) we get:

\[
(x(1 - x^2)\partial_x^2 - 2(1 - 2x^2)(d - 3)\partial_x - 4x) \Phi_n(x) = K_n(x) \tag{5.17}
\]

where the inhomogeneous term \( K_n(x) \) is given by:

\[
K_n(x) = -4x^3(-4)^n \left( H_{0n}^\epsilon(x) + \frac{1}{2} H_{0n-1}^\epsilon(x) + \frac{1}{16} H_{0n-2}^\epsilon(x) \right) + 4x(-2)^n \left( H_{0n}^\epsilon(x) + H_{0n-1}^\epsilon(x) + \frac{1}{4} H_{0n-2}^\epsilon(x) \right) - 2[2(1 - 2x^2)\partial_x + 7x] \Phi_{n-1}(x) + 12x \Phi_{n-2}(x) \tag{5.18}
\]
where \( H_{0_n} = 1 \), \( H_{0_n<0} = \Phi_{<0}(x) = 0 \) and \( \vec{0}_n \) is a vector with \( n \) zeros. The two linearly independent solutions of the homogeneous equation associated with equation (5.17) are:

\[
\begin{align*}
\Phi_1(x) &= (1 - x^2)^2 = ((1 - x)(1 + x))^2 \\
\Phi_2(x) &= 2x(1 + x^2) - (1 - x^2)^2[H_{-1}(x) + H_1(x)] \\
&= 2x(1 + x)^2 - 4x^2 - ((1 - x)(1 + x))^2[H_{-1}(x) + H_1(x)]
\end{align*}
\]

(5.19)

With these two solutions one can construct the general solution of the inhomogeneous equation (5.17) by variation of constants:

\[
\Phi_{n}(x) = \Phi_1(x) \left( c_{1,n} + \int dx \frac{-\Phi_2(x) K_n(x)}{16x^3((1 - x)(1 + x))^4} \right) + \Phi_2(x) \left( c_{2,n} + \int dx \frac{\Phi_1(x) K_n(x)}{16x^3((1 - x)(1 + x))^4} \right)
\]

(5.20)

where \( c_{i,n} \) are integration constants, which have to be fixed by relations (5.13) and (5.14) order by order. Since \( \Phi_{1,2}(x) \) and \( K_n(x) \) are all functions of \( x, \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \), it follows that the integrand contains harmonic polylogarithms, according to the definition in section 3.3. It also follows that the integration from 0 to \( x \) as in equation (5.20) of HPL’s gives again HPL’s, due to the recursive definition, so that to all orders in \( \epsilon \) the solution of \( \Phi(d, x) \) will be in terms of HPL’s. Even more, since we are interested in \( B_4(1) \), which is \( \Phi(d, x = 1) \), it follows that, to all orders in \( \epsilon \), the \( B_4(1) \) is given in terms of HPL’s with argument 1, which are harmonic/Euler-Zagier sums at infinity, which are multiple zeta values. We quote here only the first few coefficients from [37]:

\[
\begin{align*}
B_4(1) &_{d=4-2\epsilon} -2 - \frac{5}{3} \epsilon - \frac{1}{2} \epsilon^2 + O(\epsilon^3) \\
B_4(1) &_{d=3-2\epsilon} \frac{1}{\epsilon} + 2 - 4 \ln 2 + (16 - 8 \ln 2 + 4 \ln^2 2 + 4 \zeta_2) \epsilon + O(\epsilon^2)
\end{align*}
\]

(5.21)

This result of \( B_4 \) gives at the same time the result of \( J(1) \) via the equation (5.8), since \( J(1) \) is known.

### 5.3 4 Loop

At four loop level there are eight master integrals which have to be solved. First let us quote the first few orders (in 4-dim.) of those integrals \( (T6, T4, \ldots) \).
BB4) which we couldn’t solve with methods presented in this thesis, but which have been solved by method analog to the one used in case of B4(1) (from [37]):

\[
\begin{align*}
\frac{J^4}{J^5} & = T^4_{d=4-2\epsilon} = \frac{2}{3} + \frac{4}{3} \epsilon + \frac{2}{3} \epsilon^2 + O(\epsilon^3) \\
\frac{J^6}{J^4} & = T^6_{d=4-2\epsilon} = \frac{3}{2} + \frac{7}{2} \epsilon + \frac{9}{2} \epsilon^2 + O(\epsilon^3) \\
\frac{BB^4}{J^4} & = BB^4_{d=4-2\epsilon} = -1 - \frac{1}{2} \epsilon + \frac{17}{36} \epsilon^2 + O(\epsilon^3) \quad (5.22)
\end{align*}
\]

Now there are five integrals to solve, namely $T^3, T^2, BB^2, VBa, VBb$. Two of them, $T^2, BB^2$ can be solved easily. $T^2$ has already been solved in section 4.3, where it was shown that one can rewrite the massless lines into one massless line with different power, so that one can use the x-space evaluation method, which then gave:

\[
\frac{J^4}{J^3} = T^2 = \frac{8^{d-3}\Gamma^3\left(\frac{1}{2}\right)\Gamma(6-2d)\Gamma^3\left(\frac{d}{2}\right)}{\sin\left(\frac{3\pi d}{2}\right)\Gamma\left(\frac{11-3d}{2}\right)\Gamma^2\left(2-\frac{d}{2}\right)\Gamma^2(d-2)} \quad (5.23)
\]

We have solved $BB^2$ using the same method in that chapter, but here we want to illustrate another way of obtaining the result, which we will also use on $T^3$. First let us look at the difference equation for $BB^2$ [38]:

\[
BB^2(x) = \frac{8^{d-3}\Gamma^3\left(\frac{1}{2}\right)\Gamma(6-2d)\Gamma^3\left(\frac{d}{2}\right)}{\sin\left(\frac{3\pi d}{2}\right)\Gamma\left(\frac{11-3d}{2}\right)\Gamma^2\left(2-\frac{d}{2}\right)\Gamma^2(d-2)}
\]

\[
(2d - 4 - x)(3d - 6 - 2x)BB^2(x) = 2x(7 - 3d + x)BB^2(x + 1) \quad (5.24)
\]

This is a first order homogeneous difference equation, whose solution is:

\[
BB^2(x) = \left[ \prod_{i=x_0}^{x-1} \frac{(2d - 4 - i)(3d - 6 - 2i)}{2i(7 - 3d + i)} \right] BB^2(x_0) \quad (5.25)
\]

Now, a closer look at the difference equation (5.24) shows that it is only valid for $x > 0$. That means that we have to choose our $x_0$ in the solution (5.25) to be at least one:

\[
BB^2(x) = \left[ \prod_{i=1}^{x-1} \frac{(2d - 4 - i)(3d - 6 - 2i)}{2i(7 - 3d + i)} \right] BB^2(1) \quad (5.26)
\]
5.3. 4 LOOP

The problem is that what we want is $BB(1)$, so we cannot plug it in. We have to try something different. Let us formally rewrite equation (5.24) as

$$BB(1) = \frac{BB(x)}{\prod_{i=1}^{\infty} \frac{(2d-4-i)(3d-6-2i)}{2i(7-3d+i)}},$$

which at first sight doesn’t help much. Now let us use the fact from section 4.1, which states that for $x \gg 1$ the generic Feynman integral $U(x)$ can be written as

$$\lim_{x \to \infty} U(x) = \left[ \int \frac{1}{(p^2 + 1)^x} \right] \times [g(0)]$$

where, $g(0)$ is a graph with the “special” line cut out. Using this fact, we can write our equation (5.27) for large $x$ as:

$$BB(1) = \frac{J(x)B1}{\prod_{i=1}^{\infty} \frac{(2d-4-i)(3d-6-2i)}{2i(7-3d+i)}}$$

only for $x \gg 1$

where $B1$ is obtained from $BB(1)$ by cutting away the line with the attached dot, and which can be computed by x-space method to be (see eq. 4.33):

$$\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = B1 = -\frac{3\Gamma(3-d/2)\Gamma(3-d)\Gamma^2(d/2 - 1)}{\Gamma^3(1 - d/2)}.$$

By writing $J(x)$ as (and setting $J(1) = 1$)

$$J(x) = \prod_{i=1}^{x-1} \frac{-d + 2i}{2i}$$

we get for our equation (5.27):

$$BB(1) = -\frac{\prod_{i=1}^{\infty} \frac{2i - d}{2i} \prod_{i=1}^{\infty} \frac{(2d-4-i)(3d-6-2i)}{2i(7-3d+i)}}{\prod_{i=1}^{\infty} \frac{(2d-4-i)(3d-6-2i)}{2i(7-3d+i)}}$$

only for $x \gg 1$

let us now formally set $x = \infty$. With

$$\prod_{i=1}^{\infty} \frac{(2d-4-i)(3d-6-2i)}{2i(7-3d+i)} \equiv \frac{8^{3-d}\Gamma(\frac{11-3d}{2})}{\sqrt{\pi}\Gamma(6-2d)}$$

and

$$\prod_{i=1}^{\infty} \frac{2i - d}{2i} \equiv \frac{1}{\Gamma(1 - d/2)}$$

(5.33)
we get the analytic solution for $BB_2$:

$$BB_2(1) = -\frac{3\sqrt{\pi}2^{3d-7}\Gamma(5-2d)\Gamma(3-\frac{3d}{2})\Gamma(3-d)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma^4(1-\frac{d}{2})\Gamma\left(\frac{9-3d}{2}\right)} \quad (5.34)$$

Now, we should make some remarks here. The setting $x = \infty$ can be made mathematically more rigorous, namely by computing the products over polynomials to $x = 1$, which give Pochhammer symbols, and letting $x ! 1$ and using the Stirling formula for ratio of Gamma-functions $\frac{\Gamma(x-a)}{\Gamma(x)} \sim x^{-a}$. This gives that

$$J(x \to \infty) \sim x^{-\frac{d}{2}}$$

$$\lim_{x \to \infty} \prod_{i=1}^{x-1} \frac{(2d-4-i)(3d-6-2i)}{2i(7-3d+i)} \sim x^{-\frac{d}{2}} \frac{\Gamma(8-3d)}{\Gamma(5-2d)\Gamma(4-\frac{3d}{2})} \quad (5.35)$$

One can see that in equation (5.32) the $x$-dependency drops out and the $x$-independent part (times $\frac{1}{\Gamma(1-\frac{d}{2})}$ for normalization reasons) gives the solution. Let us apply the same method to $T_3$, where we will use the faster approach, knowing that, like in the example above, we can obtain the rigor result using Stirling formula. The difference equation of $T_3$ is from [38]:

$$T_3(x) = \begin{array}{c}
\bigtriangleup
\end{array}$$

$$T_3(x + 1) = \frac{(2 - d + x)(5 - 2d + x)}{x(8 - 3d + x)} T_3(x) + \frac{(d - 2x)(3d - 8)}{4x(8 - 3d + x)} J(x) B_2(1) + \frac{(d - 2)(3d - 6 - 2x)(d - 2 - x)}{2x(8 - 3d + x)(5 - 2d + x)} B_2(x)$$

$$\equiv \frac{(d - 2 - x)(5 - 2d + x)}{x(8 - 3d + x)} T(x) + G(x) \quad (5.36)$$

where $J(x)$ in $G(x)$ is already computed and $B_2(x)$ can be computed via its difference equation from [38]:

$$B_2(x) = \begin{array}{c}
\bigcirc
\end{array}$$

$$B_2(x + 1) = \frac{(d - 2 + x)(3d - 6 - 2x)}{2x(5 - 2d + x)} B_2(x) ,$$

$$B_2(1) \text{ eq.(5.4)} \equiv \frac{2^{d-3}\Gamma(4-\frac{3d}{2})\Gamma(3-d)\Gamma(d)}{\Gamma\left(\frac{7}{2}-d\right)\Gamma\left(1-\frac{d}{2}\right)} \quad (5.37)$$
and again using equation (4.10) we get:

\[ B2(x) = \frac{\Gamma(3 - \frac{3d}{2} + x)\Gamma(\frac{d}{2})\Gamma(3 - d)\Gamma(2 - d + x)}{\Gamma(5 - 2d + x)\Gamma(x)\Gamma(1 - \frac{d}{2})\Gamma(2 - \frac{d}{2})} \]  

(5.38)

so that we have:

\[ G(x) = 2^{d-5}\Gamma(4 - \frac{3d}{2})\Gamma(\frac{3-d}{2})\Gamma(1+x)\Gamma(9-3d+x) \times \binom{2^{7-2d}\Gamma(\frac{d}{2})\Gamma(3-\frac{d}{2})\Gamma(6-2d+x)}{\Gamma(\frac{d}{2})\Gamma(5-2d+x) + 2(8-3d)\Gamma(1 + x - \frac{d}{2})} \times \binom{x(8-3d+x)\Gamma(\frac{5}{2} - d)\Gamma(2 - \frac{d}{2})\Gamma(6 - 2d + x)\Gamma(3 - d + x)}{x} \]  

(5.39)

Since our equation for \( T3(x) \) is a first order difference equation, we can write down its formal solution to be:

\[
T3(x) = \prod_{i=1}^{x-1} \left( \frac{2 - d + i)(5 - 2d + i)}{i(8 - 3d + i)} \right) T3(1) \\
+ \sum_{j=1}^{x-1} \left\{ \prod_{i=j+1}^{x-1} \left( \frac{2 - d + i)(5 - 2d + i)}{i(8 - 3d + i)} \right) \times G(j) \right\} 
\]  

(5.40)

where we start from \( x = 1 \), since for \( x = 0 \) the difference equation is not defined. So we have situation like in the case of \( BB2 \). Again, we can rewrite the equation as:

\[
T3(1) = T3(x) \times \left[ \prod_{i=1}^{x-1} \left( \frac{2 - d + i)(5 - 2d + i)}{i(8 - 3d + i)} \right) \right]^{-1} \\
- \sum_{j=1}^{x-1} \left\{ \prod_{i=j+1}^{x-1} \left( \frac{2 - d + i)(5 - 2d + i)}{i(8 - 3d + i)} \right) \times G(j) \right\} \times \left[ \prod_{i=1}^{x-1} \left( \frac{2 - d + i)(5 - 2d + i)}{i(8 - 3d + i)} \right) \right]^{-1} 
\]  

(5.41)

We now set \( x = \infty \) and \( T3(\infty) = J(\infty) \times V2 \), where \( V2 = \bigotimes \) and \( J(\infty) = \frac{1}{\Gamma(1 - \frac{d}{2})} \). From that we get for the first term:

\[
J(\infty) \times \bigotimes \left[ \prod_{i=1}^{\infty} \frac{(d-2-i)(5-2d+i)}{i(8-3d+i)} \right] = 0
\]  

(5.42)
and for the second term we get:
\[
\left[ \prod_{i=1}^{\infty} \frac{(2 - d + i)(5 - 2d + i)}{i(8 - 3d + i)} \right] \approx \frac{\Gamma(1 + j)\Gamma(9 - 3d + j)}{\Gamma(6 - 2d + j)\Gamma(3 - d + j)}
\]

\[ \Rightarrow \text{second term of eq. (5.41)} = \]

\[
= - \sum_{j=1}^{\infty} \left( \frac{\Gamma(1 + j)\Gamma(9 - 3d + j)}{\Gamma(6 - 2d + j)\Gamma(3 - d + j)} \times G(j) \right) \times \left[ \frac{\Gamma(9 - 3d)}{\Gamma(6 - 2d)\Gamma(3 - d)} \right]^{-1}
\]

(5.43)

The last term is our \( T3(1) \), since the first term is zero. Actually MATHEMATICA can “perform” the sum and write the result in terms of two hypergeometric functions (which are of course per definition again sums to infinity, so it is just rewriting the sum), so that we can write the result as:

\[
T3(1) \equiv \frac{\Gamma(9 - 3d)}{\Gamma(6 - 2d)\Gamma(3 - d)} \times \left[ \sum_{j=1}^{\infty} \left( \frac{\Gamma(1 + j)\Gamma(9 - 3d + j)}{\Gamma(6 - 2d + j)\Gamma(3 - d + j)} \times G(j) \right) \times \left[ \frac{\Gamma(9 - 3d)}{\Gamma(6 - 2d)\Gamma(3 - d)} \right]^{-1} \right]
\]

(5.44)

With existing program codes one can only expand in \( \epsilon \) the hypergeometric functions of this type around integer valued numbers [12], that means we can only expand the above equation in \( d = 4 - 2\epsilon \) using the program similar to the one in the appendix, modified from [13], for the two hypergeometric functions, with the result (here up to 6-th order which is new, but in principle one can expand to arbitrary order):

\[
T3(1) \overset{d=4-2\epsilon}{=} \frac{1}{4} + \frac{1}{2} \epsilon + \left( -8 + \frac{13\zeta_3}{2} \right) \epsilon^3 + \left( -\frac{241}{4} - \frac{5\pi^4}{8} + 4\zeta_3 \right) \epsilon^4
\]

\[
+ \left( -\frac{669}{5} - \frac{\pi^4}{5} - 36\zeta_3 + \frac{693}{2}\zeta_5 \right) \epsilon^5
\]

\[
+ \left( -1636 + \frac{21\pi^4}{5} - \frac{44\pi^6}{21} - 289\zeta_3 + \frac{241}{2}\zeta_3^2 + 72\zeta_5 \right) \epsilon^6 + O(\epsilon^7)
\]

(5.45)
Again, the algorithm for the expansion of hypergeometric functions around half-integer values exists and if one could find a transformation which transforms the coefficients in hypergeometric functions in our solution so that they appear in the form needed to apply algorithm from [16] then one could also calculate the result in $d = 3 - 2\epsilon$. From [37] we know that in $d = 3 - 2\epsilon$ the solution can be expanded in terms of MZV so there should be such transformation, but we were not able to find it.

There are two MI’s left to be solved, $VBa$ and $VBb$. Let us take a look at the difference equation of $VBb$ [38]:

$$VBb6(x) = \bigodot_{\mathcal{J}}^x$$

$$0 = (d - 3 - x)(3d - 8 - 2x)VBb6(x) - 2x(7 - 2d + x)VBb6(x + 1) + a_1(x)BB4(x) + a_2(x)BB4(x) + a_3(x)J(x) + a_4(x)B4(x) + a_5(x)B4(x + 1) + a_6(x)T2(x) + a_7(x)T2(x + 1) + a_8(x)$$

$$\Rightarrow VBb6(x + 1) = \frac{(d - 3 - x)(3d - 8 - 2x)}{2x(7 - 2d + x)}VBb6(x) + G(x) \quad (5.46)$$

This is a first order difference equation, with the same property as we had in previous example, namely it is valid only for $x > 0$ (the $a_i(x)$ are all polynomials in $x$ which are too long to be given here). Using the same strategy as in case of $T3$ one can write:

$$VBb6(1) = \left[ \prod_{i=1}^{\infty} \frac{(d - 3 - i)(3d - 8 - 2i)}{2i(7 - 2d + i)} \right]^{-1} VBb6(\infty) - \sum_{j=1}^{\infty} \left\{ \prod_{i=1+j}^{\infty} \frac{(d - 3 - i)(3d - 8 - 2i)}{2i(7 - 2d + i)} \right\} G(j) \times \left[ \prod_{i=1}^{\infty} \frac{(d - 3 - i)(3d - 8 - 2i)}{2i(7 - 2d + i)} \right]^{-1}$$

$$= \frac{\Gamma(5 - \frac{3d}{2})\Gamma(4 - d)}{\Gamma(8 - 2d)\Gamma(1 - \frac{d}{2})} \bigodot \left\{ \frac{\Gamma(5 - \frac{3d}{2})\Gamma(4 - d)}{\Gamma(8 - 2d)} \sum_{j=1}^{\infty} \frac{\Gamma(1 + j)\Gamma(8 - 2d + j)}{\Gamma(5 - \frac{3d}{2} + j)\Gamma(4 - d)} G(j) \right\} \quad (5.47)$$

Now, in the inhomogeneous term $G(x)$ we have $T4(x)$ which is governed by a second order difference equation, which we cannot solve directly nor
via Laplace transform method. The same problem arises for $BB_4(x)$ and $B_4(x)$, where we know, as quoted from [37], the result at the power of the propagators being one, and can use this as the initial value, but nevertheless we were not able to solve the second order difference equations. If however, all three integrals, the $BB_4(x), B_4(x)$ and $T_4(x)$ were given, for all $x$, in terms of harmonic sums, then $VBb_6(1) = VBb$ can be expressed in terms of harmonic sums at infinity, which are multiple zeta values.

From IBP relations one can also form another difference equation for $VBb$, where the power $x$ is attached to a different propagator, namely:

$$ VBb_8(x) = \frac{\partial}{\partial t} \left[ (d - 3 - x)(2d - 6 - x)VBb_8(x) - (2(d - 4)^2) + (23 - 6d)x + 3x^2)VBb(x + 1) - 2(2d - 8 - x)(1 + x)VBb(x + 2) + G(x) \right] $$

(5.48)

where $G(x)$ is a function of (we don’t write here the explicit expression for $G(x)$ since it is too long) $BB_4(x), BB_4(x + 1)$ which we don’t have and $B_2(1), J(x)$ and $V_3(x)$ which we have. The difference equation is second order, so we can’t give a formal solution. We can try to solve it by means of Laplace transformation. Following the instruction lines in section 4.2 we get for the homogeneous part of the equation (5.48) the differential equation:

$$ 0 = \left\{ t^2[(t - 1)(2t - 1)]\partial_t^2 - t[(8 - 3d) + (6d - 14)t + (8 - 4d)t^2]\partial_t + [2(d - 3)^3 + 2t(3 - d)(d - 2) + (4d - 12)t^2] \right\} v(t) $$

(5.49)

whose solution builds the solution of the difference equation through equation (4.14). The two linearly independent solution of the homogeneous differential equation (5.49) are:

$$ v(t) \overset{d=4-2\epsilon}{=} c_1 \left( \frac{1 - 2t - 2t \ln(t - 1) + 2t^2 \ln(t - 1) + 2t \ln(t) - 2t^2 \ln(t)}{t^2(t - 1)} \right) + c_2 \frac{1}{t} \quad \text{and} \quad v(t) \overset{d=3-2\epsilon}{=} c_1 (\ln(t - 1) - \ln(t)) + c_2 $$

(5.50)

Both solutions can be written in terms of HPL’s and $f_\alpha(x)$ functions so that the solution of the difference equation for $VBb_8(x)$ could be expressed
in terms of HPL’s if and only if the differential equation associated with
the inhomogeneous part, from we which we know that it contains \( BB_4(x) \),
\( B_2(1) \), \( J(x) \), \( V_3(x) \), can also be written in terms of HPL’s and \( f_n(x) \)’s. Since, again \( B_2(1) \), \( J(x) \) and \( V_3(x) \) are known, it remains only to find out
what \( BB_4(x) \) is. Similarly, one difference equation for \( V Ba \) is:

\[
V Ba_6(x) \equiv \frac{\partial^3}{\partial x^3}
\]
\[
0 = (d - 2 - x)(3d - 8 - 2x)V Bb_6(x)
- 2x(6 - 2d + x)V Ba_6(x - 1) + G_2(x)
\] (5.51)

where in \( G_2(x) \) there are \( BB_4(x) \), \( J(x) \), \( B_2(x) \) and \( T_3(x) \). Again, if \( BB_4(x) \)
wass given in terms of harmonic/Euler-Zagier sums, then one could obtain
the solution for \( V Ba_6(x) \) also, following the same procedure like in the case
of \( T_3 \).
To conclude this chapter, let us say that we have found a method for ex-
tracting the result at \( x = 1 \) of difference equations first order which are
valid only for \( x > 0 \) and that in order to solve the two remaining integrals
analytically to arbitrary order in \( \epsilon \) we have to solve \( BB_4(x) \) for general \( x \),
which is given by the difference equation from [38]:

\[
BB_4(x) \equiv \Gamma(x) \frac{\partial^3}{\partial x^3},
\]
\[
0 = -16(x + 5 - 2d)BB_4(x + 2)
+ 2(7x^2 - 16xd + 39x + 8d^2 - 40d + 50)BB_4(x + 1)
+ (x + 2 - d)(x + 4 - 2d)(2x + 6 - 3d) + \Gamma(x + 1 - \frac{d}{2} - \frac{3(d - 2)^2}{2\Gamma(1 - \frac{d}{2})})
\] (5.52)

with the boundary conditions (\( BB_4(1) \) is given in [37]):

\[
BB_4(2) = \frac{5 - 2d}{4}BB_4(1),
\]
\[
BB_4(x \gg 1) = \frac{\Gamma(x - \frac{d}{2})}{\Gamma(1 - \frac{d}{2})}B_3(1).
\] (5.53)

Solving this difference equation for general \( x \) is the next step one has to do.
Chapter 6

Discussion and Outlook

The nested sums seem to be the natural language of perturbative QFT. As we have seen, all Feynman graphs solved can be expressed in the $\epsilon$ expansion as numbers multiplied with $Z/S$-sums, or some subclasses thereof. For those diagrams which we could not solve, we could nevertheless formulate a conjecture, which states that they, the $V Ba$ and $V B b$, will be also expressible in terms of multiple zeta values, if $BB4(x)$ was given in terms of harmonic/Euler-Zagier sums, which we strongly believe will be the case. Now the question that one might ask is whether or not one can say that all single scale Feynman graphs of a given theory will be expressible in terms of some $S/Z$-sums in $\epsilon$-expansion. There is a paper [43], in which the three loop sunrise-type one scale massive graph is shown to contain non-logarithmic transcendental numbers. Until now it is unknown whether these can be transformed to $S/Z$-sums. It is interesting to know these numbers, since they could show a way to new mathematics underlying perturbative QFT. Maybe one could then find an even more abstract class of numbers of which the $S/Z$-sums would be special cases. Anyway it is worth keeping an eye on this branch, since some integrals which before the introduction of $S/Z$-sums couldn’t have been calculated, are now almost trivial to do. At the end, one needs these numbers to know how accurate the Standard Model is and where new physics might come in.
Appendix A

Example Codes

In this appendix we present some small sample codes in FORM. For the introduction in the programming language FORM we strongly recommend the excellent Vermaseren lecture given at CAPP 05 in Zeuthen/Germany, available at http://www-zeuthen.desy.de/theory/capp2005/. Here is an example code for the expression (3.19):

```form
#include summer6.h

L F = sum(j1,1,n)*S(R(1),j1)*S(R(2,1),j1-1)*den(j1+2); 
call summer(1)
Print +f +s;
.end
```

The `summer(1)` in the program above is the main procedure, which calls all necessary sub-procedures like `synch()` for synchronization and others. The one 1 in the bracket needs to be there since there is one sum over j1 to be done. The result is:

\[
\frac{1}{2 + j1}S(R(1,1),n)\theta(n) - \frac{2}{2 + j1}S(R(1,1,1),n)\theta( - 1 + n) - \frac{1}{2 + j1}S(R(2,1),n)\theta(n) - \frac{1}{2 + j1}S(R(2,2),n)\theta(n) - \frac{5}{2 + j1}S(R(3,1),n)\theta(n) -
\]
\[
\frac{1}{2 + j1} \cdot S(R(3,1),n) \cdot \theta(n) \cdot n;
\]

where \( \theta(n) \) is one when \( n > 0 \) and zero when \( n \leq 0 \). Another simple example would be \( S_{1,-1}(N) \times S_{2,1}(N) \):

```plaintext
#include summer6.h
L F = S(R(1,-1),N) * S(R(2,1),N);
#call basis(S)
Print +f +s;
.end

FORM gives as a result:

\[
F = \\
- S(R(1,-3,1),N) + S(R(1,-1,2,1),N) \\
- S(R(1,2,-2),N) + S(R(1,2,-1,1),N) \\
+ S(R(1,2,1,-1),N) - S(R(2,1,-2),N) \\
+ S(R(2,1,-1,1),N) + 2* S(R(2,1,1,-1),N) \\
- S(R(2,2,-1),N) + S(R(3,-2),N) \\
- S(R(3,-1,1),N) - S(R(3,1,-1),N);
\]

Here is the code for the expression (3.42):

```plaintext
#include harmpol.h
S y,a,m;
CFunction k1, f(a,x);
L F1 = k1*f(1,x)*H(R(1,0,1),x)*H(R(1,1),x);
#call hbasis(H,x)
print F1;
.sort
id k1*f(a?,x?)*H(R(?m),x?) = H(R(a,?m),x);
print F1;
.end

F1 = \\
3*f(1,x)*k1*H(R(1,1,1,2),x) + 4*f(1,x)*k1*
```
H(R(1,1,2,1),x) + 3*f(1,x)*k1*H(R(1,2,1,1),x);

Time = 0.21 sec  Generated terms = 3
F1  Terms in output = 3
Bytes used = 114

F1 =
3*H(R(1,1,1,1,2),x) + 4*H(R(1,1,1,2,1),x) +
3*H(R(1,1,2,1,1),x);

An example code for $2F_1(2\varepsilon - 1, \varepsilon; 2\varepsilon; z)$ up to $\varepsilon^2$ order would be:

* Code modified from S. Moch and P. Uwer's code
* from [13]

```c
#define EXPANDEP "4"
#define TRUNCIEP "2"
#define MAXSUM "3"
#define MAXWEIGHT "2"

#include declvars.h
#include declsums.h

cf ln, Li;
S D, Q2;
S CHECK;
Autodeclare S z, l;
PolyFun, acc;

autodeclare s nu;
s s123;
Off statistics;
.global

L F = sumj(j1, 0, inf)*pow(x1, j1)*invfac(j1)*Gamma(2*ep-1+j1)*
    InvGamma(2*ep-1)*Gamma(ep+j1)*InvGamma(ep)*
    InvGamma(j1+2*ep)*Gamma(2*ep);

* convert the sums into nested ones:
```
id \sum_1(j_1,0,\infty) = \text{replace}((j_1,0)) + \sum_1(j_1,1,\infty);

id \sum_1(i_1,0,\infty) \ast \sum_2(i_2,0,\infty) = \text{replace}((i_1,0,i_2,0))
+ \sum_1(j_1,1,\infty) \ast \text{replace}((i_1,j_1,0,i_2,1))
+ \sum_1(j_1,1,\infty) \ast \text{replace}((i_1,j_1,1,i_2,0))
+ \sum_1(j_1,1,\infty) \ast \sum_2(j_2,1,j_1-1) \ast \text{replace}((i_2,j_2,i_1,j_1-j_2));

.sort;

* call the routine for Gamma-fct expansion

#include Simplify

* immediate truncation of series in ep
id ep^n0?,\geq\text{TRUNCEP}\} = 0;
.sort: trunc eps;

* call the routine for summation

#include DoSum(1,1)

repeat id \text{pow}(x_1?,n_1?)\ast\text{pow}(x_1?,n_2?) = \text{pow}(x_1,n_1+n_2);
.id pow(x_1?,0) = 1;

.SplitArg,((\infty)),\text{pow};
.id pow(x?,?,\infty) = 0;
.id pow(x?,?,0?,\infty) = 0;
.repeat id x_1*\text{den}(1 - x_1) = -1 + \text{den}(1 - x_1);
.repeat id x_1*\text{den}(x_1 + x_2) = 1 - x_2*\text{den}(x_1 + x_2);
.repeat id 1/x_1*\text{den}(x_1+x_2) = 1/x_2*(1/x_1-\text{den}(x_1+x_2));

#include ConvStoZ(S,Z)
.id Z(R(?a),X(?x),\infty) = \text{Li}(R(\text{reverse}(?a)),X(\text{reverse}(_(?x))));

Argument Li;
Argument X;
.repeat id x_1*\text{den}(x_1 + x_2) = 1 - x_2*\text{den}(x_1 + x_2);
.id x_1^{-1}\ast x_2 = 1/x_1*\text{num}(x_1+x_2)-1;
.id \text{num}(x?)\ast\text{den}(x?) = 1;
.id x_1^{-1}\ast x_2 = 1/x_1*\text{num}(x_1+x_2)-1;
.id \text{num}(x?) = x;
.id x_1^{-1}\ast x_2 = 1/x_1*\text{num}(x_1+x_2)-1;

.EndArgument;
.EndArgument;
.sort;

* immediate truncation of series in ep
.id acc(x?) = x;
.id ep^n0?,\geq\text{TRUNCEP}\} = 0;
.sort(PolyFun=acc): trunc eps;
.PolyFun;
id \quad \text{acc}(x?) = x; \\
\text{id } \text{Li}(\text{?a},x(?y)) = \text{Li(?a,?y)}; \\
\text{B ep,den;} \\
\text{.sort} \\
\text{B den ep;} \\
\text{bracket ep;} \\
\text{print +s;} \\
\text{.end;} \\

\text{and the result is:} \\
F = \\
\quad + \text{ep} \times ( \\
\quad \quad + \text{Li}(1,x_1) \\
\quad \quad - \text{Li}(1,1,x_1) \times x_1 \\
\quad ) \\
\quad + \text{ep}^2 \times ( \\
\quad \quad + \text{Li}(1,1,1,x_1) \\
\quad \quad - \text{Li}(1,1,1,1,x_1) \times x_1 \\
\quad \quad + \text{Li}(1,1,1,2,x_1) \times x_1 \\
\quad ) \\
\quad + 1 \\
\quad \quad - 1/2 \times x_1 \\
\quad ;
Appendix B

Additional Results

Using the \( \epsilon \)-expansion from the previous chapter, we can write down the result of \( \bigotimes \) up to \( O(\epsilon^2) \), where the x1 is our \( z = \frac{3}{4} \):
where

\[ L_{n}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \] (B.2)

is multiple polylogarithm. One can of course expand the \( \bigotimes \) up to arbitrary order, but since this Feynman graph doesn’t belong to the class of diagrams which are expressible through the multiple zeta values, the result would be very long. Since there are many \( \sqrt{-z} \) expressions in the result, one might wonder if the result is real as it should be. To check that we set the numerical value for \( z \) which is \( \frac{3}{4} \) in the above result and get:

\[ J^2 = \frac{3}{2} - \frac{3}{2} \epsilon + \left( -3 + \frac{\pi(\pi + 3 \ln(3))}{\sqrt{3}} - 2\sqrt{3} \ln[\frac{3 + \sqrt{3}}{2}] \right) \epsilon^2 + O(\epsilon^3). \] (B.3)

This expression looks as if it has a non-zero imaginary part, but computing it numerically using MATHEMATICA option \( \text{Chop}[\text{N}:::] \) we get:

\[ J^2 = 1.5 - 1.5\epsilon + 0.515861\epsilon^2 + O(\epsilon^3) \] (B.4)

which is real and in agreement with the result from [38].

The result of \( \bigotimes \) in terms of hypergeometric functions is:

\[ \bigotimes \bigotimes = \frac{1}{\Gamma(\frac{d}{2})} \left\{ 2^{-3(d+1)}\pi^{-1} - \frac{2d}{\Gamma\left(\frac{d}{2} - 1\right)} \left( 2^d \sqrt{\pi \Gamma(3-d)}(1 + 3\Gamma(1 - \frac{d}{2}) + \Gamma(2 - \frac{d}{2}))\Gamma\left(\frac{d}{2}\right) _3F_2\left[1, 3 - d, 2 - \frac{d}{2}; \frac{5-d}{2}, \frac{1}{2}\right] \right) \right\} \]

\[ \left( (d - 4)\Gamma\left(\frac{d}{2}\right)(2\Gamma(1 - \frac{d}{2}) + \Gamma(d - 3) + \Gamma(\frac{3d}{2} - 4)) \right) \]

\[ \left( 1 + 2\cos[\ln(3)] \right) \left( 2\Gamma\left(\frac{d}{2}\right)(1 + 3\Gamma(1 - \frac{d}{2}) + \Gamma(2 - \frac{d}{2}))\Gamma\left(\frac{d}{2}\right) _3F_2\left[1, 3 - d, 2 - \frac{d}{2}; \frac{5-d}{2}, \frac{1}{2}\right] \right) \]

\[ \left( (d - 4)\Gamma\left(\frac{d}{2}\right)(2\Gamma(1 - \frac{d}{2}) + \Gamma(d - 3) + \Gamma(\frac{3d}{2} - 4)) \right) \]

\[ 3_3F_2\left[\frac{1}{2} - 2, d - 3, \frac{3d}{2} - 4; d - \frac{5}{2}, \frac{1}{2}; \cos^{-1}[\ln(3)]\right] \sin\left[\frac{\pi}{2}\right] \} \] (B.5)

The hypergeometric functions appearing in the above result are in four dimensions all of the form which is expandable with the algorithm described in [16].
I would like to thank Prof. Dr. Mikko Laine for introducing me to this subject and for his support during my thesis. A special thanks belongs to Dr. York Schröder, on whose work much of this thesis is based on, and who always had time for a discussion, whether it was on physics, mathematics or programming issue. Further I would like to thank Dipl. Phys. K. Huebner and Dipl. Phys. O. Vogt, with whom I shared the office with, for a great atmosphere during my thesis.
Erklärung

Hiermit versichere ich, die vorliegende Arbeit mit keinen weiteren als im Literaturverzeichnis angegebenen Hilfsmitteln angefertigt zu haben.

Bielefeld, 31.03.2006
Bibliography


[25] A.B.Goncharov, 
http://www.math.uiuc.edu/K-theory/0297

[26] J.M.Borwein, D.M.Bradley, D.J.Broadhurst, P.Lisonek,
math.CA/9910045

[27] L.Euler,
Novi Comm.Acad.Sci.Petropol. 20, 140 (1775)

[28] D.Zagier,
First European Congress of Mathematics, Voll 2
Birkhauser, Boston, 497 (1994)

[29] N.Nielsen,
Nova Acta Leopoldina(Halle 90),123 (1909)


k-fold Euler/Zagier sums: a compendium of results for arbitrary k,”

cial Values of Multiple Polylogarithms,” Trans. Am. Math. Soc. 353

[33] D. Maitre, “HPL, a Mathematica implementation of the harmonic poly-

[34] K. G. Chetyrkin and F. V. Tkachov, “Integration By Parts: The Al-
gorithm To Calculate Beta Functions In 4 Loops,” Nucl. Phys. B 192


[36] Y. Schroder and A. Vuorinen, “High-precision epsilon expansions of
single-mass-scale four-loop vacuum bubbles,” JHEP 0506 (2005) 051

[37] Y.Schroder, “in preparation”

[38] Y.Schroder, “unpublished”


