The real spectrum of a product of Ginibre matrices

Nick Simm

Mathematics Department, University of Sussex, Brighton, UK

Sums and products of random matrices

August 2018, ZiF, Bielefeld University

Ginibre ensembles

Ginibre ensembles

- Let G be an $N \times N$ real matrix whose entries are *i.i.d.* standard normal random variables.

Ginibre ensembles

- Let G be an $N \times N$ real matrix whose entries are *i.i.d.* standard normal random variables.
- As a probability density on matrix space:

$$P(G) = \frac{1}{(2\pi)^{N^2/2}} \exp(-\frac{1}{2}\mathrm{Tr}(GG^{\mathrm{T}}))$$

Ginibre ensembles

- Let G be an $N \times N$ real matrix whose entries are *i.i.d.* standard normal random variables.
- As a probability density on matrix space:

$$P(G) = \frac{1}{(2\pi)^{N^2/2}} \exp(-\frac{1}{2}\mathrm{Tr}(GG^{\mathrm{T}}))$$

- Clearly *invariant under orthogonal transformations* $G \to O_1 G O_2$ where $O_1, O_2 \in O(N)$.

Ginibre ensembles

- Let G be an $N \times N$ real matrix whose entries are *i.i.d.* standard normal random variables.
- As a probability density on matrix space:

$$P(G) = \frac{1}{(2\pi)^{N^2/2}} \exp(-\frac{1}{2}\mathrm{Tr}(GG^{\mathrm{T}}))$$

- Clearly *invariant under orthogonal transformations* $G \to O_1 G O_2$ where $O_1, O_2 \in O(N)$.
- Known as the *Ginibre orthogonal ensemble (GinOE)*.

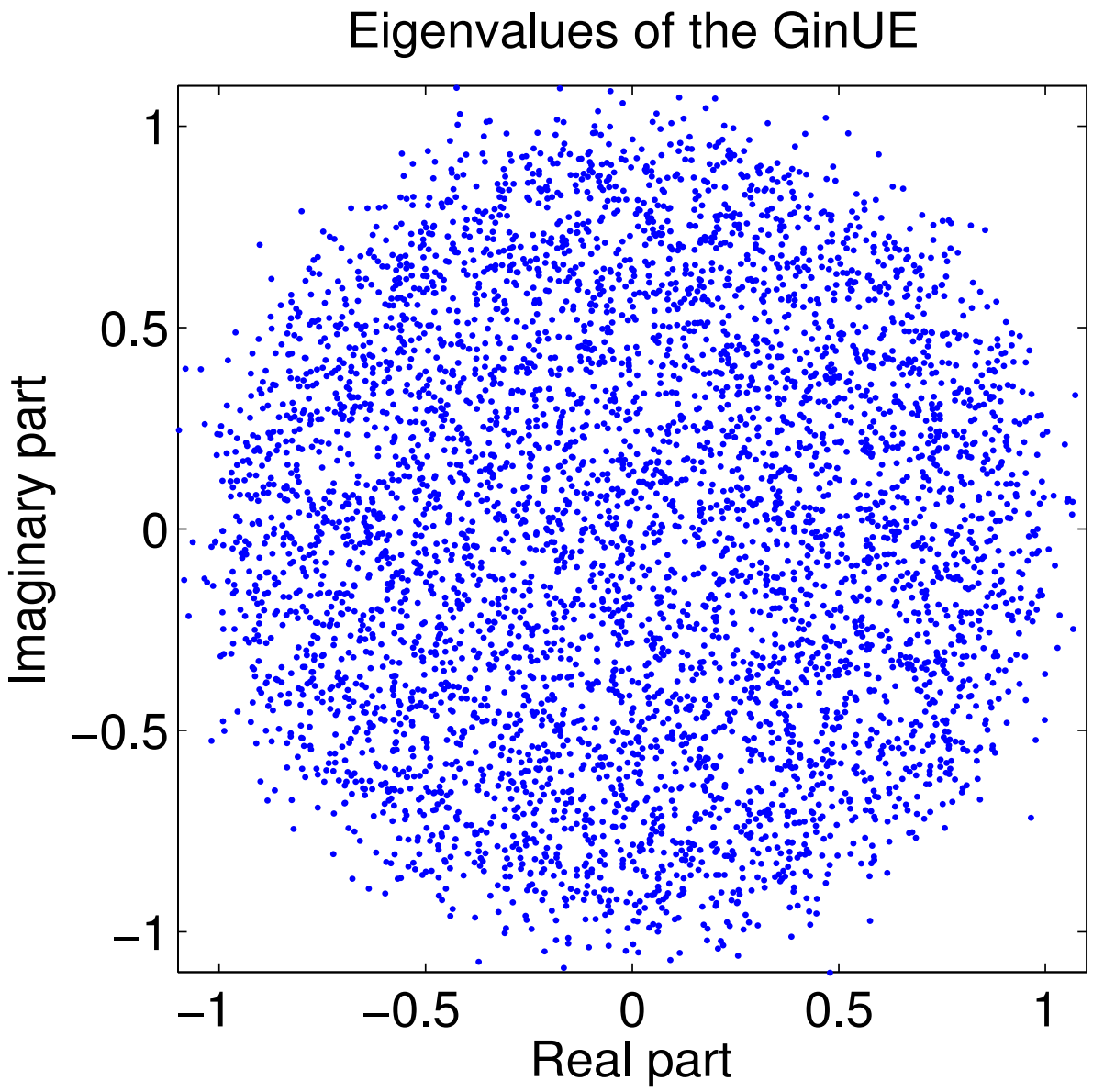
Ginibre ensembles

- Let G be an $N \times N$ real matrix whose entries are *i.i.d.* standard normal random variables.
- As a probability density on matrix space:

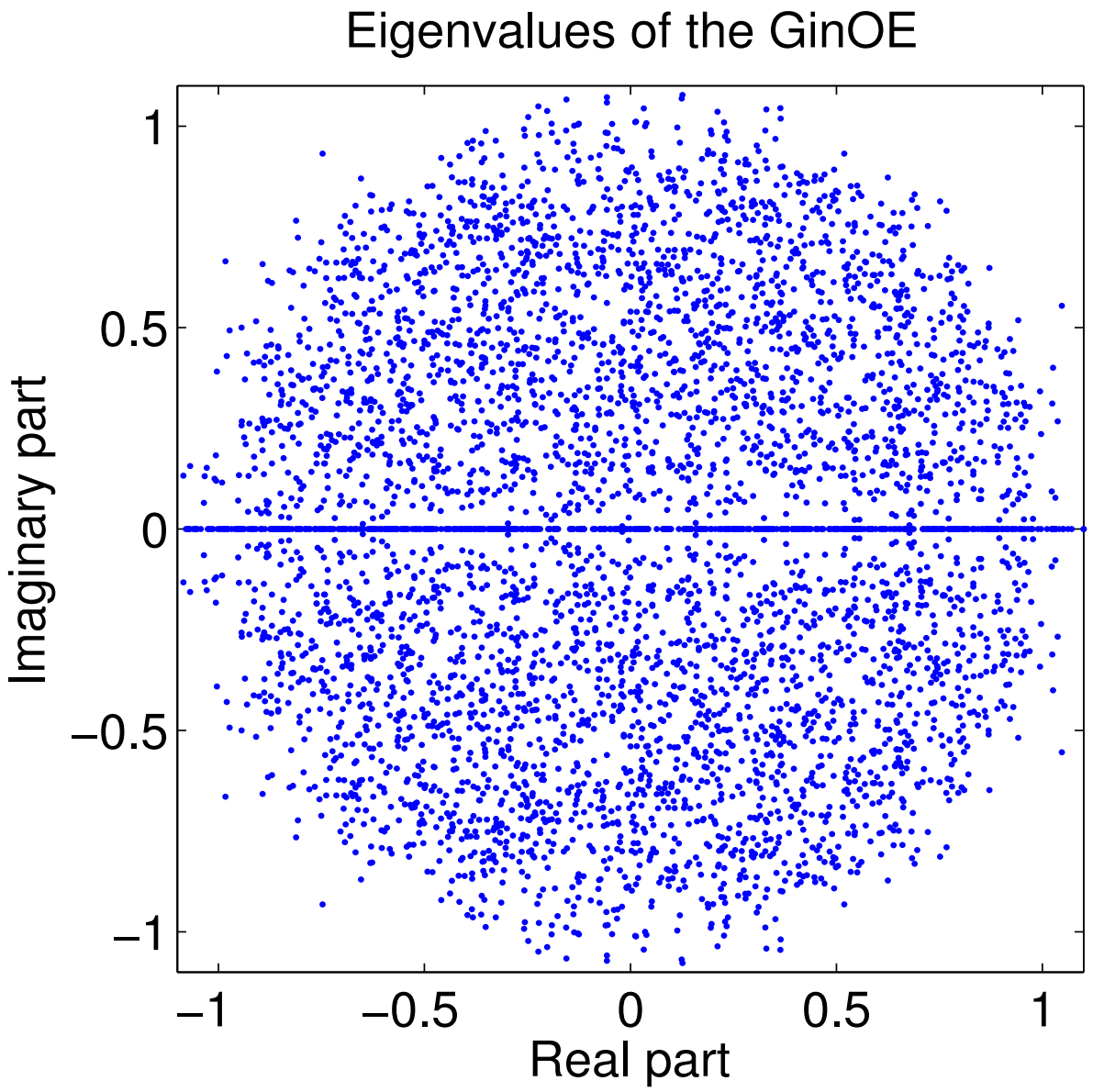
$$P(G) = \frac{1}{(2\pi)^{N^2/2}} \exp(-\frac{1}{2}\mathrm{Tr}(GG^{\mathrm{T}}))$$

- Clearly *invariant under orthogonal transformations* $G \to O_1 G O_2$ where $O_1, O_2 \in O(N)$.
- Known as the *Ginibre orthogonal ensemble (GinOE)*.
- The Ginibre *unitary* ensemble (GinUE) instead consists of complex variables whose real and imaginary parts are $\mathcal{N}(0, 1/2)$.

Eigenvalues of $G/\sqrt{N}$ where $G \sim \mathrm{GinUE}$



Eigenvalues of $G/\sqrt{N}$ where $G \sim \mathrm{GinOE}$



Early discoveries for the Ginibre ensemble

Early discoveries for the Ginibre ensemble

- Jean Ginibre's 1965 paper introduced a total of three ensembles: GinUE, GinSE and GinOE, in increasing order of difficulty.

Early discoveries for the Ginibre ensemble

- Jean Ginibre's 1965 paper introduced a total of three ensembles: GinUE, GinSE and GinOE, in increasing order of difficulty.
- Eigenvalue distribution: Ginibre completely solved the GinUE, partially solved GinSE and left GinOE unsolved.

Early discoveries for the Ginibre ensemble

- Jean Ginibre's 1965 paper introduced a total of three ensembles: GinUE, GinSE and GinOE, in increasing order of difficulty.
- Eigenvalue distribution: Ginibre completely solved the GinUE, partially solved GinSE and left GinOE unsolved.
- Lehmann and Sommers 1991. Joint PDF of GinOE complex eigenvalues $x_j + iy_j$ and $N_{\mathbb{R}}$ real eigenvalues λ_j:

$$\frac{1}{c_N}|\Delta| \exp\left(\sum_{j=1}^{N-N_{\mathbb{R}}} (y_j^2 - x_j^2) - \sum_{j=1}^{N_{\mathbb{R}}} \lambda_j^2/2\right) \prod_{j=1}^{N-N_{\mathbb{R}}} \operatorname{erf}(y_j\sqrt{2})$$

Early discoveries for the Ginibre ensemble

- Jean Ginibre's 1965 paper introduced a total of three ensembles: GinUE, GinSE and GinOE, in increasing order of difficulty.
- Eigenvalue distribution: Ginibre completely solved the GinUE, partially solved GinSE and left GinOE unsolved.
- Lehmann and Sommers 1991. Joint PDF of GinOE complex eigenvalues $x_j + iy_j$ and $N_{\mathbb{R}}$ real eigenvalues λ_j:

$$\frac{1}{c_N}|\Delta| \exp\left(\sum_{j=1}^{N-N_{\mathbb{R}}} (y_j^2 - x_j^2) - \sum_{j=1}^{N_{\mathbb{R}}} \lambda_j^2/2\right) \prod_{j=1}^{N-N_{\mathbb{R}}} \operatorname{erf}(y_j\sqrt{2})$$

- Correlation functions in terms of Pfaffian structures (2007-2008): Borodin and Sinclair, Forrester and Nagao, Sommers and Wieczorek.

How many eigenvalues of a random real matrix are real?

How many eigenvalues of a random real matrix are real?

Theorem (Edelman, Kostlan and Shub '94)

For an $N \times N$ real Ginibre matrix G, one has

$$\mathbb{E}(N_{\mathbb{R}}) = \sqrt{2N/\pi} + O(1) \qquad N \to \infty$$

and the convergence to the uniform law

$$\frac{1}{\mathbb{E}(N_{\mathbb{R}})}\mathbb{E}\left[\sum_{j=1}^{N} \delta(\lambda_j - x)\right] \to \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

How many eigenvalues of a random real matrix are real?

Theorem (Edelman, Kostlan and Shub '94)

For an $N \times N$ real Ginibre matrix G, one has

$$\mathbb{E}(N_{\mathbb{R}}) = \sqrt{2N/\pi} + O(1) \qquad N \to \infty$$

and the convergence to the uniform law

$$\frac{1}{\mathbb{E}(N_{\mathbb{R}})}\mathbb{E}\left[\sum_{j=1}^{N} \delta(\lambda_j - x)\right] \to \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Products: What is the analogue of this result for products of independent Ginibre random matrices?

How many eigenvalues of a random real matrix are real?

Theorem (Edelman, Kostlan and Shub '94)

For an $N \times N$ real Ginibre matrix G, one has

$$\mathbb{E}(N_{\mathbb{R}}) = \sqrt{2N/\pi} + O(1) \qquad N \to \infty$$

and the convergence to the uniform law

$$\frac{1}{\mathbb{E}(N_{\mathbb{R}})}\mathbb{E}\left[\sum_{j=1}^{N} \delta(\lambda_j - x)\right] \to \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Products: What is the analogue of this result for products of independent Ginibre random matrices?

Fluctuations: Variance and central limit theorem?

Products

Products

Let $G_1, \ldots, G_m$ be m independent real Ginibre matrices of size $N \times N$ and set $X_m = N^{-m/2} G_1 G_2 \ldots G_m$.

Products

Let $G_1, \ldots, G_m$ be m independent real Ginibre matrices of size $N \times N$ and set $X_m = N^{-m/2} G_1 G_2 \ldots G_m$.

Theorem (S. '17)

For every fixed $m \in \mathbb{N}$ we have

$$\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{\frac{2Nm}{\pi}} + O(\log(N))$$

Products

Let $G_1, \ldots, G_m$ be m independent real Ginibre matrices of size $N \times N$ and set $X_m = N^{-m/2} G_1 G_2 \ldots G_m$.

Theorem (S. '17)

For every fixed $m \in \mathbb{N}$ we have

$$\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{\frac{2Nm}{\pi}} + O(\log(N))$$

and the weak convergence

$$\frac{1}{\mathbb{E}(N_{\mathbb{R}}^{(m)})} \mathbb{E}\left[\sum_{j=1}^{N_{\mathbb{R}}^{(m)}} \delta(\lambda_j - \lambda)\right] \to \begin{cases} \frac{1}{2m}|\lambda|^{\frac{1}{m}-1} & |\lambda| < 1 \\ 0 & |\lambda| > 1 \end{cases}$$

as $N \to \infty$.

Products

Let $G_1, \ldots, G_m$ be m independent real Ginibre matrices of size $N \times N$ and set $X_m = N^{-m/2} G_1 G_2 \ldots G_m$.

Theorem (S. '17)

For every fixed $m \in \mathbb{N}$ we have

$$\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{\frac{2Nm}{\pi}} + O(\log(N))$$

and the weak convergence

$$\frac{1}{\mathbb{E}(N_{\mathbb{R}}^{(m)})} \mathbb{E}\left[\sum_{j=1}^{N_{\mathbb{R}}^{(m)}} \delta(\lambda_j - \lambda)\right] \to \begin{cases} \frac{1}{2m}|\lambda|^{\frac{1}{m}-1} & |\lambda| < 1 \\ 0 & |\lambda| > 1 \end{cases}$$

as $N \to \infty$.

Compare to known density for the complex eigenvalues (Burda et al., Götze and Tikhomirov, O'Rourke and Soshnikov 2010): $p(z) = \frac{1}{m\pi}|z|^{2/m-2} 1_{|z|<1}$.

Remarks

Remarks

- This theorem (but without the leading constant $\sqrt{m}$) was conjectured by Forrester and Ipsen (2016).

Remarks

- This theorem (but without the leading constant $\sqrt{m}$) was conjectured by Forrester and Ipsen (2016).
- Philosophy: $G_1 \dots G_m \sim G^m$ up to symmetry? (Burda et al. 2010)

Remarks

- This theorem (but without the leading constant $\sqrt{m}$) was conjectured by Forrester and Ipsen (2016).
- Philosophy: $G_1 \dots G_m \sim G^m$ up to symmetry? (Burda et al. 2010)
- The error term $O(\log N)$ can easily be replaced with $O(1)$. True error should be $O(N^{-1/2})$.

Remarks

- This theorem (but without the leading constant $\sqrt{m}$) was conjectured by Forrester and Ipsen (2016).
- Philosophy: $G_1 \dots G_m \sim G^m$ up to symmetry? (Burda et al. 2010)
- The error term $O(\log N)$ can easily be replaced with $O(1)$. True error should be $O(N^{-1/2})$.
- Proof still works for $m = N^\delta$ for some small $\delta > 0$. What if $m = cN$ for some large constant $c > 0$?

Remarks

- This theorem (but without the leading constant $\sqrt{m}$) was conjectured by Forrester and Ipsen (2016).
- Philosophy: $G_1 \dots G_m \sim G^m$ up to symmetry? (Burda et al. 2010)
- The error term $O(\log N)$ can easily be replaced with $O(1)$. True error should be $O(N^{-1/2})$.
- Proof still works for $m = N^\delta$ for some small $\delta > 0$. What if $m = cN$ for some large constant $c > 0$?

Idea of the proof is to compute moments and show that

$$\lim_{N\to\infty} \frac{1}{\mathbb{E}(N_{\mathbb{R}}^{(m)})} \mathbb{E}\left[\sum_{j=1}^{N_{\mathbb{R}}^{(m)}} \lambda_j^k\right] = \begin{cases} \frac{1}{1+mk}, & k \quad \text{even} \\ 0, & k \quad \text{odd} \end{cases}$$

Products of random variables

Products of random variables

Let $Y_1, \ldots, Y_m$ be i.i.d. standard Gaussians. What is the density of the product $X_m^{(N=1)} = Y_1 Y_2 \ldots Y_m$?

Products of random variables

Let $Y_1, \ldots, Y_m$ be i.i.d. standard Gaussians. What is the density of the product $X_m^{(N=1)} = Y_1 Y_2 \ldots Y_m$?

$$w_m(x) := \int_{\mathbb{R}^m} \prod_{j=1}^{m} dx_j \, e^{-x_j^2/2} \delta(x - x_1 x_2 \ldots x_m) = G_{0,m}^{m,0}\left(\begin{matrix} \overline{} \\ 0,\ldots,0 \end{matrix} \middle| \frac{x^2}{2^m} \right).$$

where the *Meijer G-function* is

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_\gamma \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, ds$$

Products of random variables

Let $Y_1, \ldots, Y_m$ be i.i.d. standard Gaussians. What is the density of the product $X_m^{(N=1)} = Y_1 Y_2 \ldots Y_m$?

$$w_m(x) := \int_{\mathbb{R}^m} \prod_{j=1}^{m} dx_j \, e^{-x_j^2/2} \delta(x - x_1 x_2 \ldots x_m) = G_{0,m}^{m,0}\left(\begin{matrix} \overline{} \\ 0,\ldots,0 \end{matrix} \middle| \frac{x^2}{2^m} \right).$$

where the *Meijer G-function* is

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_\gamma \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \, ds$$

The contour γ connects $-i\infty$ to $+i\infty$ such that *all poles* of $\Gamma(b_j - s)$ on right and $\Gamma(1 - a_k + s)$ on left.

Next: the case $N > 1$.

Products form a Pfaffian point process

Products form a Pfaffian point process

Theorem (Ipsen and Kieburg '14, Forrester and Ipsen '16)

The real eigenvalues of the matrix product $G_1 \dots G_m$ form a Pfaffian point process with correlation kernel given by

$$\mathbb{K}(x,y) = \begin{pmatrix} D(x,y) & S(x,y) \\ -S(y,x) & I(x,y) \end{pmatrix}$$

where

$$S(x,y) = \sum_{j=0}^{N-2} \frac{w_m(x)x^j}{(2\sqrt{2\pi}j!)^m}(xA_j(y) - A_{j+1}(y))$$

and

$$A_j(y) = \int_{\mathbb{R}} w_m(v)\mathrm{sgn}(y-v)v^j \, dv,$$

Products form a Pfaffian point process

Theorem (Ipsen and Kieburg '14, Forrester and Ipsen '16)

The real eigenvalues of the matrix product $G_1 \dots G_m$ form a Pfaffian point process with correlation kernel given by

$$\mathbb{K}(x,y) = \begin{pmatrix} D(x,y) & S(x,y) \\ -S(y,x) & I(x,y) \end{pmatrix}$$

where

$$S(x,y) = \sum_{j=0}^{N-2} \frac{w_m(x)x^j}{(2\sqrt{2\pi}j!)^m}(xA_j(y) - A_{j+1}(y))$$

and

$$A_j(y) = \int_{\mathbb{R}} w_m(v)\mathrm{sgn}(y-v)v^j \, dv,$$

In particular, the desired moments are just

$$M_{k,N} := \mathbb{E}\left[\sum_{j=1}^{N_{\mathbb{R}}^{(m)}} \lambda_j^k\right] = \int_{\mathbb{R}} x^k S(x,x)\, dx$$

Moments and Meijer G

Moments and Meijer G

The last integral splits in two pieces
$M_{2k,N}(m) = M^{(1)}_{2k,N}(m) - M^{(2)}_{2k,N}(m)$ where

$$M^{(1)}_{2k,N}(m) = N^{-mk} \sum_{j=0}^{(N-2)/2} \frac{2^{(2j+k)m}}{(\sqrt{\pi}(2j)!)^m}(a_{j+1,j+k+1} + a_{j+k+1,j+1})$$

$$M^{(2)}_{2k,N}(m) = N^{-mk} \sum_{j=0}^{N/2-2} \frac{2^{(2j+1+k)m}}{(\sqrt{\pi}(2j+1)!)^m}(a_{j+k+2,j+1} + a_{j+2,j+k+1})$$

Moments and Meijer G

The last integral splits in two pieces
$M_{2k,N}(m) = M^{(1)}_{2k,N}(m) - M^{(2)}_{2k,N}(m)$ where

$$M^{(1)}_{2k,N}(m) = N^{-mk} \sum_{j=0}^{(N-2)/2} \frac{2^{(2j+k)m}}{(\sqrt{\pi}(2j)!)^m}(a_{j+1,j+k+1} + a_{j+k+1,j+1})$$

$$M^{(2)}_{2k,N}(m) = N^{-mk} \sum_{j=0}^{N/2-2} \frac{2^{(2j+1+k)m}}{(\sqrt{\pi}(2j+1)!)^m}(a_{j+k+2,j+1} + a_{j+2,j+k+1})$$

Here $a_{j,k}$ is a particular case of the Meijer-G function

$$\begin{aligned} a_{j,k} &= G^{m+1,m}_{m+1,m+1}\left(\begin{matrix} 3/2-j,\ldots,3/2-j,1 \\ 0,k,\ldots,k \end{matrix} \middle| 1 \right) \\ &= \frac{1}{2\pi i} \int_\gamma \frac{(\Gamma(k-s)\Gamma(-1/2+j+s))^m}{-s}\, ds \end{aligned}$$

Saddle point analysis

Saddle point analysis

The formula

$$\left(\frac{\Gamma(k-s)\Gamma(-1/2+j+s)}{\Gamma(j+k-1/2)}\right) = \int_0^\infty \frac{t^{k-s-1}}{(1+t)^{k+j+1/2}}\,dt$$

implies that $a_{j+1,j+k+1}$ can be written

Saddle point analysis

The formula

$$\left(\frac{\Gamma(k-s)\Gamma(-1/2+j+s)}{\Gamma(j+k-1/2)}\right) = \int_0^\infty \frac{t^{k-s-1}}{(1+t)^{k+j+1/2}}\,dt$$

implies that $a_{j+1,j+k+1}$ can be written

$$\Gamma(2j+k+3/2)^m \int_1^\infty \frac{dx_m}{x_m} \prod_{l=1}^{m-1} \left[\int_0^\infty \frac{dx_l}{x_l} \frac{(x_l/x_{l+1})^{j+1/2}}{(1+x_l/x_{l+1})^{2j+k+3/2}}\right] \frac{x_1^{j+k+1}}{(1+x_1)^{2j+k+3/2}}$$

Saddle point analysis

The formula

$$\left(\frac{\Gamma(k-s)\Gamma(-1/2+j+s)}{\Gamma(j+k-1/2)}\right) = \int_0^\infty \frac{t^{k-s-1}}{(1+t)^{k+j+1/2}}\,dt$$

implies that $a_{j+1,j+k+1}$ can be written

$$\Gamma(2j+k+3/2)^m \int_1^\infty \frac{dx_m}{x_m} \prod_{l=1}^{m-1} \left[\int_0^\infty \frac{dx_l}{x_l} \frac{(x_l/x_{l+1})^{j+1/2}}{(1+x_l/x_{l+1})^{2j+k+3/2}}\right] \frac{x_1^{j+k+1}}{(1+x_1)^{2j+k+3/2}}$$

$$= \Gamma(2j+k+3/2)^m \int_1^\infty \int_{[0,\infty)^{m-1}} e^{j\Phi(\mathbf{x})} F(\mathbf{x})\,dx_1 \ldots dx_m$$

Asymptotics as $j \to \infty$ with fixed k,m: Use the classical (multi-dimensional) saddle point method.

Saddle point analysis

The formula

$$\left(\frac{\Gamma(k-s)\Gamma(-1/2+j+s)}{\Gamma(j+k-1/2)}\right) = \int_0^\infty \frac{t^{k-s-1}}{(1+t)^{k+j+1/2}}\,dt$$

implies that $a_{j+1,j+k+1}$ can be written

$$\Gamma(2j+k+3/2)^m \int_1^\infty \frac{dx_m}{x_m} \prod_{l=1}^{m-1} \left[\int_0^\infty \frac{dx_l}{x_l} \frac{(x_l/x_{l+1})^{j+1/2}}{(1+x_l/x_{l+1})^{2j+k+3/2}}\right] \frac{x_1^{j+k+1}}{(1+x_1)^{2j+k+3/2}}$$

$$= \Gamma(2j+k+3/2)^m \int_1^\infty \int_{[0,\infty)^{m-1}} e^{j\Phi(\mathbf{x})} F(\mathbf{x})\,dx_1 \ldots dx_m$$

Asymptotics as $j \to \infty$ with fixed k,m: Use the classical (multi-dimensional) saddle point method.

Because of cancellations one has to get the first sub-leading correction.

Possible alternative approach

Possible alternative approach

The correlation kernel of real eigenvalues is

$$S(x,y) = \int_{-\infty}^{\infty} (x-v)\mathrm{sgn}(y-v) w_r(x) w_r(v) \sum_{j=0}^{N-2} \frac{(xv)^j}{(j!)^m}\, dv$$

Possible alternative approach

The correlation kernel of real eigenvalues is

$$S(x,y) = \int_{-\infty}^{\infty} (x-v)\mathrm{sgn}(y-v) w_r(x) w_r(v) \sum_{j=0}^{N-2} \frac{(xv)^j}{(j!)^m}\, dv$$

Idea: Compute asymptotics of $S(x/\sqrt{N}, y/\sqrt{N})$. Then use:

$$\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} S(u/\sqrt{N}, u/\sqrt{N})\, du$$

Possible alternative approach

The correlation kernel of real eigenvalues is

$$S(x,y) = \int_{-\infty}^{\infty} (x-v)\mathrm{sgn}(y-v) w_r(x) w_r(v) \sum_{j=0}^{N-2} \frac{(xv)^j}{(j!)^m}\, dv$$

Idea: Compute asymptotics of $S(x/\sqrt{N}, y/\sqrt{N})$. Then use:

$$\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} S(u/\sqrt{N}, u/\sqrt{N})\, du$$

Difficulties: near the edge $|x| = \pm 1$ and $|uv| = \pm 1/\sqrt{N}$. Even $m = 1$! Uniformity?

This is overcome by directly integrating $S(x,x)$ in our case.

Linear statistics of random matrix models

Linear statistics of random matrix models

The linear eigenvalue statistic

$$X_N[f] = \sum_{j=1}^{N} f(z_j)$$

is usually studied in two main cases.

Linear statistics of random matrix models

The linear eigenvalue statistic

$$X_N[f] = \sum_{j=1}^{N} f(z_j)$$

is usually studied in two main cases.

1. Hermitian ensembles: <u>Bounded variance CLT</u> with the $H^{1/2}$ *noise* (*e.g.* Johansson '98):

$$\lim_{N\to\infty} \mathrm{Var}(X_N(f)) = \sum_{k=1}^{\infty} |k||c_k(f)|^2$$

Linear statistics of random matrix models

The linear eigenvalue statistic

$$X_N[f] = \sum_{j=1}^{N} f(z_j)$$

is usually studied in two main cases.

1. Hermitian ensembles: <u>Bounded variance CLT</u> with the $H^{1/2}$ *noise* (*e.g.* Johansson '98):

$$\lim_{N\to\infty} \mathrm{Var}(X_N(f)) = \sum_{k=1}^{\infty} |k||c_k(f)|^2$$

2. Non-Hermitian ensembles: <u>Bounded variance CLT</u> with GFF-type structure (Rider and Virag '07):

$$\lim_{N\to\infty} \mathrm{Var}(X_N(f)) = \int_{\mathbb{U}} |\nabla f|^2 d^2 z + \sum_{k=1}^{\infty} |k||\hat{f}(k)|^2$$

Linear statistics of random matrix models

The linear eigenvalue statistic

$$X_N[f] = \sum_{j=1}^{N} f(z_j)$$

is usually studied in two main cases.

1. Hermitian ensembles: <u>Bounded variance CLT</u> with the $H^{1/2}$ *noise* (*e.g.* Johansson '98):

$$\lim_{N\to\infty} \mathrm{Var}(X_N(f)) = \sum_{k=1}^{\infty} |k||c_k(f)|^2$$

2. Non-Hermitian ensembles: <u>Bounded variance CLT</u> with GFF-type structure (Rider and Virag '07):

$$\lim_{N\to\infty} \mathrm{Var}(X_N(f)) = \int_{\mathbb{U}} |\nabla f|^2 d^2 z + \sum_{k=1}^{\infty} |k||\hat{f}(k)|^2$$

What if we restrict to the real axis? For simplicity, consider $m = 1$.

Fluctuations of the real eigenvalue count

Fluctuations of the real eigenvalue count

Theorem (S. '15)

The variance of the total number of real eigenvalues of a real $(2n \times 2n)$ Gaussian random matrix is given by the following explicit formula

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{k=1}^{n} \frac{\Gamma(2k-3/2)}{\Gamma(2k-1)} - \frac{2}{\pi} \sum_{k_1,k_2} \frac{\Gamma(k_1+k_2-3/2)^2}{\Gamma(2k_1-1)\Gamma(2k_2-1)}}$$

where $\Gamma(x)$ is the Gamma function.

Fluctuations of the real eigenvalue count

Theorem (S. '15)

The variance of the total number of real eigenvalues of a real $(2n \times 2n)$ Gaussian random matrix is given by the following explicit formula

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{k=1}^{n} \frac{\Gamma(2k-3/2)}{\Gamma(2k-1)} - \frac{2}{\pi} \sum_{k_1,k_2} \frac{\Gamma(k_1+k_2-3/2)^2}{\Gamma(2k_1-1)\Gamma(2k_2-1)}}$$

where $\Gamma(x)$ is the Gamma function. The asymptotics are

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2-\sqrt{2}}{\sqrt{\pi}} 2\sqrt{n} + O(1)} \tag{1.1}$$

Fluctuations of the real eigenvalue count

Theorem (S. '15)

The variance of the total number of real eigenvalues of a real $(2n \times 2n)$ Gaussian random matrix is given by the following explicit formula

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{k=1}^{n} \frac{\Gamma(2k-3/2)}{\Gamma(2k-1)} - \frac{2}{\pi} \sum_{k_1,k_2} \frac{\Gamma(k_1+k_2-3/2)^2}{\Gamma(2k_1-1)\Gamma(2k_2-1)}}$$

where $\Gamma(x)$ is the Gamma function. The asymptotics are

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2-\sqrt{2}}{\sqrt{\pi}} 2\sqrt{n} + O(1)} \tag{1.1}$$

Furthermore, the CLT holds:

$$\frac{N_{\mathbb{R}} - \mathbb{E}(N_{\mathbb{R}})}{\sqrt{\mathrm{Var}(N_{\mathbb{R}})}} \xrightarrow{d} \mathcal{N}(0,1), \qquad n \to \infty.$$

Fluctuations of the real eigenvalue count

Theorem (S. '15)

The variance of the total number of real eigenvalues of a real $(2n \times 2n)$ Gaussian random matrix is given by the following explicit formula

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{k=1}^{n} \frac{\Gamma(2k-3/2)}{\Gamma(2k-1)} - \frac{2}{\pi} \sum_{k_1,k_2} \frac{\Gamma(k_1+k_2-3/2)^2}{\Gamma(2k_1-1)\Gamma(2k_2-1)}}$$

where $\Gamma(x)$ is the Gamma function. The asymptotics are

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2-\sqrt{2}}{\sqrt{\pi}} 2\sqrt{n} + O(1)} \tag{1.1}$$

Furthermore, the CLT holds:

$$\frac{N_{\mathbb{R}} - \mathbb{E}(N_{\mathbb{R}})}{\sqrt{\mathrm{Var}(N_{\mathbb{R}})}} \xrightarrow{d} \mathcal{N}(0,1), \qquad n \to \infty.$$

So no bounded variance CLT for this linear statistic.

Smooth linear statistics of the real eigenvalues

Smooth linear statistics of the real eigenvalues

Now consider the linear statistic

$$R_N[\phi] = \sum_{j=1}^{N_{\mathbb{R}}} \phi(\lambda_j)$$

Smooth linear statistics of the real eigenvalues

Now consider the linear statistic

$$R_N[\phi] = \sum_{j=1}^{N_{\mathbb{R}}} \phi(\lambda_j)$$

Theorem (Kopel '15, S. '15)

Let ϕ be either:

- *K. Any smooth test function such that for some $\delta > 0$ we have $\mathrm{supp}(\phi) \subset (-1+\delta, 1-\delta)$.*
- *S. Any even polynomial.*

Then we have the central limit theorem:

$$n^{-1/4}(R_N[\phi] - \mathbb{E}(R_N[\phi])) \xrightarrow{d} \mathcal{N}\left(0, \frac{2-\sqrt{2}}{\sqrt{\pi}} \int_{-1}^{1} \phi(x)^2 \, dx\right).$$

Smooth linear statistics of the real eigenvalues

Now consider the linear statistic

$$R_N[\phi] = \sum_{j=1}^{N_{\mathbb{R}}} \phi(\lambda_j)$$

Theorem (Kopel '15, S. '15)

Let ϕ be either:

- *K. Any smooth test function such that for some $\delta > 0$ we have $\mathrm{supp}(\phi) \subset (-1+\delta, 1-\delta)$.*
- *S. Any even polynomial.*

Then we have the central limit theorem:

$$n^{-1/4}(R_N[\phi] - \mathbb{E}(R_N[\phi])) \xrightarrow{d} \mathcal{N}\left(0, \frac{2-\sqrt{2}}{\sqrt{\pi}} \int_{-1}^{1} \phi(x)^2 \, dx\right).$$

Phil Kopel also proved this under a (fourth) moment matching hypothesis.

Proof starting point

Proof starting point

Lemma

The moment generating function of any even linear statistic is a determinant:

$$\boxed{\mathbb{E}e^{s\sum_{j=1}^{N_{\mathbb{R}}} f(\lambda_j)} = \det\left(\delta_{jk} + \frac{A[e^{s(f(x)+f(y))} - 1]_{2j,2k-1}}{\sqrt{2\pi\Gamma(2j-1)\Gamma(2k-1)}}\right)_{j,k=1}^{n}}$$

Can be extracted from a result of Sinclair (2007), combined with evenness of f.

Proof starting point

Lemma

The moment generating function of any even linear statistic is a determinant:

$$\boxed{\mathbb{E}e^{s\sum_{j=1}^{N_{\mathbb{R}}} f(\lambda_j)} = \det\left(\delta_{jk} + \frac{A[e^{s(f(x)+f(y))} - 1]_{2j,2k-1}}{\sqrt{2\pi\Gamma(2j-1)\Gamma(2k-1)}}\right)_{j,k=1}^{n}}$$

Can be extracted from a result of Sinclair (2007), combined with evenness of f. Scalar product:

$$A[\psi]_{jk} = \frac{1}{2}\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\ \psi(x)\psi(y)e^{-x^2/2-y^2/2}P_{j-1}(x)P_{k-1}(y)\mathrm{sign}(y-x)$$

where

$$P_{2j}(x) = x^{2j}$$
$$P_{2j+1}(x) = x^{2j+1} - 2jx^{2j-1}$$

Compute cumulants

Compute cumulants

Key idea: Use $\underline{\log\det = \mathrm{Tr}\log}$!

Compute cumulants

Key idea: Use $\underline{\log\det = \mathrm{Tr}\log}$!

Find that

Lemma

The pth cumulant of the linear statistic $\sum_{j=1}^{N_{\mathbb{R}}} f(\lambda_j)$ is

$$\kappa_p = p! \sum_{m=1}^{p} \frac{(-1)^{q+1}}{q} \sum_{\nu_1+\ldots+\nu_q=p} \frac{\mathrm{Tr}(M_n^{(\nu_1)}[f]\ldots M_n^{(\nu_q)}[f])}{\nu_1!\ldots\nu_q!}$$

where $M_n^{(\nu)}[f]_{j,k} = A[(f(x)+f(y))^{\nu}]_{2j,2k-1}$, $j,k = 1,\ldots,n$.

Compute cumulants

Key idea: Use $\underline{\log\det = \mathrm{Tr}\log}$!

Find that

Lemma

The pth cumulant of the linear statistic $\sum_{j=1}^{N_{\mathbb{R}}} f(\lambda_j)$ is

$$\kappa_p = p! \sum_{m=1}^{p} \frac{(-1)^{q+1}}{q} \sum_{\nu_1+\ldots+\nu_q=p} \frac{\mathrm{Tr}(M_n^{(\nu_1)}[f]\ldots M_n^{(\nu_q)}[f])}{\nu_1!\ldots\nu_q!}$$

where $M_n^{(\nu)}[f]_{j,k} = A[(f(x)+f(y))^{\nu}]_{2j,2k-1}$, $j,k = 1,\ldots,n$.

Proof proceeds by expanding the above trace. Estimates in the limit $n \to \infty$ obtained using complex analysis.

See also work of Kanzieper, Poplavskyi, Timm, Tribe, Zaboronski '15.

Compute cumulants

Key idea: Use $\underline{\log\det = \mathrm{Tr}\log}$!

Find that

Lemma

The pth cumulant of the linear statistic $\sum_{j=1}^{N_{\mathbb{R}}} f(\lambda_j)$ is

$$\kappa_p = p! \sum_{m=1}^{p} \frac{(-1)^{q+1}}{q} \sum_{\nu_1+\ldots+\nu_q=p} \frac{\mathrm{Tr}(M_n^{(\nu_1)}[f]\ldots M_n^{(\nu_q)}[f])}{\nu_1!\ldots\nu_q!}$$

where $M_n^{(\nu)}[f]_{j,k} = A[(f(x)+f(y))^{\nu}]_{2j,2k-1}$, $j,k = 1,\ldots,n$.

Proof proceeds by expanding the above trace. Estimates in the limit $n \to \infty$ obtained using complex analysis.

See also work of Kanzieper, Poplavskyi, Timm, Tribe, Zaboronski '15.

Ultimately obtain $\kappa_p = O(\sqrt{n})$ as $n \to \infty$.

Conclusion and discussion

Conclusion and discussion

We found that $\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{2Nm/\pi} + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.

Conclusion and discussion

We found that $\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{2Nm/\pi} + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.

There are many possible related questions:

- Fluctuations: Variance of $N_{\mathbb{R}}^{(m)}$ and is there a CLT? Conjecture: $\mathrm{Var}(N_{\mathbb{R}}^{(m)}) \sim \mathrm{Var}(N_{\mathbb{R}}^{(1)})|_{N \to Nm}$.
- Large deviations: What is the probability that the product has *no real eigenvalues*? Conjecture: $N \to Nm$.
- What is happening near the origin: distribution of the smallest real eigenvalue of the product?

Conclusion and discussion

We found that $\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{2Nm/\pi} + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.

There are many possible related questions:

- Fluctuations: Variance of $N_{\mathbb{R}}^{(m)}$ and is there a CLT? Conjecture: $\mathrm{Var}(N_{\mathbb{R}}^{(m)}) \sim \mathrm{Var}(N_{\mathbb{R}}^{(1)})|_{N \to Nm}$.
- Large deviations: What is the probability that the product has *no real eigenvalues*? Conjecture: $N \to Nm$.
- What is happening near the origin: distribution of the smallest real eigenvalue of the product?

Thanks for listening!

Potential postdocs or PhD students in RMT? Please contact me.