Muttalib-Borodin ensemble with general potential via 1×2 vector-valued Riemann-Hilbert problem

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Muttalib-Borodin ensemble

Inspired by the quasi-1D conductor-insulator phase transition model, Muttalib defined the n-particle ensemble whose density function is

$$\frac{1}{C}\prod_{1\leq i<j\leq n}|x_i - x_j||x_i^\theta - x_j^\theta|\prod_{i=1}^{n} e^{-nV(x_i)},$$

where the location of particles $x_1, \dots, x_n \in (0, +\infty)$ and the potential V is defined on $(0, +\infty)$. Muttalib's interest was on the partition function.

From the mathematical point of view, Borodin also studied this particle ensemble, and he solved the limiting local distribution around 0 when the potential is the Laguerre $V(x) = x$.

Random matrix relation

This particle model has a random matrix realization when the potential is Laguerre, found by Cheliotis.

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots \\ 0 & y_{22} & y_{23} & \cdots \\ 0 & 0 & y_{33} & \cdots \\ 0 & 0 & 0 & \ddots \end{pmatrix}.$$

Let $\alpha_j = \theta(j-1)$ for $j = 1, \ldots, n$, and define an $n \times n$ upper-triangular random matrix Y such that all upper-triangular entries are independent, the diagonal entries are nonnegative and satisfies $2|y_{ii}|^2 = \chi^2_{2(\alpha_i+1)}$, and the strictly upper-triangular entries are in standard complex normal distribution. Then the engenvalues of Y^*Y are a Muttalib-Borodin ensemble.

Limiting distribution around 0

As found by Borodin, we state the limiting distribution of particles around 0 as the particle number $n \to \infty$. We state the result only for integer θ, and use the language of Meijer G functions.

Since the Muttalib-Borodin ensemble is a determinantal process, we can describe it by the correlation kernel $K(x, y)$, that is, the correlation function

$$R(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{i,j=1}^{k}.$$

Denote $\nu_j = j/\theta - 1$ where $j = 1, \ldots, \theta$, we have

$$\lim_{n\to\infty} n^{-\frac{1}{\theta}-1} x^{\frac{1}{\theta}-1} K\left(\frac{\theta}{n}\left(\frac{x}{n}\right)^{\frac{1}{\theta}}, \frac{\theta}{n}\left(\frac{y}{n}\right)^{\frac{1}{\theta}}\right) = K_{\nu_1,\ldots,\nu_\theta}(y, x),$$

where (G is the Meijer G function)

$$K_{\nu_1,\ldots,\nu_\theta}(x, y) = \int_0^1 G^{1,0}_{0,\theta+1}\left(\left.\begin{matrix} - \\ 0, -\nu_1, \ldots, -\nu_\theta \end{matrix}\right| ux\right) \times G^{\theta,0}_{0,\theta+1}\left(\left.\begin{matrix} - \\ \nu_1, \ldots, \nu_\theta, 0 \end{matrix}\right| uy\right) du.$$

Universality by analogy

In a certain sense, the correlation kernel $K_{\nu_1,\dots,\nu_\theta}(x,y)$ is universal, because it occurs in other random matrix models. The best known example is the product of random Ginibre matrices: Let $G_1,\dots,G_\theta$ be independent random matrices in size $(n+\nu_1)\times(n+\nu_0),\dots,(n+\nu_\theta)\times(n+\nu_{\theta-1})$ where $\nu_0=0$, and suppose all G_k entries are independent random variables with standard complex normal distribution. Then the eigenvalues of $(G_\theta\cdots G_1)^*(G_\theta\cdots G_1)$ are a determinantal process, and the correlation kernel has the limit around 0

$$\lim_{n\to\infty}\frac{1}{n}K\left(\frac{x}{n},\frac{y}{n}\right)=K_{\nu_1,\dots,\nu_\theta}(x,y).$$

We remark that the "universality" is not very precise, since here ν_i are integers, while for Muttalib-Borodin ensemble they are not.

Universality conjecture

Although the Muttalib-Borodin ensemble does not have a random matrix interpretation for general V, we expect the limiting correlation kernel to be universal. Suppose $V(x)$ is continuous and nonzero at 0, then we expect the limit identity of the correlation kernel remains, with only the scaling factor depending on V.
In practice, we would like to assume that V is analytic, and satisfies the technical requirement that it is "regular" and "one-cut".
We tackle the problem in the usual way. First we observe that the Muttalib-Borodin ensemble is biorthogonal, that is, we can find polynomials $p_j(x)$ and $q_j(x)$ with degree $j = 0, 1, \dots,$ such that

$$\int_0^{+\infty} p_j(x) q_k(x^\theta) e^{-nV(x)} dx = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Then by the standard theory of determinantal process, we have that the correlation kernel for the Muttalib-Borodin ensemble is

$$K(x, y) = \sum_{k=0}^{n-1} p_k(x) q_k(y) e^{-n\frac{V(x)+V(y)}{2}}.$$

Universal conjecture continued

It is natural to conjecture that for all $k = \kappa n$ with $c \in (0, 1]$,

$$p_k\left(\frac{\theta}{n}\left(\frac{x}{n}\right)^{1/\theta}\right) \sim C_\kappa x^{1-1/\theta} G^{\theta,0}_{0,\theta+1}\left(\begin{array}{c} - \\ \nu_1, \ldots, \nu_\theta, \nu_0 \end{array} \middle| u_\kappa x\right),$$

$$q_k\left(\frac{\theta}{n}\left(\frac{y^\theta}{n}\right)^{1/\theta}\right) \sim C'_\kappa G^{1,0}_{0,\theta+1}\left(\begin{array}{c} - \\ -\nu_0, -\nu_1, \ldots, -\nu_\theta \end{array} \middle| u_\kappa y\right),$$

with $C_\kappa, C'_\kappa, u_\kappa$ satisfying some relations. Then the sum of $p_k(x)q_k(y)$ converges to the integral over $x^{1-1/\theta} G^{1,0}_{0,\theta+1}(u_\kappa x) G^{\theta,0}_{0,\theta+1}(u_\kappa y)$.

In this talk, we concentrate on the special case that $\theta = 2$ and $\kappa = 1$, and we want to compute the asymptotics of

$$p_n(cn^{-3/2}x) \sim CxG^{2,0}_{0,3}\left(\begin{array}{c} - \\ -\frac{1}{2}, 0, 0 \end{array} \middle| x^2\right)$$

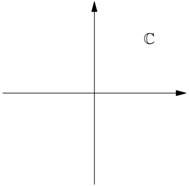
for some c and C. The computation for q_n is analogous.

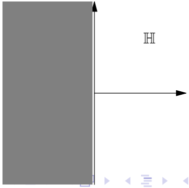
Equilibrium measure and g-functions

The "one-cut" condition is that the limiting empirical distribution of the n particles, which is called the equilibrium measure in physical term, is supported on an interval $[0, b]$. Suppose $d\mu(x) = \psi(x)dx$ is the density function of the equilibrium measure that satisfies $\psi(x)$ is continuous on $(0, b)$, $\psi(x)$ vanishes like a square root as $x \to b$ and blows up like $x^{-1/(\theta+1)}$ as $x \to 0$ (this is part of the regularity).
Define

$$g(z) = \int_0^b \log(z - y)\psi(y)dy, \quad \tilde{g}(z) = \int_0^b \log(z^2 - y^2)\psi(y)dy.$$

Note that $g(z)$ is defined on $\mathbb{C} \setminus (-\infty, b]$, while $\tilde{g}$ is defined on $\mathbb{H} \setminus (-\infty, b]$.

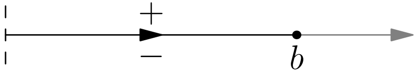




Riemann-Hilbert problem: warm-up

Consider $G(z) = g'(z)$ and $\tilde{G}(z) = \tilde{g}'(z)$ defined on $\mathbb{C}$ and $\mathbb{H}$. We have

- $G_\pm(x) + \tilde{G}_\mp(x) = V'(x)$ for $x \in (0, b)$.
- $\tilde{G}(-ix) = \tilde{G}(ix)$ for $x > 0$.
- As $z \to \infty$ in $\mathbb{C}$, $G(z) = z^{-1} + \mathcal{O}(z^{-2})$.
- As $z \to \infty$ in $\mathbb{H}$, $\tilde{G}(z) = 2z^{-1} + \mathcal{O}(z^{-3})$.



These properties comstitute a Riemann-Hilbert problem (RHP), and define G and $\tilde{G}$ uniquely. Moreover, we have a practical way to compute b and ψ by the RHP stated above. (It is due to Claeys and Romano.)
We note that the RHP is a 1×2 vector-valued one, and the difficulty lies in that G and $\tilde{G}$ have different domains. A trick to solve it is to map $\mathbb{C}$ and $\mathbb{H}$ to two complemental regions on $\mathbb{C}$, and then transform the RHP into a scalar one defined on $\mathbb{C}$.

Riemann-Hilbert prblem: set-up

Let $Y = (Y_1, Y_2)$ be defined on $\mathbb{C} \times \mathbb{H}$ such that $Y_1(z) = p_n(z)$ and

$$Y_2(z) = Cp_n(z) = \int_0^\infty \frac{p_n(x)e^{-nV(x)}}{x^2 - z^2} dx.$$

Then Y is uniquely determined by the following RHP

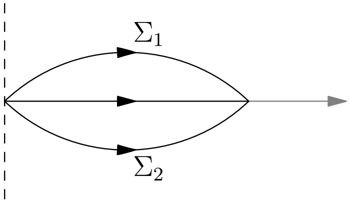
- $$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \frac{1}{2x} e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \quad x > 0.$$
- As $z \to \infty$ in $\mathbb{C}$, $Y_1(z) = z^n + \mathcal{O}(z^{n-1})$.
- As $z \to \infty$ in $\mathbb{H}$, $Y_2(z) = \mathcal{O}(z^{-2(n+1)})$.
- As $z \to 0$ in $\mathbb{H}$, $Y_2(z) = \mathcal{O}(z^{-1})$.

First transform

Define

$$\phi(z) = g(z) + \tilde{g}(z) - V(x) - \ell,$$

where ℓ is a constant making $\phi(0) = 0$. We draw two arcs Σ_1 and Σ_2 from 0 to b.



Then define $T = (T_1, T_2)$ and $S = (S_1, S_2)$ on the domain $\mathbb{C} \times \mathbb{H}$ such that

$$T = (Y_1(z)e^{-ng(z)}, Y_2(z)e^{n(\tilde{g}(z)-\ell)}),$$

and

$$S(z) = \begin{cases} T(z) & \text{outside the lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ 2ze^{-n\phi(z)} & 1 \end{pmatrix} & \text{in the lower lens,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -2ze^{-n\phi(z)} & 1 \end{pmatrix} & \text{in the upper lens.} \end{cases}$$

Jump matrix for S

We have that S satisfies the RHP

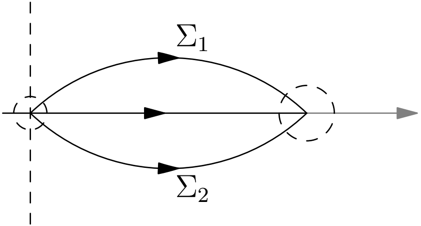
- $S_+(z) = S_-(z)J_s(z)$, where

$$J_s(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 2ze^{-n\phi(z)} & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & (2z)^{-1} \\ -2z & 0 \end{pmatrix}, & z \in (0, b), \\ \begin{pmatrix} 1 & (2z)^{-1}e^{n\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (b, +\infty). \end{cases}$$

- As $z \to \infty$ in $\mathbb{C}$, $S_1(z) \to 1$.
- As $z \to 0$ in $\mathbb{C}$, $S_1(z) = \mathcal{O}(1)$.
- As $z \to \infty$ in $\mathbb{H}$, $S_2(z) = \mathcal{O}(z^{-1})$.
- As $z \to 0$ in $\mathbb{H}$, $S_2(z) = \mathcal{O}(z^{-1})$.

Idea of the asymptotic analysis of RHP: Patchwork

We divide the whole domain $\mathbb{C} \times \mathbb{H}$ into three regions, one is a region around b, which is small but of a constant radius; one is a region around 0, which is small and with a shrinking radius depending on n; and the outer region. The RHP is further transformed in each of these three regions, with the help of the "parametrices", and then the transformed RHP has simple jumps within each region, but rather complicated jumps at the border between two regions. But we will show that the jumps, although complicated, converge to 0 *uniformly* as $n \to \infty$. Hence we derive that the final RHP converges to a constant, and can get the asymptotics of the original RHP reversely.



Global parametrix

Function $\phi(z)$ has the property that

$$\Re\phi(z) > 0 \qquad \text{for all} \quad z \in \Sigma_1 \cup \Sigma_2,$$
$$\phi(z) < 0 \qquad \text{for all} \quad z \in (b, +\infty).$$

It seems that we can approximate the jump matrix J_S into J_∞ that is $\begin{pmatrix} 0 & (2z)^{-1} \\ -2z & 0 \end{pmatrix}$ on $(0, b)$, and trivial on $(b, +\infty)$ and Σ_1, Σ_2.
The solution $P^{(\infty)}$ to the simplified RHP is called the global parametrix. It is exactly solvable, but there is a serious problem: The convergence of J_S to J_∞ is not uniform. Then the convergence of S to $P^{(\infty)}$ is dubious.
(Actually, S does not converge to $P^{(\infty)}$ at the points where J_S does not converge uniformly to J_∞, namely 0 and b.)

The first patch: Global

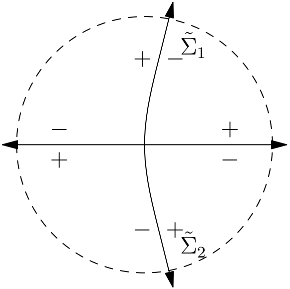
Define $R = (R_1, R_2)$ on the outer region of $\mathbb{C} \times \mathbb{H}$ such that $R_i(z) = S_i(z)/P_i^{(\infty)}(z)$. It is clear that on (a, b), the jump of R is very simple:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and on Σ_1, Σ_2, the jump of R is very small, if we do not consider the two local patches.

Search for the local parametrix at 0

To solve the problem, we resort to techniques from the large size RHP (and this is why we need to assume that θ is integer).
Let $U = (U_1, U_2, U_3)$ whose components are defined on $D_\epsilon(0) \subseteq \mathbb{C}$, a small disk centred at 0, such that



$$U_1(w) = S_1(-\sqrt{w})[P_1^{(\infty)}(-\sqrt{w})]^{-1} \quad w \in D_\epsilon(0) \setminus (\tilde{\Sigma} \cup \tilde{\Sigma}_2 \cup \mathbb{R}),$$
$$U_2(w) = S_1(\sqrt{w})[P_1^{(\infty)}(\sqrt{w})]^{-1} \quad w \in D_\epsilon(0) \setminus (\tilde{\Sigma} \cup \tilde{\Sigma}_2 \cup \mathbb{R}),$$
$$U_3(w) = S_2(\sqrt{w})[P_2^{(\infty)}(\sqrt{w})]^{-1} \quad w \in D_\epsilon(0) \setminus (\tilde{\Sigma} \cup \tilde{\Sigma}_2 \cup \mathbb{R}_+),$$

where $\tilde{\Sigma}_i = \{w \in \mathbb{C} \mid \sqrt{w} \in \Sigma_i\}$. We note that

$$P_1^{(\infty)}(z)/P_2^{(\infty)}(z) \to 2z \quad \text{as } z \to 0.$$

RHP of the local parametrix

Then U satisfies the RHP $U_+(w) = U_-(w)J_U(w)$, where

$$J_U(w) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{-n\phi(\sqrt{w})}\frac{2\sqrt{w}P_2^{(\infty)}(\sqrt{w})}{P_1^{(\infty)}(\sqrt{w})} & 1 \end{pmatrix} & w \in \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & w \in \mathbb{R}_+, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & w \in \mathbb{R}_-. \end{cases}$$

This is a quite simple RHP, and we can construct an explicit 3×3 matrix-valued function that satisfies the same RHP, with the help of Meijer G functions.

Model RHP

We let

$$y_1^{(k)}(w) = (-1)^{k-1}\sqrt{3}\cdot 2^{\frac{2k}{3}-1}\sqrt{w}\,G_{0,3}^{2,0}\left(\begin{array}{c|c} - & \frac{z^2}{4} \\ -\frac{1}{2},0,k & \end{array}\right)\Bigg|_{z^2=w,\ \arg(z)\in(\frac{\pi}{2},\frac{3\pi}{2})}$$

$$y_2^{(k)}(w) = (-1)^{k}\sqrt{3}\cdot 2^{\frac{2k}{3}-1}\sqrt{w}\,G_{0,3}^{2,0}\left(\begin{array}{c|c} - & \frac{z^2}{4} \\ -\frac{1}{2},0,k & \end{array}\right)\Bigg|_{z^2=w,\ \arg(z)\in(-\frac{\pi}{2},\frac{\pi}{2})};$$

$$y_3^{(k)}(w) = \frac{\sqrt{3}\cdot 2^{\frac{2k}{3}-1}}{2\pi i}\sqrt{w}\,G_{0,3}^{3,0}\left(\begin{array}{c|c} - & -\frac{z^2}{4} \\ -\frac{1}{2},0,k & \end{array}\right)\Bigg|_{z^2=w,\ \arg(z)\in(-\frac{\pi}{2},\frac{\pi}{2})},$$

and then define the 3×3 matrix $P^{(0)}$ by

$$(P^{(0)})_{k,1}(w) = y_1^{(k-1)}(w), \qquad (P^{(0)})_{k,3}(w) = y_3^{(k-1)}(w),$$

$$(P^{(0)})_{k,2}(w) = y_2^{(k-1)}(w) \qquad \arg(w)\in(\frac{\pi}{2},\pi)\cup(-\pi,-\frac{\pi}{2}),$$

$$(P^{(0)})_{k,2}(w) = y_3^{(k-1)}(w) - y_3^{(k-1)}(w) \qquad \arg(w)\in(0,\frac{\pi}{2}),$$

$$(P^{(0)})_{k,2}(w) = y_3^{(k-1)}(w) + y_3^{(k-1)}(w) \qquad \arg(w)\in(-\frac{\pi}{2},0).$$

Construction of the local parametrix at 0

At last we let

$$\tilde{P}_n^{(0)}(w) = \begin{pmatrix} 1 & & \\ & (cn)^{-1} & \\ & & (cn)^{-2} \end{pmatrix} P^{(0)}((cn)^3 w)$$

$$\times \begin{pmatrix} \frac{e^{-nf_1(w)}}{P_1^{(\infty)}(-\sqrt{w})} & & \\ & \frac{e^{-nf_2(w)}}{P_1^{(\infty)}(\sqrt{w})} & \\ & & \pm 2\sqrt{w}\frac{e^{nf_3(w)}}{P_2^{(\infty)}(\sqrt{w})} \end{pmatrix},$$

where

$$f_1(w) = g(-\sqrt{w}) + V(\sqrt{w}) - \ell + C,$$
$$f_2(w) = g(\sqrt{w}) - V(\sqrt{w}) - \ell + C,$$
$$f_3(w) = \tilde{g}(\sqrt{w}) - C,$$

such that C makes $f_1(0) = f_2(0) = 0$, and c depends on the potential V. This matrix solves the RHP with J_U, and it is the local parametrix.

Patches at 0 and b

Why we bother constructing a matrix-valued solution to the RHP with J_U? The reason is that a matrix can be inversed.
Consider the vector

$$U(\tilde{P}_n^{(0)})^{-1}$$

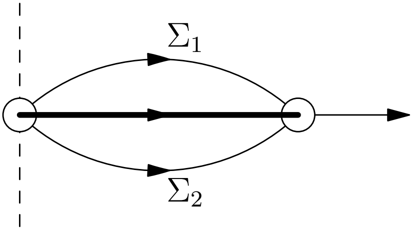
in a small neighbourhood of 0, and this is a 1×3 vector whose components are all analytic — because all the jumps are cancelled. They we can map it back to $\mathbb{C} \times \mathbb{H}$, and it is a well-defined around 0 with trivial jump.
This is not yet the patch we want. But we can further transform it into a vector on $\mathbb{C} \times \mathbb{H}$ around 0 with a jump

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on $(0, b)$. It also satisfies other requirement of the patch, but there is no time for the details. We let R in the shrinking region around 0 be this vector.
Around b, we can make a similar patch and define R there. It involves the usual 2×2 local parametrix for Airy kernel.

Final approximation

R, defined on the patches, satisfies the a RHP with the following jump cuts:



We note that the jump on $(0, b)$ is the simple one

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and on the other cuts, although the jump can be complicated, they are all small. So the solution to this RHP converges to a constant (actually identity).

Therefore the solution is done.