The real spectrum of a product of Ginibre matrices

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Sums and products of random matrices

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Let $G$ be an $N \times N$ real matrix whose entries are i.i.d. standard normal random variables. As a probability density on matrix space: $P(G) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{1}{2} \text{Tr}(GG^T)\right)$.

Clearly invariant under orthogonal transformations $G \rightarrow O_1GO_2$ where $O_1, O_2 \in O(N)$.

Known as the Ginibre orthogonal ensemble (GinOE).

The Ginibre unitary ensemble (GinUE) instead consists of complex variables whose real and imaginary parts are $N(0, 1/2)$. 
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Eigenvalues of the GinOE
Early discoveries for the Ginibre ensemble

Jean Ginibre's 1965 paper introduced a total of three ensembles: GinUE, GinSE and GinOE, in increasing order of difficulty.

Eigenvalue distribution: Ginibre completely solved the GinUE, partially solved GinSE and left GinOE unsolved.

Lehmann and Sommers 1991. Joint PDF of GinOE complex eigenvalues $x_j + iy_j$ and $N_R$ real eigenvalues $\lambda_j$:

$$\frac{1}{c^N |\Delta|} e^{\frac{1}{N} \sum_{j=1}^{N_R} \frac{\lambda_j^2}{2}} \prod_{j=1}^{N_R} \text{erf} \left( y_j \sqrt{2} \right)$$

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1 \cdot N \prod_{j=1}^{N-R} \text{erf}(y_j \sqrt{2}) 
+ \frac{1}{2} N \cdot N \sum_{j=1}^{N-R} \left(y_j^2 - x_j^2 \right) - N \sum_{j=1}^{N-R} \lambda_j^2 / 2
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How many eigenvalues of a random real matrix are real?
Theorem (Edelman, Kostlan and Shub ’94)

For an $N \times N$ real Ginibre matrix $G$, one has

$$\mathbb{E}(N_R) = \sqrt{2N/\pi} + O(1) \quad N \to \infty$$

and the convergence to the uniform law

$$\frac{1}{\mathbb{E}(N_R)} \mathbb{E} \left[ \sum_{j=1}^{N} \delta(\lambda_j - x) \right] \to \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$
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**Products:** What is the analogue of this result for products of independent Ginibre random matrices?
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Fluctuations: Variance and central limit theorem?
Products
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Let $G_1, \ldots, G_m$ be $m$ independent real Ginibre matrices of size $N \times N$ and set $X_m = N^{-m/2} G_1 G_2 \ldots G_m$. 

Theorem (S. '17) For every fixed $m \in \mathbb{N}$ we have $E \left( \sum_{j=1}^m \delta(\lambda_j - \lambda_j) \right) \rightarrow \left\{ \frac{1}{2} \biggm| \lambda \biggm| - \frac{1}{2} \right\}$ as $N \rightarrow \infty$.

Compare to known density for the complex eigenvalues (Burda et al., Götze and Tikhomirov, O'Rourke and Soshnikov 2010): $p(z) = \frac{1}{m \pi} \frac{|z|^{-2}}{|z| < 1}$. 
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\frac{1}{\mathbb{E}(N_R^{(m)})} \mathbb{E} \left[ \sum_{j=1}^{N_R^{(m)}} \delta(\lambda_j - \lambda) \right] \rightarrow \begin{cases} 
\frac{1}{2m} |\lambda|^{\frac{1}{m}-1} & |\lambda| < 1 \\
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Remarks

This theorem (but without the leading constant $\sqrt{m}$) was conjectured by Forrester and Ipsen (2016).

Philosophy: $G_1 \ldots G_m \sim G_m$ up to symmetry? (Burda et al. 2010)

The error term $O(\log N)$ can easily be replaced with $O(1)$. True error should be $O(N^{-1/2})$.

Proof still works for $m = N\delta$ for some small $\delta > 0$. What if $m = cN$ for some large constant $c > 0$?

Idea of the proof is to compute moments and show that

$$\lim_{N \to \infty} \frac{1}{E(N(m)R)} \sum_{j=1}^{N(m)} \lambda_{k,j} = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$
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\lim_{N \to \infty} \frac{1}{\mathbb{E}(\mathcal{N}_R^{(m)})} \mathbb{E} \left[ \sum_{j=1}^{\mathcal{N}_R^{(m)}} \chi_j^k \right] = \begin{cases} 
\frac{1}{1+mk}, & k \text{ even} \\
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Products of random variables

Let $Y_1, \ldots, Y_m$ be i.i.d. standard Gaussians. What is the density of the product $X = Y_1 Y_2 \ldots Y_m$?

$$w_m(x) := \int \mathbb{R}^m \prod_{j=1}^m dx_j e^{-x_j^2/2} \delta(x - x_1 x_2 \ldots x_m) = G_{m,0}^{0,m}(0,\ldots,0|\bigg| x^2_1 x^2_2 \ldots x^2_m).$$

where the Meijer G-function is

$$G_{m,n}^{p,q}(a_1,\ldots,a_p|b_1,\ldots,b_q|z) = \frac{1}{2\pi i} \int_{\gamma} \prod_{j=1}^m \Gamma(b_j - s) \prod_{n=1}^p \Gamma(1 - a_j + s) \prod_{q=m+1}^{p+n} \Gamma(1 - b_j + s) z^s ds$$

The contour $\gamma$ connects $-i\infty$ to $+i\infty$ such that all poles of $\Gamma(b_j - s)$ on right and $\Gamma(1 - a_k + s)$ on left.

Next: the case $N > 1$. 
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Let \( Y_1, \ldots, Y_m \) be i.i.d. standard Gaussians. What is the density of the product \( X_{m}^{(N=1)} = Y_1 Y_2 \ldots Y_m \)?
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Theorem (Ipsen and Kieburg '14, Forrester and Ipsen '16)

The real eigenvalues of the matrix product $G_1 \ldots G_m$ form a Pfaffian point process with correlation kernel given by

$$K(x, y) = D(x, y)S(x, y) - S(y, x)I(x, y)$$

where

$$S(x, y) = N - 2 \sum_{j=0}^{\infty} w_m(x) x_j (2 \sqrt{2/\pi})^m A_j(y)$$

and

$$A_j(y) = \int_{\mathbb{R}} w_m(v) \text{sgn}(y-v) v_j dv.$$
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In particular, the desired moments are just

$$M_{k,N} := \mathbb{E} \left[ \sum_{j=1}^{N_R(m)} \chi_j^k \right] = \int_{\mathbb{R}} x^k S(x, x) \, dx$$
Moments and Meijer G

The last integral splits in two pieces

\[ M_2 k, N(\text{m}) = M(1) 2 k, N(\text{m}) - M(2) 2 k, N(\text{m}) \]

where

\[ M(1) 2 k, N(\text{m}) = N - mk(\sqrt{\pi} (2j + k)! m(a_j + 1, j + k + 1, j + 1)) \]

\[ M(2) 2 k, N(\text{m}) = N - mk N/2 - 2 \sum_{j=0}^2 (2j + 1 + k) m(a_j + 2, j + 2, j + 1) \]

Here \( a_j, k \) is a particular case of the Meijer-G function

\[ a_j, k = G_{m+1, m+1}^{m+1, m+1}(3/2 - j, ..., 3/2 - j, 10, ..., k | | 1) = 1/2 \pi i \int_\gamma (\Gamma(k-s)\Gamma(-1/2+j+s)) m-s \, ds \]
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\[ M_{2k,N}(m) = M_{2k,N}^{(1)}(m) - M_{2k,N}^{(2)}(m) \] where

\[ M_{2k,N}^{(1)}(m) = N^{-mk} \sum_{j=0}^{(N-2)/2} \frac{2^{(2j+k)m}}{(\sqrt{\pi}(2j)!)^m} (a_{j+1,j+k+1} + a_{j+k+1,j+1}) \]

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a_{j,k} = G_{m+1,m}^{m+1,0} \left( \begin{array}{c} 3/2-j,\ldots,3/2-j,1 \\ 0,k,\ldots,k \end{array} \right | 1 \right) = \frac{1}{2\pi i} \int_{\gamma} \frac{(\Gamma(k-s)\Gamma(-1/2+j+s))^{m}}{-s} ds
\]
Saddle point analysis

The formula
\[
\Gamma(k - s) \Gamma(-1/2 + j + s) \Gamma(j + k - 1/2) = \int_0^\infty t^{k-s-1} \left(1 + \frac{t}{k+j+1/2}\right)^{k+j+3/2} dt
\]
implies that $a_{j+1}$, $j+k+1$ can be written
\[
\Gamma(2j+k+3/2) m \int_1^\infty dx m x^{m-1} \prod_{l=1}^m \left[ \int_0^\infty dx (x^l + 1)^{j+1/2} \left(1 + \frac{x^l}{x^l+1}\right)^{-j-k-3/2} x^{j+k+1} \right]
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Asymptotics as $j \to \infty$ with fixed $k$, $m$: Use the classical (multi-dimensional) saddle point method. Because of cancellations one has to get the first sub-leading correction.
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\Gamma(2j + k + 3/2)^m \int_1^\infty \frac{dx_m}{x_m} \prod_{l=1}^{m-1} \left[ \int_0^\infty \frac{dx_l}{x_l} \frac{(x_l/x_{l+1})^{j+1/2}}{(1 + x_l/x_{l+1})^{2j+k+3/2}} \right] \frac{x_1^{j+k+1}}{(1 + x_1)^{2j+k+3/2}}
\]

\[
= \Gamma(2j + k + 3/2)^m \int_1^\infty \int_{[0,\infty)^{m-1}} e^{i\Phi(x)} F(x) \, dx_1 \ldots dx_m
\]

Asymptotics as \( j \to \infty \) with fixed \( k, m \): Use the classical (multi-dimensional) saddle point method.

Because of cancellations one has to get the first sub-leading correction.
Possible alternative approach

\begin{equation}
S(x, y) = \int \limits_{-\infty}^{\infty} (x - v) \text{sgn}(y - v) w_r(x) w_r(v) N - 2 \sum_{j=0}^{m} (xv)^j (j!) \, dv
\end{equation}

Idea: Compute asymptotics of $S(x/\sqrt{N}, y/\sqrt{N})$. Then use:

\begin{equation}
E(N(m) R) = \sqrt{\frac{m}{N}} \int \limits_{-\infty}^{\infty} S(u/\sqrt{N}, u/\sqrt{N}) \, du
\end{equation}

Difficulties: near the edge $|x| = \pm 1$ and $|uv| = \pm 1/\sqrt{N}$. Even $m = 1!$ Uniformity? This is overcome by directly integrating $S(x, x)$ in our case.
Possible alternative approach

The correlation kernel of real eigenvalues is

\[
S(x, y) = \int_{-\infty}^{\infty} (x - v) \text{sgn}(y - v) w_r(x) w_r(v) \sum_{j=0}^{N-2} \frac{(xv)^j}{(j!)^m} dv
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$$\mathbb{E}(N^{(m)}_\mathbb{R}) = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} S(u/\sqrt{N}, u/\sqrt{N}) du$$

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Linear statistics of random matrix models

The linear eigenvalue statistic $X_N[f] = \sum_{j=1}^{N} f(z_j)$ is usually studied in two main cases.

1. Hermitian ensembles: Bounded variance CLT with the $H^{1/2}$ noise (e.g. Johansson '98):
   \[
   \lim_{N \to \infty} \text{Var}(X_N[f]) = \sum_{k=1}^{\infty} |k||\hat{c}_k[f]|^2
   \]

2. Non-Hermitian ensembles: Bounded variance CLT with GFF-type structure (Rider and Virag '07):
   \[
   \lim_{N \to \infty} \text{Var}(X_N[f]) = \int U |\nabla f|^2 \, d^2z + \sum_{k=1}^{\infty} |k||\hat{f}_k[k]|^2
   \]

What if we restrict to the real axis? For simplicity, consider $m = 1$. 
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Theorem (S. '15)

The variance of the total number of real eigenvalues of a real $(2^n 	imes 2^n)$ Gaussian random matrix is given by the following explicit formula

$$\text{Var}(N_R) = 2 \sqrt{2} \sqrt{\pi} \sum_{k=1} \frac{\Gamma(2k-3/2)}{\Gamma(2k-1)} - 2 \pi \sum_{k_1, k_2} \frac{\Gamma(k_1+k_2-3/2)}{\Gamma(2k_1-1)\Gamma(2k_2-1)}$$

where $\Gamma(x)$ is the Gamma function.

Furthermore, the CLT holds:

$$N_R - \mathbb{E}(N_R) \sqrt{\text{Var}(N_R)} \xrightarrow{d} \mathcal{N}(0, 1), \quad n \to \infty.$$
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So no bounded variance CLT for this linear statistic.
Smooth linear statistics of the real eigenvalues

Now consider the linear statistic

\[ R_N[\phi] = \sum_{j=1}^{N} \phi(\lambda_j) \]

Theorem (Kopel '15, S. '15)

Let \( \phi \) be either:

▶ K. Any smooth test function such that for some \( \delta > 0 \) we have \( \text{supp}(\phi) \subset (-1 + \delta, 1 - \delta) \).

▶ S. Any even polynomial.

Then we have the central limit theorem:

\[ n^{-1/4} \left( R_N[\phi] - E(R_N[\phi]) \right) \xrightarrow{d} N(0, 2 - \sqrt{2} \sqrt{\pi} \int_{-1}^{1} \phi(x)^2 \, dx) \]

Phil Kopel also proved this under a (fourth) moment matching hypothesis.
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\[ \frac{n - 1}{4} (R_N[\phi] - E(R_N[\phi])) \rightarrow N(0, 2 - \sqrt{2}/\sqrt{\pi} \int_{-1}^{1} \phi(x)^2 dx) \]

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Proof starting point
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Lemma
The moment generating function of any even linear statistic is a determinant:

\[ \mathbb{E} e^{s \sum_{j=1}^{N} f(\lambda_j)} = \det \left( \delta_{jk} + \frac{A[e^{s(f(x)+f(y))} - 1]_{2j,2k-1}}{\sqrt{2\pi} \Gamma(2j - 1) \Gamma(2k - 1)} \right)_{j,k=1} \]

Can be extracted from a result of Sinclair (2007), combined with evenness of \( f \).
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\]

Can be extracted from a result of Sinclair (2007), combined with evenness of \( f \). Scalar product:

\[
A[\psi]_{jk} = \frac{1}{2} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \psi(x)\psi(y)e^{-x^2/2-y^2/2} P_{j-1}(x) P_{k-1}(y) \text{sign}(y-x)
\]

where

\[
P_{2j}(x) = x^{2j} \]
\[
P_{2j+1}(x) = x^{2j+1} - 2jx^{2j-1}
\]
Compute cumulants

Key idea: Use log det = Tr log

Find that

Lemma

The $p$th cumulant of the linear statistic $\sum_{R=1}^{N} f(\lambda_j)$ is

$\kappa_p = p! \sum_{\nu_1 + \ldots + \nu_q = p} \nu_1! \ldots \nu_q! \text{Tr} \left( M(\nu_1) n_{[f]} \ldots M(\nu_q) n_{[f]} \right)$

where $M(\nu) n_{[f]} j_k, k = A \left( f(x) + f(y) \right)_{\nu_{2j_2 + \nu_2}} j_k, k = 1, \ldots, n$.

Proof proceeds by expanding the above trace. Estimates in the limit $n \to \infty$ obtained using complex analysis.

See also work of Kanzieper, Poplavskyi, Timm, Tribe, Zaboronski '15.

Ultimately obtain $\kappa_p = O(\sqrt{n})$ as $n \to \infty$. 
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The $p^{th}$ cumulant of the linear statistic $\sum_{j=1}^{N_R} f(\lambda_j)$ is

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where $M_n^{(\nu)}[f]_{j,k} = A[(f(x) + f(y))^\nu]_{2j,2k-1}$, $j, k = 1, \ldots, n$. 

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Conclusion and discussion

We found that $E(N^m) = \sqrt{2N}\pi/\log(N) + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.

There are many possible related questions:

▶ Fluctuations: Variance of $N^m R$ and is there a CLT? Conjecture: $\text{Var}(N^m R) \sim \text{Var}(N^1 R)$ as $N \to N^m$.

▶ Large deviations: What is the probability that the product has no real eigenvalues? Conjecture: $N \to N^m$.

▶ What is happening near the origin: distribution of the smallest real eigenvalue of the product?

Thanks for listening!

Potential postdocs or PhD students in RMT? Please contact me.
Conclusion and discussion

We found that $\mathbb{E}(N^{(m)}_R) = \sqrt{2Nm/\pi} + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.
Conclusion and discussion

We found that $\mathbb{E}(N_R^{(m)}) = \sqrt{2Nm/\pi} + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.

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  Conjecture: $\text{Var}(N_R^{(m)}) \sim \text{Var}(N_R^{(1)})|_{N \to Nm}$.

- Large deviations: What is the probability that the product has no real eigenvalues? Conjecture: $N \to Nm$.

- What is happening near the origin: distribution of the smallest real eigenvalue of the product?
Conclusion and discussion

We found that $\mathbb{E}(N_{\mathbb{R}}^{(m)}) = \sqrt{2Nm/\pi} + O(\log(N))$ and proved weak convergence of the real eigenvalue distribution.

There are many possible related questions:

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