The thermodynamic and the continuum limit of quenched screening masses

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Introduction

In nature four fundamental forces are known: gravitation, the electromagnetic force, the weak and the strong interactions.

Historically first electromagnetism and weak interactions have successfully been merged to the electroweak force. A single quantum field theory referred to as the GLASHOW-SALAM-WEINBERG electroweak theory (GSW) successfully describes such different observable effects as sunlight and $\beta$-decay, where the different interaction lengths are due to the big mass differences in the coupling bosons, i.e. massless photons and quite heavy W- and Z-bosons respectively. The GSW theory is formulated as a non-abelian gauge theory of $SU(2) \times U(1)$.

The theory of strong interactions, describing forces that bind quarks to form for example protons and neutrons, is referred to as $QCD$, quantum-chromo-dynamics. The name relates to the three different charges that are named after the prime colours red, blue and green. The gauge bosons that mediate the interactions are called gluons. The gluons, in contrast to e.g. photons, self-interact and can form bound states; the non-abelian structure of QCD is also strongly believed to cause the quark confinement.

GSW and QCD combined form the theoretical foundation for the standard model of elementary particles, which has undergone precise experimental and theoretical tests.

Lattice QCD at finite temperatures

Taking a closer look at QCD, the strong interactions show three quite unique features: At low temperatures quarks are bound to particles of zero colour charge and cannot be separated without creating quark-antiquark pairs that compensate for the separated charges. This property is known as confinement.

A second connected principle is asymptotic freedom; with higher temperatures the effective coupling between quarks and gluons becomes weaker. In fact, at high temperatures QCD undergoes a phase transition to a deconfined phase and the low temperature particle structure dissolves into a quark-gluon-plasma.

Thirdly the QCD lagrangian possesses a chiral symmetry for massless fermions. At low temperatures this symmetry is spontaneously broken, with pions acting as Goldstone bosons, at the transition from confinement to deconfinement the chiral symmetry is restored.
Whenever long range phenomena are investigated, for instance when one wants to calculate properties of quark bound states or close to a phase transition where correlation lengths become large, pertubative methods cannot be applied; lattice discretisation schemes and numerical methods have to be used instead to perform calculations. Different discretisation schemes with different advantages and drawbacks are available, and of course a numerical lattice approach which is bound to finite volume and finite lattice spacing introduces systematic errors into calculations.

Thermodynamic and continuum limit

For this thesis, two widely used lattice actions, the clover-improved Wilson-action and the standard staggered action, have been systematically compared by calculating meson screening masses for two temperatures above the critical temperature $T_c$ of the deconfinement phase transition.

Meson screening masses are a basic tool to study QCD at high temperature. In the plasma state of the deconfined phase the spatial meson correlators can be used to calculate screening lengths which are directly given as the inverse of the meson mass obtained from the correlator in the spatial direction (hence the name screening mass).

The comparison was motivated by the observation that, so far published [1], meson masses calculated for Wilson and staggered fermions differ at finite volume and spacing; it is of great interest to study how systematic errors account for these differences. Both actions possess discretisation errors that are quadratic in the lattice spacing, but feature different advantages and drawbacks; the most pronounced is that the staggered action partially preserves the chiral symmetry at the cost of flavour mixing while the Wilson action breaks the chiral symmetry.

To quantify and compensate the systematic errors of a discretisation scheme, extrapolations to infinite lattice volume (thermodynamic limit) and zero lattice spacing (continuum limit) can be performed. Meson screening masses for both actions have been calculated at two temperatures $T/T_c = 1.5$ and $T/T_c = 3.0$. At each temperature three lattice spacings were chosen and at each of these spacings calculations were performed at four different lattice volumes so that extrapolations towards infinite volume and zero lattice spacing could be carried out.

For Wilson fermions, meson correlators at all needed lattice sizes where already calculated by Fröhlich [2]. To obtain staggered correlators the numerical software used had to be adapted to the staggered action and the calculations had to be performed. Additionally, due to the staggered flavour mixing and a more noisy signal in the vector and axialvector channels, quite extensive statistical analysis had to be developed and performed; for consistency, this analysis was applied to all Wilson and staggered correlators.
Structure of this thesis

This thesis is subdivided into three main chapters and a conclusion. In the first chapter, *theory*, a brief introduction to lattice QCD is given: The free Wilson action and the free staggered action are introduced as solutions to the fermion doubler problem, then the formulation of gauge fields and interacting fermions on the lattice is given. Meson operators and correlators are defined and finite temperatures on the lattice are introduced, which lead to the division into spatial and temporal correlators and pole and screening masses respectively.

In the second chapter, *methods*, the numerical and statistical methods necessary to obtain the meson screening masses are discussed: A short description of the conjugate gradient algorithm used to invert the fermion matrix is given, then the structure of the statistical process used to obtain meson screening masses for each ensemble is laid out. Also the ansatz for the infinite volume and continuum extrapolations is motivated and an overview over the software and computer systems used to perform the calculations is given.

In the third chapter, *results*, first a way to fix the quark mass on the lattice is motivated and analysed, then the different staggered meson channels that exist due to the flavour mixing are discussed. After that, infinite volume and continuum extrapolations for Wilson and staggered screening masses of the scalar, pseudoscalar, vector and axialvector meson are carried out and compared.
1 Theory

1.1 Motivation for the lattice formulation of QCD

The main feature that motivated the lattice approach to QCD is confinement: It is not possible to isolate quarks, or more generally, to create free particles with a non-zero colour charge. When attempting to separate bound quarks, the confining potential ensures that at a certain distance the creation of a new quark-antiquark-pair is energetically favourable to further separation.

A confining potential cannot be treated by usual perturbative methods. This lead to the development of the lattice formulation of QCD by Wilson in 1974.

1.2 Lattice formulation for a scalar field

A lattice formulation for a given field theory is deduced by making use of the path integral formalism. Given the euclidian\(^1\) lagrangian \(L\) of a free scalar field \(\phi\), for instance, the action \(S\) and the generating functional \(Z\) are defined as:

\[
L = \frac{1}{2}(\partial_{\mu}\phi)^2 + \frac{1}{2}(m\phi)^2
\]
\[
S = \int d^4x \ L
\]
\[
Z = \int [d\Phi] \ e^{-S}
\]

The generating functional \(Z\) itself can be deduced from the Greens function. In general it cannot be calculated (since it would mean to integrate over infinite dimensions), so the field has to be discretized first and a continuum limit has to be taken after the integration.

\(^1\)The euclidian choice of the lagrangian \(L\) leads to a euclidian metric (with imaginary time). This is preferable in numerical simulations because the exponential term in \(Z\) is not oscillating in the euclidian formulation.
With $a$ as the distance between two lattice sites, a vector of form $n$ pointing to the lattice
sits and $\delta \mu$ denoting a step to the next neighbouring site in direction $\mu$ the derivative
$\partial_\mu$ can be rewritten to a discrete lattice version
\[
\partial_\mu \phi(n) \to \frac{[\phi(n + \delta_\mu) - \phi(n)]}{a} \tag{1.1}
\]
and the second order derivative in $L$ becomes symmetric on the lattice
\[
\partial_\mu^2 \phi(n) \to \sum_\mu \left[ \phi(n - \delta_\mu) + \phi(n + \delta_\mu) - 2\phi(n) \right] / a^2 \tag{1.2}
\]
Using this lattice derivative and defining $\sum_{[m,n]}$ to run over all neighbouring sites the
action $S$ reduces to a sum and the generating functional $Z$ becomes well defined
\[
S = \frac{a^4}{2} \left\{ \sum_{[m,n]} [\phi(m) - \phi(n)]^2 / a^2 \right\} + \frac{1}{2} \sum_n m^2 \phi(n)^2
\]
\[
Z = \int (\prod_n d\phi_n) e^{-S}
\]
Because of its quadratic form the action can be written as a matrix ($M$), also sources
can be easily added by a term $J(n)$ per lattice site
\[
S[J] = \sum_{m,n} [\phi(m) M_{mn} \phi(n)] - \sum_n J(n) \phi(n)
\]
\[
Z[J] = \int (\prod_n d\phi_n) \exp \left\{ -\sum_{m,n} \phi(m) M_{mn} \phi(n) + \sum_n J(n) \phi(n) \right\}
\]
and the lattice version of the generating functional with sources can be rewritten to
\[
Z[J] = \frac{1}{Z[0]} \exp \left\{ \sum_{m,n} J(m) (M^{-1})_{mn} J(n) \right\}
\]
where $Z[0] = \frac{1}{\det(M)}$. Now the propagator or two-point-function can be defined
\[
< \phi(x) \phi(y) > = \int [d\phi] \phi(x)\phi(y) e^{-S}
\]
and by differentiating with respect to the sources, its lattice version reduces to the inverse
of the matrix $M$
\[
< \phi(m) \phi(n) > = (M^{-1})_{mn}
\]
\[ ^2 \mu \text{ and } \nu \text{ are used by convention to denote directions on the lattice. Usually (in 4 euclidian dimensions)} \]
\[ \mu = 4 \text{ denotes the time direction.} \]
\[ ^3 \text{The given action } S \text{ is the simplest choice for a scalar lattice action. Other more complex terms are possible as long as they reproduce the same continuum } (a \to 0 \text{ limit) physics; in numerical simulations (with finite } a) \text{ improved actions are often used to achieve more precise results.} \]
For this non-interacting case the matrix inversion can be done analytically in Fourier space and the result

\[
< \phi(x) \phi(y) > = a^2 \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \sum_{\mu} \left[ \frac{1}{2} \sin^2 \frac{p \mu a}{2} \right] + m^2
\]

can be used by taking the limit \( a \to 0 \) to arrive at the continuum propagator

\[
< \phi(x) \phi(y) > = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} m^2 + p^2
\]

For the scalar case the approach to use the simplest lattice derivative and lattice action works without problems. For fermions, however, problems arise when calculating the propagator in the same simple way.

### 1.3 Lattice formulation for free fermions

For free fermions, the basic approach is the same. The scalar field \( \phi \) is replaced by a spinor \( \Psi \) and euclidian forms for \( L, S \) and \( Z \) read

\[
L = \bar{\Psi} (\not{\partial} + m) \Psi = \bar{\Psi} (\gamma_\mu \partial_\mu + m) \Psi \\
S = \int d^4x \ L = \int d^4x \ \bar{\Psi} (\gamma_\mu \partial_\mu + m) \Psi \\
Z = \int [d\Psi d\bar{\Psi}] e^{-S}
\]  

(1.3)

However, the lagrangian is more complex now: The spinor has four components and the short hand form \( \not{\partial} \) denotes a sum over the \( \gamma \)-matrices coupling these components (this also introduces new symmetries).

The derivative \( \not{\partial} \) in \( L \) is now of first-order (instead of second order as in the scalar case); to keep the lattice derivative symmetric\(^4\) both neighbouring sites are included:

\[
\partial_\mu \Psi(n) \to \frac{1}{2a} [\Psi(n + \delta_\mu) - \Psi(n - \delta_\mu)]
\]  

(1.4)

All following steps to calculate the propagator are analogous to the scalar case. The lattice action \( S \) reduces to a sum that can be written as a matrix because of its quadratic form:

\[
S = \sum_{n,\mu} \frac{1}{2} [\bar{\Psi}(n) \gamma_\mu \Psi(n + \delta_\mu) - \bar{\Psi}(n) \gamma_\mu \Psi(n - \delta_\mu)] + m \sum_n [\bar{\Psi}(n) \Psi(n)]
\]  

(1.5)

\[
S = \sum_{m,n} \bar{\Psi}(m) M_{mn} \Psi(n)
\]

\[
Z = \int [d\Psi d\bar{\Psi}] e^{-S}
\]

\(^4\)A symmetric lattice derivative is desirable on a finite lattice to reduce discretisation errors.
Sources $\eta$, $\bar{\eta}$ are added to the action and the generating functional $Z[\eta, \bar{\eta}]$ with sources is defined

$$S[\eta, \bar{\eta}] = \sum_{m,n} \left[ \bar{\Psi}(m) M_{mn} \Psi(n) \right] - \sum_n \bar{\eta}(n) \Psi(n) + \eta(n) \bar{\Psi}(n)$$

$$Z[\eta, \bar{\eta}] = \frac{1}{Z[0]} \exp \left\{ - \sum_{m,n} \bar{\eta}_m M^{-1}_{mn} \eta_n \right\}$$

(1.6)

where $Z[0] = \det(M)$. The fermion two-point-function also reduces to the inverse of the matrix $M$ by differentiating $Z[\eta, \bar{\eta}]$ with respect to the sources

$$\langle \Psi(m) \bar{\Psi}(n) \rangle = (M^{-1})_{mn}$$

(1.7)

Problems arise in the continuum limit.

1.4 The fermion doubler problem

Calculating the propagator is again possible in the free case by a Fourier transform to invert the matrix $M$. The result then is

$$\langle \Psi(x), \bar{\Psi}(y) \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{1}{\gamma_\mu a^2} \sin(p_\mu a) + m \epsilon^{ip(x-y)}$$

but taking the continuum limit $a \to 0$ there are now 16 contributions where $\sin(p_\mu a) = 0$ while there is only one (the argument $p_\mu a$ approaching zero) in the scalar case. This is a direct consequence of the first order derivative in the fermion case in contrast to the second order derivative in the scalar case.

These doubler contributions can be avoided by changing the so-called naive fermion action $S$ (1.5) to a more complex form with the same continuum limit. Since the doublers are a consequence of the lagrangian’s symmetries, it is however not possible to construct an action that avoids fermion doublers and preserves all symmetries at the same time. This has been shown under very general assumptions by Nielsen and Ninomiya [3].

The two most popular actions to handle fermion doublers are the Wilson action and the Kogut-Susskind or staggered action.

1.5 The free Wilson action

The Wilson action is a quite direct approach to avoid the fermion doublers. Since the unwanted contributions arise from the outer regions of the Brillouin zone (where $p \to \pm \pi a$ and so $\sin(pa) = 0$) a second order derivative term suppressing these contributions is added to the naive action.
The naive action receives the term
\[ S^{(W)} = S^{(\text{naive})} - \frac{r}{2} \sum_n \Psi(n) \, \partial_\mu^2 \Psi(n) \]
where the derivative \( \partial_\mu^2 \Psi(n) \) is defined as in the scalar case (1.2). The Wilson parameter \( r \) is a free parameter that can be chosen between 0 and 1, where 0 reproduces the naive action and 1 is the common choice. The free Wilson action can be written as
\[ S^{(W)} = \frac{1}{2} \sum_{n,\mu} \left[ \bar{\Psi}(n)(r - \gamma_\mu)\Psi(n + \delta_\mu) + \bar{\Psi}(n)(r + \gamma_\mu)\Psi(n - \delta_\mu) \right] 
+ (m + 4r) \sum_n \bar{\Psi}(n)\Psi(n) \]
and by using a Fourier transform as before, the matrix form of the Wilson action can be inverted to give the two-point-function
\[ < \Psi(x) \, \Psi(y) >= \int d^4p \, \frac{-i}{(2\pi)^4} \sum_\mu \left[ 1 - \sum_\alpha \gamma_\alpha \frac{1}{a} \sin(p_\alpha a) + m + \frac{2r}{a} \sum_\mu \sin^2(p_\mu a) \right] e^{ip(x-y)} \]
where \( \sin^2(p_\mu a / 2) \) suppresses the doublers at \( p = \pm \pi a \) and vanishes in the continuum limit.

Defining the hopping parameter \( \kappa \) as \( \kappa = \frac{1}{2am + 8r} \) and using the normalisation freedom of actions, it can be brought into the more familiar form
\[ S^{(W)} = \sum_n \bar{\Psi}(n)\Psi(x) - \kappa \sum_{n,\mu} \left[ \bar{\Psi}(n)(r - \gamma_\mu)\Psi(n + \delta_\mu) + \bar{\Psi}(n)(r + \gamma_\mu)\Psi(n - \delta_\mu) \right] \]
The Wilson action solves the doubler problem without imposing restrictions on the number of flavours, at the cost of breaking the chiral symmetry completely.

### 1.6 The free staggered action

The doubler problem can also be solved if the poles in the outer region of the Brillouin zone are avoided in the Fourier transformation. To achieve this the effective lattice spacing is changed by redistributing the spinor degrees of freedom.

Starting from the continuum action (with \( \alpha \) and \( \beta \) as spinor index)
\[ S = \int d^4x \sum_{a,\beta} \bar{\Psi}_a(\gamma_\mu \partial_\mu + m)_{a,\beta} \Psi_\beta \]
the naive lattice action is found (as already given in (1.5))
\[ S = \frac{1}{2} \sum_{n,\mu} \sum_{a,\beta} \bar{\Psi}_a(n) \, \gamma_\mu^\alpha \gamma_\mu^\beta \Psi_\beta(n + \delta_\mu) - \bar{\Psi}_a(n) \, \gamma_\mu^\alpha \gamma_\mu^\beta \Psi_\beta(n - \delta_\mu) + m \sum_n \sum_\alpha \bar{\Psi}_\alpha(n)\Psi_\alpha(n) \]
To solve the doubler problem the field variables $\Psi(n)$ are spin-diagonalized by changing to a new set of field variables $\chi(n)$. The $\chi$-field is a linear combination of the field $\Psi$:

$$
\Psi_\alpha(n) = T_{\alpha\beta}(n) \chi_\beta(n) \\
\bar{\Psi}_\alpha(n) = \bar{\chi}_\beta(n) T^\dagger_{\alpha\beta}(n)
$$

where the $4 \times 4$-matrices $T(n)$ can be defined to satisfy

$$
T^\dagger(n) \gamma_\mu T(n + \delta_\mu) = \eta(n) \mathbb{I}
$$

and, with the lattice vector components $n = (n_1, n_2, n_3, n_4)$, the phases $\eta$ are defined as

$$
\eta_\mu(n) = (-1)^{n_1 + \cdots + n_\mu}, \quad \eta_{n,1} = 1
$$

The action in terms of the $\chi$-field now reads

$$
S = \frac{1}{2} \sum_\beta \left\{ \sum_{n,\mu} \eta_\mu(n) [\bar{\chi}(n) \chi(n + \delta_\mu) - \bar{\chi}(n) \chi(n - \delta_\mu)] + m \sum_n \bar{\chi}(n) \chi(n) \right\}
$$

and is still a (rewritten) naive lattice action. However, since the scalar phases $\eta$ replaced the Dirac matrices, the four components $\chi_\beta$ are no longer coupled but are just summed up by $\sum_\beta$. This sum can be omitted (setting $\beta = 1$) so the degree of freedom per lattice site reduces from 4 to 1. The staggered action reads

$$
S^{(stag.)} = \frac{1}{2} \sum_{n,\mu} \eta_\mu(n) [\bar{\chi}(n) \chi(n + \delta_\mu) - \bar{\chi}(n) \chi(n - \delta_\mu)] + m \sum_n \bar{\chi}(n) \chi(n) \quad (1.8)
$$

Now a way to link the scalar $\chi$-field back to spinor fields $\Psi_\alpha$ has to be found. First a new numbering scheme is defined: The lattice is divided into a coarse lattice of $2^4$-sided hypercubes and the field variables on this coarse lattice are defined as $2^4 = 16$ component vectors $\chi_\rho(N)$, where the coordinate $N$ points to each hypercube and the vector $\rho$ points to 1 of the 16 sites inside the hypercube:

$$
\chi_\rho(N) = \chi(2N + \rho)
$$

Here one of the main drawbacks of the staggered formulation becomes obvious: Since each hypercube contains 16 degrees of freedom that have to be linked to spinor components, only a system with 4 flavours$^5$ can be described. Each hypercube is linked to a 4-flavour spinor $\Psi_\alpha$ by a linear transformation

$$
\Psi(N) = \Gamma \cdot \chi(N) \quad \Psi_\alpha(N) = \sum_\rho \Gamma_{\alpha f, \rho} \chi_\rho(N) \quad (1.9)
$$

$^5$The staggered scheme can also be applied to a different number of dimensions (where an odd number of dimensions needs a more complicated approach) and more general assumptions for the $T$-matrices and the phases $\eta_\mu$ are possible.

$^6$The naive and the Wilson action do not depend on the number of flavours, the staggered action however does: in the 4-dimensional system the number of flavours is fixed to 4.
where the $4 \times 4$-matrix $\Gamma$ can be defined as

$$\Gamma_{\alpha f, \rho} = \frac{1}{2} (\gamma^1 \gamma^2 \gamma^3 \gamma^4)_{\alpha f}$$

(1.10)

The fact that the 16 entries of $\Psi^f_\alpha$ can be identified as spinor and flavour components is not obvious, it is derived by comparing the contributions in the continuum limit [4].

Transforming the derivatives and making use of the $\gamma$-matrix properties, the staggered action can be written in terms of the $\Psi^f_\alpha$-fields in matrix notation ($\Delta\mu$ is the derivative on the coarse/blocked lattice)

$$S^{(\text{stag.})} = \sum_{N,\mu} (2a)^4 \bar{\Psi}(N) \left[ (\gamma_\mu \otimes 1) \Delta\mu + a(\gamma_5 \otimes \gamma^*_\mu \gamma_5)(\Delta\mu)^2 \right] \Psi(N)$$

$$+ 2m \sum_N \bar{\Psi}(N) 1 \otimes 1 \Psi(N)$$

where the left matrices in $\gamma_\mu \otimes 1$ and $\gamma_5 \otimes \gamma^*_\mu \gamma_5$ act in Dirac space and the right matrices act in flavour space. The Wilson action for a 4-flavour system written in the same form reads:

$$S^{(\text{Wilson})} = \sum_{n,\mu} a^4 \bar{\Psi}(n) \left[ (\gamma_\mu \otimes 1) \partial_\mu - \frac{ar}{2} (1 \otimes 1)(\partial_\mu)^2 \right] \Psi(n)$$

$$+ m \sum_N \bar{\Psi}(n) 1 \otimes 1 \Psi(n)$$

Both actions include a second order derivative term that suppresses the doublers and vanishes linearly with $a$ in the continuum limit. But while the simple term $\frac{ar}{2} 1 \otimes 1$ in the Wilson action breaks the chiral symmetry completely for finite lattice spacings, the term $a(\gamma_5 \otimes \gamma^*_\mu \gamma_5)$ in the staggered action conserves parts of it as a $U(1) \times U(1)$-symmetry:

$$\Psi(N) \rightarrow e^{ia(\gamma_5 \otimes \gamma_5)} \Psi(N)$$

As a consequence, at low temperatures $T < T_c$ a Goldstone pion exists for staggered fermions at finite lattice spacings while this is not the case in the Wilson formulation that breaks the chiral symmetry. As a drawback, however, the second order derivative term $\gamma_5 \otimes \gamma^*_\mu \gamma_5$ leads to the flavour mixing in the staggered scheme.

1.7 Gauge fields

Until here, the scalar and fermion fields were non-interacting. For an interacting QCD theory non-abelian gauge fields mediating the fermion interaction have to be introduced.
The continuum fermion action

\[ S = \int d^4x \, \bar{\Psi}^i (\not\partial - m) \Psi^i \]

shall be invariant under a local gauge transformation \( G(x) \in SU(N) \)

\[ \Psi(x) \rightarrow G(x) \Psi(x) \quad \quad \bar{\Psi}(x) \rightarrow \bar{\Psi}(x) G^{-1}(x) \]

where the \( N \times N \)-matrices \( G(x) \) act on a \( N \)-component spinor \( \Psi^n \) (since \( N \) is identified as the number of QCD colours it is set to \( N = 3 \), so \( i \) becomes the spinors colour index).

Deducing a derivative to arrive at a gauge invariant action, it is found that \( \not\partial \) has to be extended to the covariant version \( D^\mu = \gamma^\mu (\partial^\mu + ig_0 A^\mu) \), where \( g_0 \) is the bare coupling constant and \( A^\mu \) is the \( N \times N \)-matrix valued vector potential representing the gluon field.

With the covariant derivative the action’s fermionic part

\[ S_F = \int d^4x \, \bar{\Psi} \left( D^\mu - m \right) \Psi = \int d^4x \, \bar{\Psi} \gamma^\mu (\partial^\mu + ig_0 A^\mu) \Psi + m \bar{\Psi} \Psi \]

becomes invariant under local gauge transformations. Of course the action also has an kinetic term. It can be written as

\[ S_G = \frac{1}{4} \int d^4x \, F^B_{\mu\nu} F^{B}_{\mu\nu} \]

where the field strength tensor \( F_{\mu\nu} \) is given by \( (B, C, D \in \{1, \ldots, 8\} \) denote the colour indices of the underlying \( SU(3) \) Lie-algebra)

\[ F^B_{\mu\nu} = \partial^\mu A^B_{\nu} - \partial^\nu A^B_{\mu} - g_0 f^{BCD}_{\mu} A^C_{\nu} A^D_{\mu} \]

The self-interaction term \( g_0 f^{BCD}_{\mu} A^C_{\nu} A^D_{\mu} \) (where \( f^{ABC} \) is structure constant) is an important feature of QCD (because it is strongly believed to cause the quark confinement) and a direct consequence of the non-abelian structure; the field strength tensor of an abelian theory (like QED) takes the well known form \( F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \).

For a lattice formulation it has to be taken into account, that the derivative will become a finite difference involving not only \( \Psi(x) \) but two distinct points \( \Psi(x) \) and \( \Psi(y) \). The bilinear form

\[ \bar{\Psi}(x)\Psi(y) \rightarrow \bar{\Psi}(x) G^{-1}(x)G(y) \Psi(y) \]

is already gauge invariant under a global transformation \( (G(x) = G(y)) \), but for local gauge invariance a factor compensating the different local gauges has to be added. With the matrix valued vector potential \( A^\mu \) the parallel transporter compensating these gauge differences has been introduced as

\[ U(x, y) = \exp \left\{ ig_0 \int_x^y dz_\mu A^\mu(z) \right\} \]
Applying a gauge transformation to $U(x, y)$ results in

$$U(x, y) \rightarrow G(x)U(x, y)G^{-1}(y)$$

so a bilinear form including $U(x, y)$ becomes gauge invariant

$$\bar{\Psi}(x)U(x, y)\Psi(y) \rightarrow \bar{\Psi}(x)G^{-1}(x)\left[G(x)U(x, y)G^{-1}(y)\right]G(y)\Psi(y) = \bar{\Psi}(x)U(x, y)\Psi(y)$$

### 1.8 Lattice formulation of gauge fields

Now a lattice formulation of the gauge fields has to be found. The steps are analogous to the continuum formulation: The action has to be gauge invariant under a local transformation $G(n) \in SU(3)$ applied to the field variables $\Psi(n) \rightarrow G(n)\Psi(n)$.

The parallel transporter $U(x, y)$ can be easily defined to connect two neighbouring lattice sites $n, n + \delta_\mu$ (also referred to as link).

$$U(n, n + \delta_\mu) = \exp \left\{ ieaA_\mu(n + \frac{1}{2}\delta_\mu) \right\}, \quad U(n + \delta_\mu, n) = U^\dagger(n, n + \delta_\mu)$$

Adding these links to bilinear forms as in the continuum case yields a gauge invariant action. The naive fermion action (1.5) transforms to

$$S_{F}^{\text{(naive)}} = \sum_{n, \mu} \left[ \bar{\Psi}(n) \gamma_\mu U(n, n + \delta_\mu) \Psi(n + \delta_\mu) - \bar{\Psi}(n) \gamma_\mu U(n, n - \delta_\mu) \Psi(n - \delta_\mu) \right]$$

$$+ m \sum_n [\bar{\Psi}(n)\Psi(n)]$$

In order to obtain a gauge invariant kinetic term, the plaquette is introduced

$$\square_{\mu \nu}(n) = U(n, n + \delta_\mu) U(n + \delta_\mu, n + \delta_\mu + \delta_\nu) U(n + \delta_\mu + \delta_\nu, n + \delta_\nu) U(n + \delta_\nu, n)$$

$$= U(n, n + \delta_\mu) U(n + \delta_\mu, n + \delta_\mu + \delta_\nu) U^\dagger(n + \delta_\nu, n + \delta_\mu + \delta_\nu) U^\dagger(n, n + \delta_\nu)$$

which describes a path along four links connecting the site $n$ to its next neighbours in $\mu$- and $\nu$-direction. For each plaquette an action $S_\square$ can be defined as

$$S_{\square}(\mu, \nu(n)) = \text{Tr} \left\{ 1 - \frac{1}{6} (\square_{\mu \nu}(n) + \square_{\nu \mu}(n)) \right\}$$

and summing ($\sum_\square$) over each unique plaquette, an interaction term $S_G$ is found

$$S_G = \frac{6}{g_0^2} \sum_\square S_{\square}(\mu, \nu(n))$$

which reproduces the right continuum limit [4] and where the coefficient $\frac{6}{g_0^2}$ is found by comparing to the continuum form.
1.9 Interacting Wilson fermions

Finding a gauge invariant version of the Wilson action is easy since it does not differ much from the naive action. Adding the link-term $U$ to the bilinears the fermionic part reads

$$S_{F}^{(W)} = \sum_{n} \bar{\Psi}(n)\Psi(n) - \kappa \sum_{n,\mu} \left[ \bar{\Psi}(n) (r - \gamma_{\mu}) U(n, n + \delta_{\mu}) \Psi(n + \delta_{\mu}) + \bar{\Psi}(n + \delta_{\mu}) (r + \gamma_{\mu}) U(n + \delta_{\mu}, n) \Psi(n) \right]$$

There is a drawback using this action in numerical simulations: For finite lattice spacings $a$ the Wilson action has an error of order $O(a)$ while the naive and the standard staggered action have an error of order $O(a^2)$. The clover improvement\(^7\) can be applied to the mass term to counter the error and change its order back to $O(a^2)$. The mass term gains a contribution

$$\bar{\Psi}(n)\Psi(x) \rightarrow \bar{\Psi}(n) \left[ 1 - \frac{1}{2} i c_{SW} \sigma_{\mu\nu}(n) \right] \Psi(n)$$

where $\sigma_{\mu\nu}$ are given by $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}]$ and the factor $c_{SW}$ is calculated non-perturbativly to \(^6\)

$$c_{SW} = \frac{1 - 0.656 g^2 - 0.154 g^4 - 0.054 g^6}{1 - 0.922 g^2}$$

The full clover improved Wilson action

$$S_{F}^{(W)} = \sum_{n} \bar{\Psi}(n) \left[ 1 - \frac{1}{2} i c_{SW} \sigma_{\mu\nu}(n) \right] \Psi(n) - \kappa \sum_{n,\mu} \left[ \bar{\Psi}(n) (r - \gamma_{\mu}) U(n, n + \delta_{\mu}) \Psi(n + \delta_{\mu}) + \bar{\Psi}(n + \delta_{\mu}) (r + \gamma_{\mu}) U(n + \delta_{\mu}, n) \Psi(n) \right]$$

is used for all calculations involving Wilson fermions presented in this thesis.

1.10 Interacting staggered fermions

For staggered fermions, the link term $U$ couples to the $\chi$-fields and the staggered action (1.8) becomes gauge invariant

$$S^{(stag.)} = \frac{1}{2} \sum_{n,\mu} \eta_{\mu}(n) [\bar{\chi}(n)U(n, n + \delta_{\mu})\chi(n + \delta_{\mu}) - \bar{\chi}(n)U(n, n - \delta_{\mu})\chi(n - \delta_{\mu})] + m \sum_{n} \bar{\chi}(n)\chi(n)$$

\(^7\)The name clover improvement refers to the shape of a clover that the summed plaquettes in the improvement term take.
The link term is of course also included in the transformation linking \( \chi \)-fields to the physical \( \Psi \)-fields:

\[
\Psi(N) = \Gamma \cdot U \chi(N) \quad \longrightarrow \quad \Psi^f_{\alpha}(N) = \sum_{\rho} \Gamma_{\alpha f, \rho} U_{\rho}(N) \chi_{\rho}(N)
\]  

(1.11)

This becomes important if operators are to be calculated using the \( \Psi \)-fields: An operator local in the \( \Psi \)-field, \( O(n) = O(\bar{\Psi}(n), \Psi(n)) \), does not depend on the links in the naive or Wilson formulation. Rewriting the operator to the staggered formulation, the \( \Psi \)-field becomes a linear combination of 16 sites \( \rho \) (all belonging to one hypercube on the coarse lattice) linked by \( U_{\rho}(N) = U(2N, 2N + \delta_{\rho}) \), so the operator can still depend on the links inside each hypercube\(^8\).

In contrast to the Wilson action the standard staggered action has an error of order \( O(a^2) \) on the finite lattice, so to compare discretisation errors between both actions no improvement for the staggered action is needed.

### 1.11 Meson fields

The construction of meson operators proceeds, for Wilson fermions, as in the continuum. For instance, the pseudoscalar meson operator reads \( \bar{\Psi}(n) \gamma_5 \Psi(n) \) and the vector meson is given by \( \bar{\Psi}(n) \gamma_{\mu} \Psi(n) \) (where the correlation function will be summed over \( \mu = 1, 2, 3 \)).

For staggered fermions, the situation is a bit more complicated. In terms of the quark fields \( \Psi \), which carry spinor as well as flavour indices, the meson field \( M \) can be defined as

\[
M_{DF}(N) = \bar{\Psi}(N) \begin{bmatrix} \Gamma_D & \Gamma_F^* \end{bmatrix} \Psi(N)
\]

where the matrix \( \Gamma_D \) acts on the Dirac and the matrix \( \Gamma_F \) acts on the flavour index. Now the meson field can be rewritten in terms of the \( \chi \)-field by applying equation (1.9)

\[
M_{DF}(N) = \sum_{\rho \rho'} \chi(2N + \rho) \Gamma_{\rho}^{*\alpha f} \begin{bmatrix} \Gamma_D^{\alpha \alpha'} & \Gamma_F^{*f' f} \end{bmatrix} \Gamma_{\rho'}^{\alpha' f'} \chi(2N + \rho')
\]

where the product of the four \( \Gamma \)-matrices links the 16 hypercube sites. Note that for the interacting case equation (1.11) would have to be used and the general meson field would receive contributions from the links inside the hypercubes. These contributions can however be avoided by restricting the meson field to contributions local in the \( \chi \)-fields (which leads to less numerical effort) as follows:

\(^8\)In numerical simulations operators depending on links are often unfavourable because they increase noise and computational effort.
The $\Gamma$-matrix product is transformed and rewritten as a trace over the matrices

$$
\Gamma*^{\alpha f}_D \Gamma^*_F \Gamma^{' \alpha f'}_D = \Gamma*^{\alpha f}_D \Gamma^*_F \Gamma^{' \alpha f'}_D = \text{Tr} \left\{ \Gamma*^{\alpha f}_D \Gamma^*_F \Gamma^{' \alpha f'}_D \right\}
$$

and to arrive at a local meson field the flavour index is restricted to $\Gamma_F = \Gamma_D$. The trace now reads:

$$
\Gamma_D = \Gamma_F \rightarrow \text{Tr} \left\{ \Gamma*^{\alpha f}_D \Gamma^*_F \Gamma^{' \alpha f'}_D \right\}
$$

Using the definition $\Gamma*^{\alpha f}_D = \frac{1}{2}(\gamma^1 \gamma^2 + \gamma^3 \gamma^4)_{\alpha f}$ (given in (1.10)) and the trace theorem of gamma matrices $\text{Tr}\{\gamma^{1n+1}\} = 0$ leads to huge simplifications: Only the terms $\rho = \rho'$ contribute so the meson field becomes local; no link variables $U$ contribute to this field in the interacting case. Also with

$$
\Gamma_D \Gamma_D = (-)^{D\bar{\rho}} \Gamma_D \Gamma_D
$$

where $\bar{\rho}$ denotes the short hand form $\bar{\rho} = \rho_1 + \cdots + \rho_n$ the trace term for the local contributions can be reduced to a simple phase factor depending only on $D$ and $\bar{\rho}$.

Calculating the meson field now only involves a sum over the hypercube sites where some summands (depending on the Dirac index $D$) are multiplied by $-1$:

$$
M_D(N) = \sum_{\rho} (-)^{D\bar{\rho}} \chi(2N + \rho) \chi(2N + \rho)
$$

1.12 Meson correlators

To extract the meson spectrum from a lattice simulation, the meson correlation function or correlator is used.

It can be defined as

$$
Q_{AC}(N) = \langle M_A(N) M_C(0) \rangle
$$

and summing over all spatial components $\sum_{\bar{N}}$ (with $\bar{N} = (1,2,3)$) to project onto zero momentum the temporal correlation function reads

$$
Q_{AC}(N_4) = \sum_{\bar{N}} \langle M_A(\bar{N}, N_4) M_C(\bar{0}, 0) \rangle
$$
Staggered action

Now, for the staggered action, selection rules can be found for combinations of $A$ and $C$ where $Q_{AC} \neq 0$. Applying a spatial axial inversion (inverting $N = 1, 2, 3$) it can be found that spin is conserved ($J_A = J_C$) while parity can be inverted ($P_A = -P_C$). Using

$$Q_{AC}(N) = Q_{CA}(-N) = (-)^{|A+C|} Q_{CA}(N)$$

it is found that the parity-conserving correlation function is even, the parity inverting correlation function is odd on the staggered lattice.

Here a second drawback of staggered fermions becomes obvious: The correlation function contains particle pairs as an oscillating an a non-oscillating contribution. This leads to a more complicated process when analysing numerical data.

To actually calculate the correlation function in a simulation it can be formulated directly in terms of the $\chi$-fields and, using symmetries, it can be simplified to save a lot of computing time:

Using (1.12) to rewrite (1.14) the correlator becomes a four-point function on the staggered $\chi$-fields

$$Q_{AC}(N_4) = \frac{1}{16^2} \sum_{\bar{N}\rho\rho'} (-)^{A\rho+C\rho'} \left\langle \bar{\chi}(2N + \rho) \chi(2N + \rho) \bar{\chi}(\rho') \chi(\rho') \right\rangle$$

The Matthews-Salam formula (all following indices $i, j, k, l$ denote colour)

$$\left\langle \chi_i(x) \bar{\chi}_j(x') \right\rangle = G_{ij}(x, x')$$

(1.15)

allows to reduce the 4-point function $Q_{AC}(N_4)$ to a combination of 2-point functions $G_{ij}(x, x')$ that can be identified as fermion propagators on the $\chi$-field. It can now be used to split the correlator into a connected and a disconnected part

$$\left\langle \bar{\chi}(2N + \rho) \chi(2N + \rho) \bar{\chi}(\rho') \chi(\rho') \right\rangle = -G_{ij}(n', n') G_{ki}(n, n) + G_{kj}(n, n') G_{li}(n', n)$$

$$\text{disconnected part} + \text{connected part}$$

Except for flavour singlets the disconnected part does not contribute. Using only the connected part and the relation $G_{ij}^*(n, n') = (-)^{|n+n'|} G_{ji}(n', n)$ to rewrite $Q$ the correlator reduces to a much simpler term (the trace acts in colour space)

$$Q_{AC}(N_4) = \frac{1}{16^2} \sum_{\bar{N}\rho\rho'} (-)^{A\rho+C\rho'+|n+n'|} \left\langle \text{Tr} \left\{ G(2N + \rho, \rho') G^\dagger(2N + \rho, \rho') \right\} \right\rangle$$

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To further lower the computational effort the invariance under parity transformation is used

\[ Q_{AC}(N_4) = -\frac{1}{32} \sum_{N_{p\rho'}} (-)^{Ap+p\rho'(A_4+C_4)} \langle \text{Tr} \left| G \left( 2(N, (-)\rho' N_4) + \rho, 0 \right) \right|^2 \rangle \]

and finally \( Q \) is rewritten in terms of \( \chi \)-fields on the fine lattice to \( H \) (where \( \zeta = (-)^{\bar{\alpha}n} \) is a phase factor)

\[ H_{\alpha}(n_4) = -\frac{1}{32} \sum_{\bar{N}} \zeta(n) \langle \text{Tr} |G((\bar{\alpha}, n_4), 0)|^2 \rangle \quad (1.16) \]

Now the meson correlation function has been reduced to a sum and trace over the two-point function on the lattice. Numerically the inversion of the very big but sparsely populated fermion matrix leads to this two-point function (1.7) and is the main computational effort in the lattice simulation.

As a last step, the relation between the correlator and the meson mass has to be found and the different mesons (possible choices for \( \bar{\alpha} \) in \( H_{\alpha} \)) have to be identified as physical particles.

In the continuum the mass is related to the poles of the correlators by \( p_0 = \pm \sqrt{\vec{p}^2 + m^2} \). On a lattice with finite spacing however this relation becomes

\[ p_0 = \text{arcsinh} \left( \pm \sqrt{\sin^2(\bar{\vec{p}}) + m^2} \right) \]

Projecting to zero momentum (\( \vec{p} = 0 \)) and with the mixed correlation function of the staggered formulation the following relation between the mass and the correlator is found

\[ G(t) = A_{no} \cdot e^{-m_{no} \frac{N_t}{2}} \cdot \cosh \left[ m_{no} \cdot \left( \frac{N_t}{2} - t \right) \right] \\
+ (-)^t A_{ox} \cdot e^{-m_{ox} \frac{N_t}{2}} \cdot \cosh \left[ m_{ox} \cdot \left( \frac{N_t}{2} - t \right) \right] \quad (1.17) \]

Depending on the phase factor \( \zeta(n) \) in (1.16) and the oscillating or non-oscillating part of the correlation function different particles are found and identified as physical particle states.
Wilson action

Now, for the Wilson action, the four-point-function

\[ Q(n) = \langle \bar{\Psi}(n) \Gamma \Psi(0) \bar{\Psi}(0) \Gamma \Psi(n) \rangle \]

found when inserting the Wilson meson field operators into (1.13) can also be reduced to a two-point function of the fermion fields. As for the staggered action, the Matthews-Salam formula is used to split the four-point function into a connected and a disconnected part, where the connected part which is of interest for the calculations in this thesis reads

\[ Q(n_4) = \sum \langle \text{Tr} \left( G((\vec{n}, n_4), 0) \Gamma \gamma_5 G^\dagger((\vec{n}, n_4), 0) \gamma_5 \Gamma \right) \rangle \]

(1.18)

The \( \Gamma \)-matrices set the particle content of the correlator: The scalar meson \( a_0 \) is found for \( \Gamma = 1 \), the pseudoscalar meson \( \pi \) is found for \( \Gamma = \gamma_5 \). The transversal vector meson \( \rho \) is found by setting \( \Gamma = \gamma_1, \gamma_2, \gamma_3 \) and summing over the resulting correlators. The axialvector meson is found in the same way by setting \( \Gamma = \gamma_1\gamma_5, \gamma_2\gamma_5, \gamma_3\gamma_5 \) and also summing over the correlators.

1.13 Temperature on the lattice

For a lattice simulation at finite temperature of course temperatures on the lattice have to be introduced. This will also set a physical scale that has not been explicitly introduced until now.

The partition function of a quantum system at finite temperature in equilibrium is given by

\[ Z(T) = \text{Tr} \left\{ e^{-\hat{H}/T} \right\} \]

which can also be written as a path integral

\[ Z(T, V) = \int \mathcal{D}\phi \exp \left\{ - \int_0^{1/T} d\tau \int_V d^3x L_E[\phi(x, \tau)] \right\} \]

Comparing this expression to the path integral used to deduce a lattice formulation of fermion fields (1.3), it is found, that the temporal extend is bound to and restricted by the inverse temperature \( \beta = 1/T \).

For a finite lattice \( N^3 \sigma \times N_\tau \) where the temporal extent \( N_\tau \) is significantly smaller than the spatial extent \( N_\sigma \) \( (N_\tau \ll N_\sigma) \) this leads to a relation between the temperature and the temporal extent:

\[ N_\tau \cdot a = \beta = \frac{1}{T} \]

\[ V = (N_\sigma a)^3 \]
Also, since the time dimension $\tau$ is now unique, different correlators can be calculated in either the temporal ($N_4 = N_\tau$) or spatial ($N_3 = N_z$ by convention) direction. Rotating the correlator given in (1.17), the temporal direction is exchanged with the spatial $z$-direction. The spatial correlator

$$G(z) = A_{n_0} \cdot e^{-m_{n_0,scr.} \frac{N_z}{2}} \cdot \cosh \left[ m_{n_0,scr.} \cdot \left( \frac{N_z}{2} - z \right) \right]$$

$$+ (-)^z A_{o_\tau} \cdot e^{-m_{o_\tau,scr.} \frac{N_z}{2}} \cdot \cosh \left[ m_{o_\tau,scr.} \cdot \left( \frac{N_z}{2} - z \right) \right]$$

(1.19)

takes the same form with exchanged indices $\tau$ and $z$. The mass, however, is a screening mass $m_{scr.}$. It can differ from the mass characterising the exponential decay in the temporal direction. Moreover, because the temporal direction $\tau$ is distinct on the lattice the spatial correlator (summing over $x, y, \tau$) has less symmetry than the temporal correlator (summing only over the spatial directions $x, y, z$) which will lead to different screening masses of the transverse (T) and the longitudinal (L) vector and axialvector polarisations.

All analyses in this thesis are done by using spatial correlators; for reference the eight staggered channels investigated and their particle content are given in table 1.1.

<table>
<thead>
<tr>
<th>Phasefactor</th>
<th>$T$</th>
<th>$J^{PC}$</th>
<th>Particles</th>
<th>Channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>($-)^{t+y+t}$</td>
<td>$\gamma_3 \gamma_7$</td>
<td>$\pi^{0+}$</td>
<td>$\pi^{0+}$</td>
<td>$M_1$</td>
</tr>
<tr>
<td>($-)^{0}$</td>
<td>$\gamma_5 \gamma_7$</td>
<td>$\pi^{0+}$</td>
<td>$\pi^{0-}$</td>
<td>$M_2$</td>
</tr>
<tr>
<td>($-)^{y+t}$</td>
<td>$\gamma_7 \gamma_3$</td>
<td>$\rho_2^{0+}$</td>
<td>$\rho_2^{0+}$</td>
<td>$M_3$</td>
</tr>
<tr>
<td>($-)^{t+y}$</td>
<td>$\gamma_7 \gamma_4 \gamma_5$</td>
<td>$\rho_2^{0+}$</td>
<td>$\rho_2^{0+}$</td>
<td>$M_4$</td>
</tr>
<tr>
<td>($-)^{y}$</td>
<td>$\gamma_7$</td>
<td>$\rho_1^{0+}$</td>
<td>$\rho_1^{0+}$</td>
<td>$M_5$</td>
</tr>
<tr>
<td>($-)^{t}$</td>
<td>$\gamma_4 \gamma_7 \gamma_2$</td>
<td>$\rho_1^{0+}$</td>
<td>$\rho_1^{0+}$</td>
<td>$M_6$</td>
</tr>
</tbody>
</table>

Table 1.1: Particle content in the different staggered meson channels

Fixing a temperature also fixes the physical scale on the lattice, since it sets the distance between lattice points ($a$). With $a$ set by $T$ and $N_\tau$ now $197\text{MeV} \approx (1\text{fm})^{-1}$ can be used to convert values between their physical and their lattice units. The lattice temperatures in this thesis are given in relation to the critical temperature $T_C = 270\text{MeV}$ as $\frac{T}{T_C} \approx 1.5$ or $\frac{T}{T_C} \approx 3.0$. 

26
2 Methods

A lot of problems in lattice gauge theories do not have an analytic solution. The lattice structure of the theory however lays out a quite straight forward way to implement numerical simulations. Three main areas of these numerical methods contribute to the lattice simulation results in this thesis:

First an ensemble of link fields $U_\mu(n)$ has to be generated for each given combination of temperature $T$ and lattice size $N^3_\sigma \times N_\tau$. A specific link field in one of these ensembles is usually referred to as a lattice configuration. Since the configurations have been prepared and were already used for the analyses by Fröhlich [2], only a short description of the algorithm to obtain them will be given here.

Secondly, for each configuration the fermion matrix $M$ has to be inverted to calculate the quark propagator and the meson correlation functions. The inversion is done by using the conjugate gradient algorithm.

Thirdly, the obtained correlators have to be statistically analysed. Since the configurations are statistically correlated and the signal in some of the meson channels is noisy, the fitting procedure and error analysis of (1.19) to extract the meson masses is quite extensive. After meson masses for each lattice size $N^3_\sigma \times N_\tau$ are found, infinite volume and continuum extrapolations can be carried out.

2.1 Generating the link fields

Fixing only the macroscopic values lattice size $(N_\sigma, N_\tau)$ and temperature $(T)$, an infinite number of configurations with these properties could be found. To choose a finite ensemble for which the fermion matrices will be inverted, configurations with a large contribution to the partition function $Z$ (small action $S_E$) have to be generated; this importance sampling is generally done via a Markov chain algorithm.

All configurations used in this thesis have been generated using the quenched approximation, which means the fermion determinant $\det M$ in (1.6) is set to 1. For the simulated particles the quenched approximation means to ignore the contribution of virtual quark loops. Computationally this leads to a huge speedup in the creation of the configurations while for the meson screening masses discussed in this thesis the error is expected to be below 5%.
The algorithm applied to generate the ensembles used in the thesis is the pseudo heat bath algorithm by Cabibbo and Marinari [7, 8] followed by up to 5 overrelaxation steps [9, 10]. After inverting the fermion matrix for each configuration, the correlation function can be found by averaging over all correlators in an ensemble; since the algorithm performs an importance sampling, the correlators for each configuration can simply be averaged without weighting them with $S_E$. For an error analysis, however, the correlation between different configurations has to be taken into account.

The number of configurations per ensemble used in this thesis is given in table 2.1. The numbers differ for some lattice sizes because between the Wilson results of Fröhlich [2] and the staggered result in this thesis new configurations have been computed. Also the number of Wilson results might differ between this thesis and Fröhlich because more Wilson correlators have been computed in the meantime.

### 2.2 Conjugate gradient matrix inverter

For each configuration the fermion matrix $M$ has to be inverted to obtain the propagator (1.7) necessary to calculate the meson correlator (1.16). This inversion can be done by solving the equation $M\chi = \Phi$, such that the solution $\chi_i = M^{-1}_{ij}\Phi_j$ is the required quark propagator $\chi_i = M^{-1}_{0i}$ if $\Phi$ is a point source $\Phi_j = \delta_{j0}$. The solution of the system of linear equations is obtained by means of the conjugate gradient algorithm [11].

Since this algorithm needs a hermitian, positive definite matrix ($M^\dagger = M$) to operate, the equation to solve is altered to

$$M^\dagger M\chi = M^\dagger \Phi \quad (2.1)$$

Now the solution $\chi$ is found by choosing a start vector $\chi(0)$, setting $d(0) = r(0) = M^\dagger\Phi - A\chi(0)$ where $A = M^\dagger M$ and iterating

$$\chi(i+1) = \chi(i) + \alpha(i)d(i)$$

<table>
<thead>
<tr>
<th>$N_\sigma/N_\tau$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>8</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T/T_C$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>60</td>
<td>59</td>
<td>61</td>
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<td>80</td>
<td>208</td>
<td>190</td>
<td>140</td>
<td>108</td>
</tr>
</tbody>
</table>

Table 2.1: Number of configurations per ensemble
where \( \alpha(i) \), \( d(i) \) and the residual \( r(i) \) are defined as

\[
\alpha(i) = \frac{r^\dagger(i)r(i)}{d^\dagger(i)Ad(i)} \quad r(i+1) = r(i) - \alpha(i)d(i) \\
d(i+1) = r(i+1) + \frac{r^\dagger(i+1)r(i+1)}{r^\dagger(i)r(i)}d(i)
\]

Each iteration step reduces the number of remaining dimensions by one (the iteration steps form a Krylov space) so an exact solution (in exact arithmetic) for an \( N \times N \)-matrix is found after \( N \) steps. However, even for a small lattice \( 16^3 \times 8 \) this would lead to \( > 90000 \) iterations.

Fortunately, reaching an approximate solution for a given norm of the residual requires much less iterations. Evaluating the precision necessary to calculate correlators on lattice with large \( N_\sigma \) (the correlator values around \( N_\sigma/2 = 64 \) reach \( 10^{-14} \)), the residual for all calculations in this thesis was set to \( ||r|| < 10^{-23} \). At the high temperatures investigated this residual can be reached with comparatively few iteration steps, which is due to the fact that high temperatures suppress zero modes in the fermion matrix. The given residual is reached after 100 to 300 iteration steps at \( T/T_c = 3.0 \) and 300 to 600 iteration steps at \( T/T_c = 1.5 \) (tab. 2.3).

For the staggered action only involving next neighbour interactions, the inversion can be speeded up by an even-odd decomposition to reorganize the matrix \( M^\dagger M \). Defining a lattice site \( n \) as even or odd if \( n_1 + n_2 + n_3 + n_4 = \text{even/odd} \), each next neighbour to an even site is odd, so the lattice derivative (1.4) at an even site only involves contributions from odd sites. This can be used to reorder the fermion matrix \( M \), so it can be written as

\[
\begin{pmatrix}
  m & D_{oe} \\
  D_{eo} & m
\end{pmatrix}
\begin{pmatrix}
  \chi_e \\
  \chi_o
\end{pmatrix}
= \begin{pmatrix}
  \Phi_e \\
  \Phi_o
\end{pmatrix}
\]

Equation (2.1) then becomes blockdiagonal and the solution \( \chi_e \) is obtained from

\[
(m^2 - D_{eo}D_{oe})\chi_e = \Phi_e - D_{eo}\Phi_o
\]

and as soon as \( \chi_e \) is known, the solution for the odd sites, \( \chi_o \) can be reconstructed by

\[
m\chi_o + D_{oe}\chi_e = \Phi_o
\]

so half of the computational effort is saved.
Another very advantageous feature of the conjugate gradient algorithm is that it only needs the results of the (matrix × vector) operation, so there is no need to store\(^1\) the full matrix; instead the sparse matrix entries are calculated on the fly from the quark mass and the link fields.

### 2.3 Fitting and error analysis

After the correlators for each meson channel (Table 1.1) have been calculated for a set of configurations, the meson masses can be extracted by fitting (1.19) to the obtained data.

First, since the configurations are statistically correlated, a general form of the jackknife algorithm is used for fitting and error analysis. Given a jackknife block size of \(m\), where \(m\) has to be bigger than the correlation length of two correlators in an ensemble, all correlators \(c_i(z)\) (where the total number \(I\) of correlators should be a multiple of \(m\)) are divided into jackknife subsets (2.2). After testing different block sizes it was found that fairly constant results can be obtained between \(m = 6\) and \(m = 10\) so the jackknife block size was fixed to \(m = 8\) for all analysis in this thesis.

Each subset \(C_n(z)\) contains all correlators but a block of size \(m\), so it can be defined as

\[
C_n(z) = \{c_1(z), \ldots, c_{(n-1)m}(z), c_{(n,m)+1}(z), \ldots, c_I(z)\}
\]

(2.2)

Until here, this is the well known definition of the jackknife algorithm. Now as a second step, for each subset \(C_n(z)\) an average \(\overline{C_n}(z)\) over all correlators in the subset and an according variance \(\sigma_n(z)\) are calculated. Then the ansatz \(G(z)\) (1.19) is fitted against these values; this is done in the usual way by minimising

\[
\sum_{z \in \delta'} \frac{(\overline{C_n}(z) - G(z))^2}{\sigma_n(z)}
\]

using the LEVENBERG-MARQUARD algorithm (the range \(\delta'\) of the fitted \(z\)-values will be determined below). The fit gives two mass values \(m_{n,\delta',\text{oz}}\) and \(m_{n,\delta',\text{no}}\) for each subset \(C_n\). Averaging over these masses and calculating their jackknife error to \(\Delta = \frac{N-1}{N} \sum (m_n - \bar{m})^2\) (where \(N\) is the number of jackknife blocks) finally gives \(m_{\delta',\text{oz}}\) and \(m_{\delta',\text{no}}\).

As a third and last step the range \(\delta'\) can be varied: Since for a finite lattice size the correlator contains exited states (especially at points near to \(z = 0\) and \(z = N_{\sigma}\)), the mass fit is usually not done for all \(N_{\sigma}\) points but restricted to a fit window of for

\(^1\)Storage space (computer memory) for \(\chi\) and the link field \(U\) scales with the number of lattice sites \(N = N_3 \cdot N_{\sigma}\). Storing the matrix \(M\) would scale with \(O(N^2)\) and put the lattice size under a significant constraint.
example \( \delta' \in \left[ \frac{1}{4} N_\sigma, \frac{3}{4} N_\sigma \right] \). This window can be varied to \( \delta' \in \left[ \frac{1}{4} N_\sigma - \delta, \frac{3}{4} N_\sigma + \delta \right] \) where \( \delta \in [-2; 2] \), giving a pair of masses \( m_{\delta, oz}/no \) with jackknife errors \( \Delta m_{\delta, oz}/no \) for each \( \delta \).

The final mass is now calculated by an error-weighted average

\[
m_{oz/no} = \sum_\delta \left( \frac{m_{\delta, oz}/no}{\Delta m_{\delta, oz}/no} \right) \cdot \sum_\delta \Delta m_{\delta, oz}/no
\]

and its error is given by a weighted standard deviation.

Combining all these steps is necessary to obtain masses precise enough for continuum extrapolations in every meson channel. Assembling this method to calculate the meson masses was the major effort for the data analysis in this thesis.

The fitting procedure was used for every staggered meson channel. For result consistency it was also applied (with a corresponding function \( G(z) \)) to the Wilson correlators already calculated by Fröhlich \cite{2}.

### 2.4 The infinite volume and continuum limit

To arrive at a continuum meson screening mass \( m \), the lattice volume \((N_\sigma \cdot a)^3\) has to be taken to infinity and the lattice spacing \( a \) has to be taken to zero. For a given temperature \( T \), a common inverse aspect ratio \( \theta = \frac{N_\tau}{N_\sigma} \) describes lattices of constant physical volume\(^2\), so it is used as a scale for lattice volume plots and the volume extrapolation fits.

Now first an ansatz to fit the extrapolations towards infinite volume has to be found. The analytically known free spatial continuum correlator for pions reads \cite{12}

\[
G_\pi(z) = \frac{3T}{2\pi z^2 \sinh(2\pi T z)} \left[ 1 + 2\pi T z \coth(2\pi T z) \right]
\]

and deriving this term to obtain an effective mass, it is found that the screening mass depends linearly on the inverse spatial separation \( z \) in leading order and the pion mass becomes \( 2\pi \) in the limit \( z \to \infty \), i.e. the infinite volume limit:

\[
m(z) = 2\pi T \left\{ \frac{1}{x} \left[ 2 + x \coth(x) \right] + \frac{1}{1 + x \coth(x)} \left( \frac{x}{\sinh^2(x)} - \coth(x) \right) \right\}
\]

\[
\frac{m(z)}{T} = 2\pi \left( 1 + \frac{1}{x} + ... \right)
\]

Here, with \( x = 2\pi T z \) and by setting \( z \) to \( z = N_\sigma/4 \) this equation can be used to derive the lattice screening masses volume dependence as

\[
\frac{m}{T} = 2\pi \left( 1 + \frac{2}{\pi N_\tau} + ... \right)
\]

\(^2\)Doubling \( N_\tau \) at constant \( T \) halves the spacing \( a \) so \( N_\sigma \) has to be doubled to match the same volume.
The spatial correlation function for free fermions on the lattice can be calculated either by numerically inverting the free case fermion matrix or by semi-analytically calculating the correlation function. The resulting effective masses show the predicted linear volume dependence as soon as a minimal volume is reached (fig. 2.1). Of course they also include a lattice spacing dependence, which for finite lattice spacing lead to infinite volume masses that differ from $2\pi T$. While perturbative calculations [13] show a positive next to leading order correction, so that the masses approach $2\pi$ from above, we will show by means of a systematic analysis of finite volume and finite lattice spacing effects that on the lattice at the temperatures investigated the high temperature limit $2\pi$ is approached from below.

Knowing the linear volume dependence for the free case and taking into account the relation $m \sim 1 + c \cdot (N_v/N_\sigma)^3$, known from numerical simulation to hold approximately at low temperatures [14], it is reasonable to choose the ansatz

$$m_a(\theta) = m_{a,\text{vol}} \cdot (1 + c_a \sigma^b)$$  \hspace{1cm} (2.3)

To arrive at a continuum mass $m_{\text{cont}}$, the lattice spacing dependence of $m_{a,\text{vol}}$ has to be found. In the free case this can be done analytically ($m \sim (1 - \gamma a^2)$) while in the interacting case the order $\mathcal{O}(a^2)$ of the error in the standard staggered and clover-improved Wilson action suggests the same relation (only with an unknown sign):

$$m_{\text{vol}} = m_{\text{cont}} \cdot (1 \pm \lambda a^2)$$

This is also reflected in the spacing dependence of the free case masses that show a clear relation to $a^2$.

This relation can also be used to extrapolate $a \to 0$ at a fixed aspect ratio. Effects that only depend on the lattice spacing and not on the finite volume can be eliminated in
this way without having to calculate a whole set of different aspect ratios. This can for
instance be used to extrapolate Wilson and staggered masses to a common meson mass
at finite volume.

2.5 Computing resources

To obtain the staggered correlators, a parallelized matrix inverter and software to sta-
tistically analyse the correlators had to be written. Also the requirements in terms of
computing resources become quite high for bigger lattice sizes. This section is meant to
give a short overview over the more technical issues of the utilised software and hard-
ware.

Implementation of the matrix inverter

The software for the staggered fermion matrix inverter has been implemented in FOR-
TRAN, the base code is in constant development and was already used to obtain the
Wilson correlators in [2] that are compared to the staggered results in this thesis.

The programm utilises MPI\textsuperscript{3} [21] for parallelisation. The lattice is subdivided into equal
4-dimensional cubes that are distributed onto the MPI nodes, where the distribution
scheme can be set up at program start to match machine and communication network
characteristics; in general cubic local lattices are favourable to minimize communication
overhead. The only restriction imposed on this distribution is that local lattices have to
be of even size along every dimension.

Two major changes have been introduced to the code base: The obtained propagators
are no longer written to disk, but the correlators are directly calculated after the matrix
inversion and the propagators are discarded afterwards. This change of paradigm is due
to the fact that in recent years computing power is rising faster than disk I/O speed and
that I/O operations are still harder to parallelize to improve performance. In light of
this development it is more favourable to recalculate the propagators if needed than to
store them.

Also the remaining input/output operation of reading in the link fields has been paral-
lelized by utilizing MPI file I/O, which gives three advantages:

The first and most obvious is speed. Since the distribution scheme of the link fields is
simple and known at program start, an optimized storage subsystem can easily speed
up the distribution. In fact the read-in process on the JUGENE-System became up to

\textsuperscript{3}Message passing interface, a standard defining a programming interface (for e.g. Fortran and C)
to allow hardware- and implementation-independent parallel programming. MPI enables the same
parallelized code to be run on all standard conforming MPI systems (from small PC-clusters up
to e.g. the JUGENE supercomputer), where the underlying MPI implementation can make use of
system-specific optimizations (special network hardware, shared memory, etc.)
7 times faster, where the factor is rising with the link field size and of course with the number of MPI nodes.

A second advantage is more unique to the software at hand: The serial input module reads in chunks of the link field through the root node and uses standard MPI functions to distribute the fields. These chunks are x-y-planes of the global link field, so the maximum global lattice size is bound by the maximum local lattice size on the root node \( (N^2_{\sigma} \text{max.}}_{\text{global}} \leq (N^3 \times N_{\tau}) \text{max.}_{\text{local}} \); implementing a parallel input/output function is an easy way to overcome this limit.

The third advantage is compatibility: The MPI file access routines allow for a consistent use of defined MPI data types and a standard on disk representation, so no manual conversions are necessary, if files are used on different machines (endians, memory structures, etc.) At the current state however a native machine-dependent on-disk structure is used for backwards compatibility and both input methods are still available for systems without MPI I/O support.

Both improvements were ported back into the Fortran/MPI code used for the Wilson fermion action.

### Computing resource requirements and hardware

Trying to estimate the needed computing resources, memory and storage requirements are easily calculable, while for CPU performance test runs (tab. 2.3) are usually necessary for precise results.

Of course link field storage space scales with the number of sites/links as \( N^3_{\sigma} \times N_{\tau} \), where a site requires 4 links (one per dimension \( \mu \)) and a link consists of a \( 3 \times 3 \) unitary complex matrix so 6 complex values (each taking up 16 byte at double precision) have to be stored; over all, 384 bytes per lattice site are needed to store the link field.

At runtime at least the field variable \( \chi \) (one complex number per color, 48 byte) has to be stored additionally to the link field, which yields 432 bytes per site. For performance reasons\(^4\), also the full link field (with all 9 reconstructed matrix entries) and the pre-computed right hand side term \( M^{\dagger} \Psi \) of the conjugate gradient are stored, which leads to 672 bytes per lattice site. Storage space and memory requirement for different lattice sizes can be found in table 2.2.

\(^4\)Since in current developments the speed of floating point operations improves much more than memory access times it would be interesting to test how reconstructing the link matrices on the fly would influence performance, especially since reducing memory consumption also reduces the number of computing nodes needed and thereby communication overhead.
Table 2.2: Storage/memory requirements (in MB) for different lattice sizes

<table>
<thead>
<tr>
<th>$N^3 \times N^3_\sigma$</th>
<th>$N^3_\sigma \cdot N_\tau$</th>
<th>storage link fields</th>
<th>runtime memory min.</th>
<th>runtime memory max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16^3 \times 16$</td>
<td>65536</td>
<td>24</td>
<td>27</td>
<td>42</td>
</tr>
<tr>
<td>$32^3 \times 16$</td>
<td>524288</td>
<td>192</td>
<td>216</td>
<td>336</td>
</tr>
<tr>
<td>$48^3 \times 16$</td>
<td>1769472</td>
<td>648</td>
<td>729</td>
<td>1134</td>
</tr>
<tr>
<td>$64^3 \times 16$</td>
<td>4194304</td>
<td>1536</td>
<td>1728</td>
<td>2688</td>
</tr>
<tr>
<td>$96^3 \times 16$</td>
<td>14155776</td>
<td>5184</td>
<td>5832</td>
<td>9072</td>
</tr>
<tr>
<td>$128^3 \times 16$</td>
<td>33554432</td>
<td>12288</td>
<td>13824</td>
<td>21504</td>
</tr>
</tbody>
</table>

The Jugene-system

While development and test runs with small lattices are possible on every current desktop machine and test runs for the implementation of MPI-I/O were performed on a local 16-node x86-cluster, all runs to obtain the correlators actually used in this thesis were done on the Jugene-system in Jülich.

The Jugene-system (fig. 2.2) is currently the biggest installation of a BlueGene/P-series supercomputer [15] built by IBM. In 2009 it was extended from 16 to 72 racks and is currently the fastest computing cluster in Europe benchmarked with the Linpack test [16]. Hardware and software, e.g. the operating system and the compilers, as well as the communication network are optimized to be used with massively parallel MPI programs.

The smallest unit of a BlueGene/P-system is a computing node equipped with a quad-core PowerPC-450 CPU at a clock rate of 850 MHz and 2 GB of memory (fig. 2.3). While the CPU is not as fast as a current desktop CPU, it features a very low power consumption that leads to low waste heat and the physical size of a computation node itself is very small. The CPUs run a minimalistic POSIX-compliant single user operating system that can be reloaded at the start of each new computation job. The computing nodes are mounted on so-called node-cards, where 32 computing nodes make up the smallest unit of the system that is user-accessible for test runs. Test runs are started very quickly, usually within half an hour, but are limited to a wall clock time of 30 minutes.

Midplanes are groups of 16 node cards, so they feature a total of 512 CPUs with 2048 cores and 1 TB system memory. The midplane structure is important for the communication network layout: On a single midplane, the nodes communicate through a very fast mesh network, where the network structure cannot be influenced by the user. However, if a job is to be run on more than one midplane, the network layout between the midplanes can be set when the job is scheduled. Especially if low network latency is important a matching network layout should be chosen for performance reasons. A midplane is also the smallest unit that is user-accessible for productive runs, where productive runs are computing jobs that need a wall clock time between 30 minutes and 12 hours.
Figure 2.2: The JUGENE system at the John-von-Neumann computing center in Jülich

Figure 2.3: A BLUEGENE/P computing node (left) and computing node card (right)
Two midplanes make up a rack, which is an actual cabinet in the computing center. Each rack also includes the cooling and power supply systems and network hardware to interconnect the midplanes. Computing time on the Jugene-system is accounted in rack-days, where the wall clock time of each computing job is multiplied with the number of racks it uses.

Since CPU and memory requirements for this simulation are not very high in comparison to the capabilities of the Jugene-system, only a midplane was necessary to perform the calculations and only a combined wall clock time of one rack-day was needed. Average wall clock times for reading in and inverting configurations can be found in table 2.3.

It would also be possible to perform these calculations on a smaller cluster (as long as it features a fast MPI communication network such as Infiniband), but all ensembles would have to be copied from the storage subsystem in Jülich; in addition, estimates for a 16-node PC cluster like the system in Bielefeld show computing times of roughly two weeks for the big ensembles.

**Statistical analysis**

To accomplish the statistical analysis of the correlators, first shell scripts to extract the data and gnuplot to carry out the fits were used. However, as performing correlator fits within the jackknife subset proved to yield good results and the need to average over different fit windows became evident, a self-contained C-programm combining all analysis steps for each ensemble was written. This lead to a significant speedup, since between 30 and 200 fits to up to 100 data points had to be performed per ensemble.

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Lattice size</th>
<th>read in time</th>
<th>invert $T/T_c = 1.5$ time</th>
<th>invert $T/T_c = 3.0$ time</th>
<th>steps</th>
<th>steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>$32^3 \times 8$</td>
<td>3.1 sec</td>
<td>3.8 sec</td>
<td>2.1 sec</td>
<td>309</td>
<td>169</td>
</tr>
<tr>
<td>128</td>
<td>$64^3 \times 8$</td>
<td>5.8 sec</td>
<td>18.6 sec</td>
<td>7.5 sec</td>
<td>459</td>
<td>184</td>
</tr>
<tr>
<td>512</td>
<td>$64^3 \times 16$</td>
<td>3.0 sec</td>
<td>10.7 sec</td>
<td>6.7 sec</td>
<td>478</td>
<td>300</td>
</tr>
<tr>
<td>512</td>
<td>$128^3 \times 16$</td>
<td>13.5 sec</td>
<td>69.0 sec</td>
<td>36.4 sec</td>
<td>591</td>
<td>311</td>
</tr>
</tbody>
</table>

Table 2.3: Approximate wall clock times and average conjugate gradient steps needed to read in and invert a configuration on the Jugene system.
3 Results

This chapter holds the results for infinite volume and continuum extrapolations as well as lattice spacing extrapolations for the pseudoscalar, scalar, vector and axialvector meson screening masses.

Since to obtain the screening masses all parameters of the fermion matrix have to be set to perform the inversion, an overview of this parameters and how they were set is given first.

3.1 Quark mass and lattice parameters

Aside from the link fields used for Wilson as well as staggered fermions, further parameters have to be fixed to define the full fermion matrix. For the Wilson fermions this is the hopping parameter $\kappa$; it has been set close to the zero temperature critical value, details can be found in [17]. The values for $\kappa$ as well as $C_{SW}$ are given in table 3.2.

For staggered fermions, the lattice quark mass $\hat{m}$ has to be fixed. Since the lattice theory is dimensionless and physical units are merely defined through the lattice spacing $a$, a quark mass given in MeV has to be converted to a lattice quark mass $\hat{m} = m_q \cdot a$, depending on the spacing $a$.

Using the matching constant between the lattice and the $\overline{MS}$ scheme given by Kilcup, Gupta and Sharpe [18]

$$Z^U_P = 1 + \frac{\alpha_{MS}(q^*)}{4\pi} \left( -\gamma_P^{(0)} \ln(q^*a) + c_P^U \right) + O(a^2)$$

and values for $g_{MS}^2(1/a)$ by Stickan [17] the ratio between $m_{MS} = m_{MS}(2 \text{ GeV})$ and $\hat{m}$ can be found for each lattice spacing $a$ as

$$\frac{m_{MS}}{\hat{m}} = \left( \frac{g_{MS}^2(2 \text{ GeV})}{g_{MS}^2(1/a)} \right)^{\frac{1}{4\pi}} \left[ 1 - \frac{g_{MS}^2(1/a)}{16\pi^2} \cdot 39.1414 \right] \cdot 200 \text{ MeV} \quad (3.1)$$

The scale $\mu = 2 \text{ GeV}$ is the traditional reference scale where quark masses are quoted. The running coupling has been promoted from $\mu = \frac{1}{a}$ to this scale by two-loop perturbation theory. The resulting factors to convert between $m_{MS}$ and $\hat{m}$ are collected in
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{m} [\text{GeV}]$</td>
<td>1.370</td>
<td>0.899</td>
<td>0.587</td>
<td>0.438</td>
<td>0.342</td>
<td>0.241</td>
<td>0.144</td>
<td>0.095</td>
</tr>
<tr>
<td>$m_{\overline{MS}} \cdot a/\hat{m}$</td>
<td>0.758</td>
<td>0.923</td>
<td>1.038</td>
<td>1.118</td>
<td>1.197</td>
<td>1.285</td>
<td>1.457</td>
<td>1.563</td>
</tr>
</tbody>
</table>

Table 3.1: $m_{\overline{MS}}$ to $\hat{m}$ at different $\beta$

Figure 3.1: Quark mass ($m_{\overline{MS}}$) dependence of the pseudoscalar screening mass

table 3.1. The last row of this table gives the factor between a lattice quark mass calculated by fixing $m_{\overline{MS}} (2 \text{ GeV})$ and one which is calculated by fixing $m_q$ and calculating $\hat{m}_q = m_q \cdot a$.

The relation between $m_{\overline{MS}}$ and $\hat{m}$ was intended to be used to calculate $\hat{m}$ at the different lattice parameters at a fixed $m_{\overline{MS}}$ of 10 MeV. Unfortunately, through an error in the calculation of the right-hand-side of (3.1), slightly wrong quark masses $\hat{m}$ were used in the simulations. The (corrected) $m_{\overline{MS}}$ values corresponding to the masses $\hat{m}$ actually used are given in the last column of table 3.2, which also summarises the lattice parameters $N_\tau$, $N_\sigma$, $T/T_C$ with the corresponding values for $\beta$, lattice spacing ($a$) and Wilson hopping parameter ($\kappa$) for later reference.

It is now of some interest how the meson screening masses depend on the quark mass $m_{\overline{MS}}$. For this purpose the screening mass of the pseudoscalar pion has been obtained for different quark masses $m_{\overline{MS}}$ in the range between 0.5 MeV and 40 MeV (Figure 3.1) at different lattice sizes.

At $T/T_c = 1.5$ quark masses above 30 MeV lead to a small but noticeable increase in the pion screening mass for all lattice sizes. Quark masses below 30 MeV only influence the pion mass on the $N_\tau^3 \times N_\sigma = 64^3 \times 8$ lattice, where a large lattice spacing and a large volume are given. Under these conditions the pion screening mass shows a small but clear linear dependence on $m_q$ (the bended line in the logarithmic plot is a linear fit). At $T/T_c = 3.0$ no quark mass dependence can be found for any tested lattice size.
In light of the nearly absent quark mass dependence in the pseudoscalar channel, quark masses in the range up to 15 MeV should not lead to differences in the meson screening masses. Fixing $m_{\text{MS}}$ could however prove useful in calculations more sensitive to the quark mass.

### 3.2 The staggered meson channels

For this thesis, screening masses for four different mesons, the scalar meson $a_0$, the pseudoscalar meson $\pi$, the vector meson $\rho$, and the axialvector meson $a_1$ were compared. For the Wilson action, where there is no flavour mixing, each meson correlator can be found by setting the corresponding $\Gamma$-matrices in (1.18).

For the staggered action, contributions from the scalar meson are only found in one of the meson channels, $M_1$ (tab. 1.1). The pseudoscalar meson is found as the only particle in channel $M_2$. It could also give a non-oscillating contribution to the channel $M_1$, but no such contribution is found as soon as a certain lattice volume is reached.

For the staggered vector and axialvector mesons the situation is a bit more complex: The axialvector meson $a_1$ contributes to the channels $M_3$, $M_4$, and $M_5$, where the contributions to $M_3$ and $M_4$ are the transversal $a_{1,T}$ which is studied in this thesis. Since the axialvector channels are a lot more noisy than the scalar or pseudoscalar channel, the correlators from both channels are combined to effectively double the number of samples. This is a first step in the statistical analyses, necessary to reduce the statistical errors when extracting the masses.

In the channels $M_3$ and $M_4$ also a non-oscillating contribution from the $\rho_{2,T}$ is found, but with rising lattice volume, the masses for this contribution do not fall off towards or below $2\pi$ and the amplitudes get lower. The errors for the $\rho_2$ masses quickly become too high (even larger than the masses themselves), so no extrapolations for the $\rho_2$ can be done.
The vector meson is found as the non-oscillating contribution in the channels $M_6$, $M_7$, and $M_8$, where the transversal $\rho_{1,T}$ is found in the channels $M_6$ and $M_7$. Also here the correlators of $M_6$ and $M_7$ are combined and the second contribution, the oscillating $b_1$, cannot be fitted with errors low enough to allow extrapolations.

### 3.3 The pseudoscalar meson $\pi$

The pseudoscalar meson is the most clear channel for both Wilson and staggered fermions. There is no physical particle to contribute an oscillating component to the staggered correlator and in fact the oscillating amplitude is fitted to 0 for all staggered correlators. This very clear signal leads to only minimal errors so the extrapolations work without any problems.

![Figure 3.2: Volume extrapolation of pseudoscalar screening masses at $T/T_c = 1.5$](image)

![Figure 3.3: Volume extrapolation of pseudoscalar screening masses at $T/T_c = 3.0$](image)

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**Volume extrapolations**

All pseudoscalar meson masses show a clear volume dependence. Fitting (2.3) to the obtained masses it is found that the ansatz to choose the same exponent for all three values of $N_{\tau}$ works well; as suspected the exponential component does not strongly depend on the lattice spacing. More surprisingly, the exponent is also the same for Wilson and staggered fermions, so differences in the volume behaviour result either from the prefactors or the infinite volume mass; for the prefactors, however, only values with large errors can be obtained from the fits. The volume extrapolation fits converge very well for both Wilson and staggered fermions at both temperatures. Even though the error margins for the masses, especially at larger volumes, are very small, the fitted curves intersect all data points in or near their error margins.

Comparing the masses at the biggest volume $N_{\sigma}/N_{\tau} = 8$ to the infinite volume extrapolation, Wilson and staggered fermions show the same behaviour: At $T/T_c = 1.5$ (fig. 3.2) the masses at $N_{\sigma}/N_{\tau} = 8$ are within the error margin of the extrapolated masses, the volume dependence for aspect ratios below $1/4$ is very shallow. At $T/T_c = 3.0$ (fig. 3.3) the volume dependence is more pronounced and the masses at $N_{\sigma}/N_{\tau} = 8$ are not much but noticeably higher than the extrapolated masses, the volume dependence is steeper. This is also reflected in the strong change of the exponent from about 3.1 at $T/T_c = 1.5$ to about 2.2 $T/T_c = 3.0$. All fit results are summarized in table 3.3.

Comparing Wilson and staggered volume behaviour, two differences are evident: The first and most pronounced property is that at both temperatures all Wilson masses (at finite volume as well as in the volume extrapolations) are higher than the staggered masses. This difference is again small but given the good error margins clearly noticeable; the gap reduces with raising lattice volume and with falling lattice spacing, in the infinite volume extrapolation and at $N_{\tau} = 16$ the masses are almost equal within margins of error. As a second point, the Wilson masses fall off a bit steeper at $T/T_c = 1.5$ than the staggered masses while there is no such difference at $T/T_c = 3.0$.

**Lattice spacing extrapolations**

After the infinite volume is reached, extrapolations in the lattice spacing, $a \to 0$, are carried out to reach the continuum. Also extrapolations in the lattice spacing at finite volume are performed.

At a first glance (fig. 3.2 and 3.3) the staggered pseudoscalar masses show a clear lattice spacing dependence, proportional to $a^2$ at infinite volume as well as at all aspect ratios below $N_{\sigma}/N_{\sigma} < 1/2$. This holds for both $T/T_c = 1.5$ and $T/T_c = 3.0$. The Wilson masses show no such clear $a$-dependence at $T/T_c = 1.5$, at $T/T_c = 3.0$ the masses for the large spacing $N_{\tau} = 8$ deviates from the masses at $N_{\tau} = 12$ and $N_{\tau} = 16$, but there is no clear proportion to $a^2$. 

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Extrapolating in the lattice spacing at infinite volume the missing $a^2$-dependence of the Wilson masses at $T/T_c = 1.5$ leads to continuum masses lower than for the staggered fermions, though every single Wilson mass is heavier than the corresponding staggered mass (fig. 3.4). Carrying out the $a^2$-extrapolation at finite volume staggered and Wilson masses become equal at $N_{\tau}/N_{\sigma} = 1/3$ and $N_{\tau}/N_{\sigma} = 1/4$, but already diverge at aspect ratios $N_{\tau}/N_{\sigma} = 1/8$ where the masses are near to the infinite volume limit. Generally, the $a^2$-dependence of the Wilson fermions is weaker than for the staggered fermions.

At the higher temperature $T/T_c = 3.0$, staggered and Wilson masses become equal in the continuum extrapolation as well as in lattice spacing extrapolations at finite volume. But also here, the slope of the $a^2$ dependence is steeper for the staggered fermions. At $N_{\tau}/N_{\sigma} = 1/4$, the masses extrapolated in $a^2$ reach the analytic high temperature and infinite volume limit of $2\pi$, but at other aspect ratios the limit is not reached (Figures 3.4).

All lattice spacing extrapolations yield very low errors. At higher aspect ratios and in the continuum extrapolation the statistical errors are so low that even rounding errors in the fitting process have to be taken into account. The errors given in tables 3.3 and 3.4 are, however, only the statistical errors taken from the fitting results.
Figure 3.4: Lattice spacing dependence of the pseudoscalar screening masses at different aspect ratios

Table 3.4: Results of the $a \rightarrow 0$ extrapolation for pseudoscalar masses at $N_T/N_\sigma = 1/3, 1/4, 1/8$ and at infinite volume.
The signal for the scalar meson $a_0$ is also very clear. Though the pseudoscalar pion could contribute a non-oscillating part to the staggered correlator, neither amplitude nor mass for the non-oscillating part can be fitted within reasonable errors.

The extrapolated scalar and pseudoscalar masses are degenerated, the masses are nearly the same at every lattice size $N^3 \sigma \times N^\tau$, (Table 3.5). For staggered fermions this degeneracy is very exact: Already at $T/T_c = 1.5$ and of course also at $T/T_c = 3.0$ the difference is within 50% of the statistical errors, so scalar and pseudoscalar meson can be considered as equal. For Wilson fermions the degeneracy does not fall in the error margins at some of the $N^\sigma \gg N^\tau$ aspect ratios at $T/T_c = 1.5$, but they become equal within the margin of error in the infinite volume and lattice spacing extrapolations.

Plots for Wilson and staggered infinite volume extrapolation (fig. 3.5 and 3.6) as well as a table (3.6) with lattice spacing extrapolation results are given. Of course the interpretation of these values is the same as for the pseudoscalar masses, so it is not repeated here.

There are a few exceptions: Some correlators have a very low statistical error that is too small to hold the mass difference. However, a statistical error below 0.1% is not longer significant because it is even lower than the accumulated numerical errors in the fitting process.
Table 3.6: Results of the $a \to 0$ extrapolation for scalar masses at $N_r/N_\sigma = 1/3, 1/4, 1/8$ and at infinite volume.

<table>
<thead>
<tr>
<th></th>
<th>$T/T_c = 1.5$</th>
<th></th>
<th>$T/T_c = 3.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_r/N_\sigma = 1/3$</td>
<td>6.000 ± 0.001</td>
<td>5.991 ± 0.005</td>
<td>6.475 ± 0.064</td>
</tr>
<tr>
<td>$N_r/N_\sigma = 1/4$</td>
<td>5.878 ± 0.047</td>
<td>5.811 ± 0.043</td>
<td>6.299 ± 0.015</td>
</tr>
<tr>
<td>$N_r/N_\sigma = 1/8$</td>
<td>5.709 ± 0.018</td>
<td>5.772 ± 0.001</td>
<td>6.135 ± 0.017</td>
</tr>
<tr>
<td>$N_r/N_\sigma \to 0$</td>
<td>5.680 ± 0.013</td>
<td>5.761 ± 0.001</td>
<td>6.083 ± 0.016</td>
</tr>
</tbody>
</table>

Figure 3.5: Volume extrapolation of scalar screening masses at $T/T_c = 1.5$

Figure 3.6: Volume extrapolation of scalar screening masses at $T/T_c = 3.0$
Screening masses for both staggered and Wilson vector mesons are found, as for the scalar and pseudoscalar mesons, by fitting (2.1) to the obtained correlators, where the number of correlators is doubled for the staggered action by combining the channels $M_6$ and $M_7$. Infinite volume extrapolations for both actions are, however, only possible through varying the fit window ($\frac{N_\sigma}{4} \pm 2$) This procedure, now used for all meson correlators in this thesis, was first applied to extract vector masses smooth enough to allow the infinite volume extrapolation fit to converge.

**Volume extrapolations**

As the pseudoscalar/scalar screening masses, the vector meson masses show a clear volume dependence at both temperatures for both actions. Again (2.3) is fitted against the obtained masses with a fixed exponent $b$ for all lattice spacings at one temperature. The fits converge very well for both actions at both temperatures, table 3.7 summarises the results for all fitted parameters, figures 3.7 ($T/T_c = 1.5$) and 3.8 ($T/T_c = 3.0$) show the graphs.

The exponent $b$ changes from about 3 at $T/T_c = 1.5$ to about 2 at $T/T_c = 3.0$ for both actions. They are equal within the margin of error to the exponents found for the pseudoscalar/scalar mesons, but the error margins are higher.

The errors for the prefactors $c_a$ are lower than in the pseudoscalar channel; a clear dependence on the lattice spacing where the prefactors become smaller with smaller spacing.
and on the temperature where the prefactors become smaller with rising temperature can be found. Within the error margins the prefactors are equal for both actions, the large errors of the pseudoscalar prefactors do not allow for a significant comparison.

Comparing, as for the pseudoscalar, the masses at the biggest available volume ($N_N/N_T = 8$) to the infinite volume extrapolation, the same behaviour as for the pseudoscalar is found: At $T/T_c = 1.5$ the infinite volume masses are nearly equal to the extrapolated masses, at $T/T_c = 3.0$ the extrapolation leads to masses slightly lower than at the largest volume. This can be attributed to the change in the exponent $b$ and, in contrast to the pseudoscalar (where the error margins did not allow an assessment), to the change in the prefactors $c_a$ from about 1.2 at $T/T_c = 1.5$ to about 0.75 at $T/T_c = 3.0$.

Comparing Wilson and staggered volume extrapolations it is found that at $T/T_c = 3.0$ the Wilson masses are slightly higher than the staggerd masses. At $T/T_c = 1.5$ the
Wilson masses are heavier for $N_\tau = 8$ but lighter for $N_\tau = 16$, so a closer look at the lattice spacing dependence is necessary.

**Lattice spacing extrapolations**

As for the pseudoscalar channel, extrapolations in the lattice spacing at infinite volume and at constant aspect ratios are carried out. All results are summarised in table 3.8.

The continuum extrapolation at infinite volume (fig. 3.9) shows a clear lattice spacing dependence proportional to $a^2$ for the staggered vector meson and for the Wilson vector meson at $T/T_c = 3.0$ and both masses become equal in the extrapolation. Again, the spacing dependence is a bit weaker for the Wilson action. At $T/T_c = 1.5$, however, no lattice spacing dependence for the Wilson vector meson is found and in fact the Wilson mass does not match the staggered mass in this case.

Trying to extrapolate in lattice spacing at a fixed aspect ratio (fig. 3.9) gives results which vary more than in the pseudoscalar case: At $N_\tau/N_\sigma = 1/8$ the results are quite similar to the ones obtained from the volume extrapolation though the resulting masses are a bit higher since the volume dependence is not as flat as in the pseudoscalar channel.

At $N_\tau/N_\sigma = 1/4$ only the masses at $T/T_c = 3.0$ show a clear lattice spacing dependence, for $T/T_c = 1.5$ the spacing dependence for the staggered masses is lost; however, since there is no spacing dependence for any Wilson vector meson mass at $T/T_c = 1.5$, now the Wilson and staggered masses are equal within their margin of error at the three different lattice spacings. Lattice spacing extrapolations at $N_\tau/N_\sigma = 1/4$ can be carried out for the pseudoscalar channel (though the results are higher than in the continuum), they do, however, not work for the vector channel.

At $N_\tau/N_\sigma = 1/3$ the lattice spacing dependence is also not longer proportional to $a^2$ for both Wilson and staggered masses at $T/T_c = 3.0$. The masses are equal at each spacing within their margin of error, but these error margins are not as low as for the extrapolated masses.

Also, while lattice spacing extrapolations for the pseudoscalar masses at $N_\tau/N_\sigma = 1/8$ yield results comparable to the continuum extrapolation at infinite volume, the vector meson masses still show a difference between the spacing extrapolations done at $N_\tau/N_\sigma = 1/8$ and at infinite volume.

The errors for the extrapolations done at higher volumes and in the infinite volume limit are again very low, if not as low as for the pseudoscalar masses.

In comparison to the pseudoscalar channel, the continuum extrapolations lead to vector meson screening masses higher than the pseudoscalar masses at both temperatures, so the pseudoscalar and the vector meson mass are clearly not yet degenerated at $T/T_c = 3.0$. 

50
Figure 3.9: Lattice spacing dependence of the vector meson screening masses at different aspect ratios

<table>
<thead>
<tr>
<th>( T/T_c ) = 1.5</th>
<th>( T/T_c ) = 3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_f/N_\sigma = 1/3 )</td>
<td>6.326 ± 0.020</td>
</tr>
<tr>
<td>( N_f/N_\sigma = 1/4 )</td>
<td>6.169 ± 0.095</td>
</tr>
<tr>
<td>( N_f/N_\sigma = 1/8 )</td>
<td>6.011 ± 0.019</td>
</tr>
<tr>
<td>( N_f/N_\sigma \to 0 )</td>
<td>6.006 ± 0.024</td>
</tr>
</tbody>
</table>

Table 3.8: Results of the \( a \to 0 \) extrapolation for vector masses at \( N_f/N_\sigma = 1/3, 1/4, 1/8 \) and at infinite volume.
<table>
<thead>
<tr>
<th>$N_r/N_\sigma$</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
<th>1/8</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
<th>1/8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{EE}$ $N_x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.210</td>
<td>0.033</td>
<td>0.010</td>
<td>0.002</td>
<td>0.002</td>
<td>0.000</td>
<td>0.001</td>
<td>0.011</td>
</tr>
<tr>
<td>1.5</td>
<td>0.048</td>
<td>0.003</td>
<td>0.001</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>16</td>
<td>-0.013</td>
<td>0.006</td>
<td>0.003</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>08</td>
<td>0.164</td>
<td>0.032</td>
<td>0.005</td>
<td>0.049</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>3.0</td>
<td>0.027</td>
<td>0.004</td>
<td>0.020</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>16</td>
<td>0.021</td>
<td>0.002</td>
<td>0.041</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>08</td>
<td>0.583</td>
<td>0.255</td>
<td>0.217</td>
<td>0.123</td>
<td>0.007</td>
<td>0.002</td>
<td>0.022</td>
<td>0.572</td>
</tr>
<tr>
<td>1.5</td>
<td>0.225</td>
<td>0.058</td>
<td>0.076</td>
<td>0.213</td>
<td>0.001</td>
<td>0.004</td>
<td>0.013</td>
<td>0.072</td>
</tr>
<tr>
<td>16</td>
<td>0.122</td>
<td>0.221</td>
<td>0.139</td>
<td>0.468</td>
<td>0.002</td>
<td>0.005</td>
<td>0.010</td>
<td>0.060</td>
</tr>
<tr>
<td>08</td>
<td>0.585</td>
<td>0.349</td>
<td>0.118</td>
<td>3.535</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
</tr>
<tr>
<td>3.0</td>
<td>0.164</td>
<td>0.093</td>
<td>0.527</td>
<td>0.158</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>16</td>
<td>0.193</td>
<td>0.084</td>
<td>2.003</td>
<td>0.018</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 3.9: Differences between vector and axialvector masses and relation of the difference to the averaged error margin $\Delta m$ of both masses

3.6 The axialvector meson $a_1$

As the scalar and pseudoscalar screening masses, the vector and axialvector masses are clearly degenerated. Table 3.9 shows the absolute mass differences and the relation of this differences to the error margin. Since masses for all finite volumes as well as consequently the extrapolated masses are equal for the vector and axialvector mesons, the discussion for vector mesons also applies for the axialvector mesons and is not repeated here.
4 Conclusion

In this thesis infinite volume i.e. thermodynamic and continuum limits of Wilson and staggered meson screening masses were carried out and compared. Obtaining masses at finite volume and spacing precise enough to perform these extrapolations proved difficult, especially for the staggered action. After substantial statistical analysis to reduce errors however very precise extrapolations could be done. Summarising the results, a lot of noticeable characteristics were found:

The pseudoscalar and scalar meson masses are degenerate for almost every finite volume and in every extrapolation. This holds for both temperatures $T/T_c = 1.5$ and $T/T_c = 3.0$ as well as for Wilson and staggered fermions. The same effect can be found for the axialvector and vector meson, which are also degenerate at almost all finite volumes, in all extrapolations, and at both temperatures.

Comparing Wilson and staggered masses for the scalar/pseudoscalar meson, it is found, that at $T/T_c = 3.0$ the masses become equal in the continuum extrapolation. Also extrapolations in the lattice spacing at fixed aspect ratios of $N_σ/N_τ = 4$ and $N_σ/N_τ = 8$ lead to the same masses and the masses of the large lattice $128^3 \times 16$ are equal. The same relation can be found for the vector/axialvector meson screening masses, which also become equal in the continuum, in lattice spacing extrapolations at fixed aspect ratio, and on the $128^3 \times 16$ lattice. For the lower temperature of $T/T_c = 1.5$ the results do not clearly show equal masses for the Wilson and the staggered action.

Comparing the slopes of the $a^2$-extrapolations, it is found that the lattice spacing dependence is noticeable weaker for the Wilson mesons than for the staggered mesons. This generally holds for the axialvector/vector mesons and the scalar/pseudoscalar mesons, where the overall spacing dependence as well as the differences between Wilson and staggered action are higher for the scalar/pseudoscalar mesons. Taking a look at both temperatures, especially the nearly absent spacing dependence of the Wilson fermions at $T/T_c = 1.5$, lead to continuum masses that differ from the staggered continuum masses; here further investigations could prove useful.

Comparing the scalar/pseudoscalar screening masses to the axialvector/vector masses, it is found that they are clearly not degenerated at both $T/T_c = 1.5$ and $T/T_c = 3.0$. In all extrapolations the axialvector/vector mesons are heavier than the pseudoscalar/scalar mesons; in the lattice spacing extrapolation both, the vector and the scalar meson masses, approach the free continuum mass of $2\pi$ from below, where as mentioned above the slope is steeper for the pseudoscalar/scalar channel.
Over all, a comprehensive comparison between the thermodynamic and continuum limits of Wilson and staggered screening masses has been performed for this thesis. The results give a certain insight which differences between Wilson and staggered calculations can be related to different systematic errors of both actions. It is of importance, that in the continuum extrapolation the Wilson and the staggered action yield the same results; if this was not the case, the actions would feature fundamental differences that would have to be investigated.

However, since the continuum extrapolation show the meson screening masses to become equal, the results allow for an assessment of the lattice volumes and spacing (and in turn of course the numerical/computational effort) needed, if consistent results were to be obtained for the Wilson and the staggered action.
## A Tables of screening masses

### Screening masses of the scalar meson $a_0$ (stagg. M1)

<table>
<thead>
<tr>
<th>$N_T$</th>
<th>$N_g$</th>
<th>Wilson $T/T_c = 1.5$</th>
<th>Wilson $T/T_c = 3.0$</th>
<th>stagg. $T/T_c = 1.5$</th>
<th>stagg. $T/T_c = 3.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>6.581 ± 0.290</td>
<td>7.054 ± 0.217</td>
<td>5.992 ± 0.133</td>
<td>6.757 ± 0.146</td>
</tr>
<tr>
<td>8</td>
<td>24</td>
<td>5.946 ± 0.092</td>
<td>6.461 ± 0.073</td>
<td>5.594 ± 0.061</td>
<td>6.229 ± 0.067</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>5.758 ± 0.023</td>
<td>6.184 ± 0.031</td>
<td>5.451 ± 0.025</td>
<td>5.969 ± 0.023</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>5.691 ± 0.009</td>
<td>5.957 ± 0.002</td>
<td>5.330 ± 0.006</td>
<td>5.772 ± 0.004</td>
</tr>
<tr>
<td>12</td>
<td>24</td>
<td>6.469 ± 0.158</td>
<td>6.964 ± 0.124</td>
<td>6.373 ± 0.165</td>
<td>7.005 ± 0.153</td>
</tr>
<tr>
<td>12</td>
<td>36</td>
<td>5.977 ± 0.050</td>
<td>6.370 ± 0.047</td>
<td>5.821 ± 0.036</td>
<td>6.272 ± 0.039</td>
</tr>
<tr>
<td>12</td>
<td>48</td>
<td>5.842 ± 0.008</td>
<td>6.276 ± 0.024</td>
<td>5.681 ± 0.006</td>
<td>6.139 ± 0.011</td>
</tr>
<tr>
<td>12</td>
<td>96</td>
<td>5.710 ± 0.002</td>
<td>6.067 ± 0.000</td>
<td>5.576 ± 0.005</td>
<td>5.984 ± 0.004</td>
</tr>
<tr>
<td>16</td>
<td>32</td>
<td>6.551 ± 0.082</td>
<td>7.003 ± 0.087</td>
<td>6.450 ± 0.089</td>
<td>6.967 ± 0.122</td>
</tr>
<tr>
<td>16</td>
<td>48</td>
<td>5.987 ± 0.031</td>
<td>6.469 ± 0.019</td>
<td>5.891 ± 0.022</td>
<td>6.421 ± 0.023</td>
</tr>
<tr>
<td>16</td>
<td>64</td>
<td>5.813 ± 0.020</td>
<td>6.271 ± 0.007</td>
<td>5.715 ± 0.009</td>
<td>6.201 ± 0.013</td>
</tr>
<tr>
<td>16</td>
<td>128</td>
<td>5.698 ± 0.004</td>
<td>6.089 ± 0.001</td>
<td>5.662 ± 0.001</td>
<td>6.049 ± 0.003</td>
</tr>
</tbody>
</table>

### Screening masses of the pseudoscalar meson $\pi$ (stagg. M2)

<table>
<thead>
<tr>
<th>$N_T$</th>
<th>$N_g$</th>
<th>Wilson $T/T_c = 1.5$</th>
<th>Wilson $T/T_c = 3.0$</th>
<th>stagg. $T/T_c = 1.5$</th>
<th>stagg. $T/T_c = 3.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>6.705 ± 0.378</td>
<td>7.161 ± 0.283</td>
<td>5.975 ± 0.133</td>
<td>6.705 ± 0.165</td>
</tr>
<tr>
<td>8</td>
<td>24</td>
<td>5.878 ± 0.102</td>
<td>6.467 ± 0.083</td>
<td>5.590 ± 0.061</td>
<td>6.229 ± 0.067</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>5.698 ± 0.029</td>
<td>6.180 ± 0.033</td>
<td>5.449 ± 0.025</td>
<td>5.969 ± 0.023</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>5.662 ± 0.016</td>
<td>5.957 ± 0.004</td>
<td>5.311 ± 0.006</td>
<td>5.772 ± 0.004</td>
</tr>
<tr>
<td>12</td>
<td>24</td>
<td>6.501 ± 0.178</td>
<td>6.983 ± 0.136</td>
<td>6.371 ± 0.147</td>
<td>7.005 ± 0.153</td>
</tr>
<tr>
<td>12</td>
<td>36</td>
<td>5.854 ± 0.062</td>
<td>6.372 ± 0.049</td>
<td>5.818 ± 0.036</td>
<td>6.272 ± 0.039</td>
</tr>
<tr>
<td>12</td>
<td>48</td>
<td>5.837 ± 0.013</td>
<td>6.273 ± 0.024</td>
<td>5.687 ± 0.017</td>
<td>6.139 ± 0.011</td>
</tr>
<tr>
<td>12</td>
<td>96</td>
<td>5.704 ± 0.002</td>
<td>6.063 ± 0.000</td>
<td>5.574 ± 0.004</td>
<td>5.984 ± 0.004</td>
</tr>
<tr>
<td>16</td>
<td>32</td>
<td>6.557 ± 0.084</td>
<td>7.018 ± 0.090</td>
<td>6.450 ± 0.089</td>
<td>6.982 ± 0.114</td>
</tr>
<tr>
<td>16</td>
<td>48</td>
<td>5.983 ± 0.032</td>
<td>6.469 ± 0.019</td>
<td>5.885 ± 0.025</td>
<td>6.422 ± 0.024</td>
</tr>
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<td>5.709 ± 0.016</td>
<td>6.258 ± 0.009</td>
<td>5.711 ± 0.009</td>
<td>6.206 ± 0.009</td>
</tr>
<tr>
<td>16</td>
<td>128</td>
<td>5.693 ± 0.002</td>
<td>6.088 ± 0.001</td>
<td>5.662 ± 0.001</td>
<td>6.049 ± 0.003</td>
</tr>
</tbody>
</table>
Screening masses of the vector meson $\rho$ (stagg. M6+M7)

<table>
<thead>
<tr>
<th>$N_\tau$</th>
<th>$N_\sigma$</th>
<th>Wilson $T/T_c = 1.5$</th>
<th>Wilson $T/T_c = 3.0$</th>
<th>stagg. $T/T_c = 1.5$</th>
<th>stagg. $T/T_c = 3.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>$7.000 \pm 0.285$</td>
<td>$7.259 \pm 0.221$</td>
<td>$7.032 \pm 0.229$</td>
<td>$7.308 \pm 0.245$</td>
</tr>
<tr>
<td>8</td>
<td>24</td>
<td>$6.340 \pm 0.123$</td>
<td>$6.676 \pm 0.082$</td>
<td>$6.290 \pm 0.130$</td>
<td>$6.667 \pm 0.074$</td>
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<td>32</td>
<td>$6.070 \pm 0.047$</td>
<td>$6.347 \pm 0.038$</td>
<td>$6.070 \pm 0.042$</td>
<td>$6.319 \pm 0.029$</td>
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<tr>
<td>8</td>
<td>64</td>
<td>$6.007 \pm 0.016$</td>
<td>$6.132 \pm 0.024$</td>
<td>$5.887 \pm 0.022$</td>
<td>$6.082 \pm 0.010$</td>
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<td>12</td>
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<td>$6.235 \pm 0.004$</td>
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<td>$6.196 \pm 0.004$</td>
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<tr>
<td>16</td>
<td>32</td>
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<td>$6.851 \pm 0.095$</td>
<td>$7.158 \pm 0.092$</td>
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<tr>
<td>16</td>
<td>48</td>
<td>$6.327 \pm 0.026$</td>
<td>$6.665 \pm 0.020$</td>
<td>$6.310 \pm 0.027$</td>
<td>$6.649 \pm 0.022$</td>
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<tr>
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<td>$6.422 \pm 0.023$</td>
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<td>$6.429 \pm 0.016$</td>
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<tr>
<td>16</td>
<td>128</td>
<td>$6.020 \pm 0.002$</td>
<td>$6.262 \pm 0.003$</td>
<td>$6.070 \pm 0.002$</td>
<td>$6.253 \pm 0.004$</td>
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Screening masses of the axialvector meson $a_1$ (stagg. M3+M4)

<table>
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<tr>
<th>$N_\tau$</th>
<th>$N_\sigma$</th>
<th>Wilson $T/T_c = 1.5$</th>
<th>Wilson $T/T_c = 3.0$</th>
<th>stagg. $T/T_c = 1.5$</th>
<th>stagg. $T/T_c = 3.0$</th>
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<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>$7.210 \pm 0.434$</td>
<td>$7.423 \pm 0.340$</td>
<td>$7.033 \pm 0.229$</td>
<td>$7.308 \pm 0.245$</td>
</tr>
<tr>
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<td>24</td>
<td>$6.373 \pm 0.140$</td>
<td>$6.707 \pm 0.098$</td>
<td>$6.290 \pm 0.130$</td>
<td>$6.668 \pm 0.074$</td>
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<tr>
<td>8</td>
<td>32</td>
<td>$6.080 \pm 0.046$</td>
<td>$6.352 \pm 0.040$</td>
<td>$6.071 \pm 0.042$</td>
<td>$6.319 \pm 0.029$</td>
</tr>
<tr>
<td>8</td>
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<td>$6.009 \pm 0.015$</td>
<td>$6.082 \pm 0.004$</td>
<td>$5.898 \pm 0.015$</td>
<td>$6.082 \pm 0.010$</td>
</tr>
<tr>
<td>12</td>
<td>24</td>
<td>$7.049 \pm 0.229$</td>
<td>$7.226 \pm 0.172$</td>
<td>$6.934 \pm 0.171$</td>
<td>$7.292 \pm 0.148$</td>
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<tr>
<td>12</td>
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<td>$6.307 \pm 0.059$</td>
<td>$6.564 \pm 0.048$</td>
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<td>$6.509 \pm 0.053$</td>
</tr>
<tr>
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<td>$6.195 \pm 0.025$</td>
<td>$6.403 \pm 0.035$</td>
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<tr>
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<td>$6.041 \pm 0.004$</td>
<td>$6.235 \pm 0.004$</td>
<td>$6.057 \pm 0.012$</td>
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<td>$7.158 \pm 0.092$</td>
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<tr>
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<td>$6.311 \pm 0.027$</td>
<td>$6.649 \pm 0.022$</td>
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<tr>
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<td>$6.107 \pm 0.019$</td>
<td>$6.463 \pm 0.018$</td>
<td>$6.079 \pm 0.025$</td>
<td>$6.429 \pm 0.016$</td>
</tr>
<tr>
<td>16</td>
<td>128</td>
<td>$6.021 \pm 0.002$</td>
<td>$6.262 \pm 0.003$</td>
<td>$6.070 \pm 0.002$</td>
<td>$6.253 \pm 0.004$</td>
</tr>
</tbody>
</table>
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Erklärung

Erklärung nach §18 Absatz 7 der Diplomprüfungsordnung der Fakultät für Physik der Universität Bielefeld:

Hiermit versichere ich, dass ich die vorliegende Diplomarbeit selbständig erarbeitet und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

Marcel Müller
Bielefeld, den 26.11.2009