

# Large complex correlated Wishart matrices: Local behavior of the eigenvalues

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Random product matrices, ZiF – Bielefeld, 2016

# Tracy-Widom's law

- Largest eigenvalue of random hermitian matrix models
- Non-colliding processes
- Longest increasing subsequence
- Random tilings (Aztec diamond & Hexagone)
- KPZ equation (with  $\delta_0$  initial condition)

# Tracy-Widom's law

- Tracy-Widom's law: law of the largest particle of the determinantal point process on  $\mathbb{R}$  with kernel

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}$$

- If  $X \sim$  Tracy-Widom, then for any  $s \in \mathbb{R}$ ,

$$\mathbb{P}(X \leq s) = \det(I - K_{\text{Ai}})_{L^2(s, \infty)}.$$

It can be expressed in terms of a solution of Painlevé II equation [Tracy-Widom '94]

# The matrix model

**Complex correlated Wishart matrix:**

$$\mathbf{W}_N = \frac{1}{N} \mathbf{G} \boldsymbol{\Sigma} \mathbf{G}^*$$

where

$\mathbf{G}$  :  $N \times n$  matrix with independent  $\mathcal{N}_{\mathbb{C}}(0, 1)$  entries

$\boldsymbol{\Sigma}$  :  $n \times n$  symmetric positive definite matrix (determinist)

**Asymptotic regime:**  $n, N \rightarrow \infty$  and  $n/N \rightarrow \gamma \in (0, \infty)$ .

## References

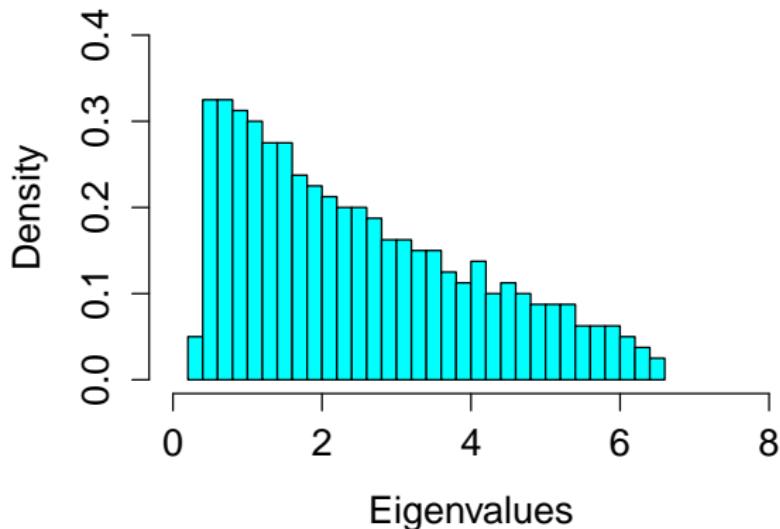
- ① Large complex correlated Wishart matrices: Fluctuations and asymptotic Independence at the edges, with [Walid Hachem](#) and [Jamal Najim](#). Annals of probability (2016)
- ② Large complex correlated Wishart matrices: The Pearcey kernel and expansion at the hard edge, with Walid Hachem and Jamal Najim. Electronic Journal of Probability (2016)
- ③ A [survey](#) on the eigenvalues local behavior of large complex correlated Wishart matrices, with Walid Hachem and Jamal Najim. ESAIM: Proceedings and surveys (2015)

Non-correlated case:  $\Sigma = I_n$

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**Example:**  $N = 400$ ,  $n = 1000$  ( $\gamma > 1$ )

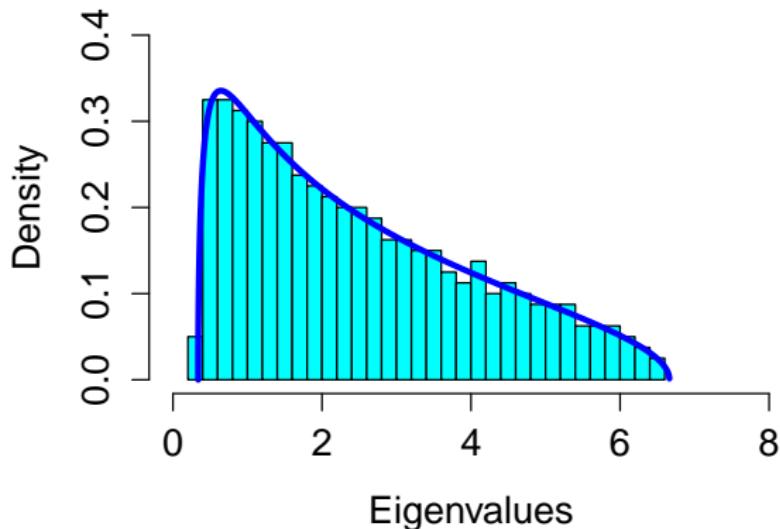


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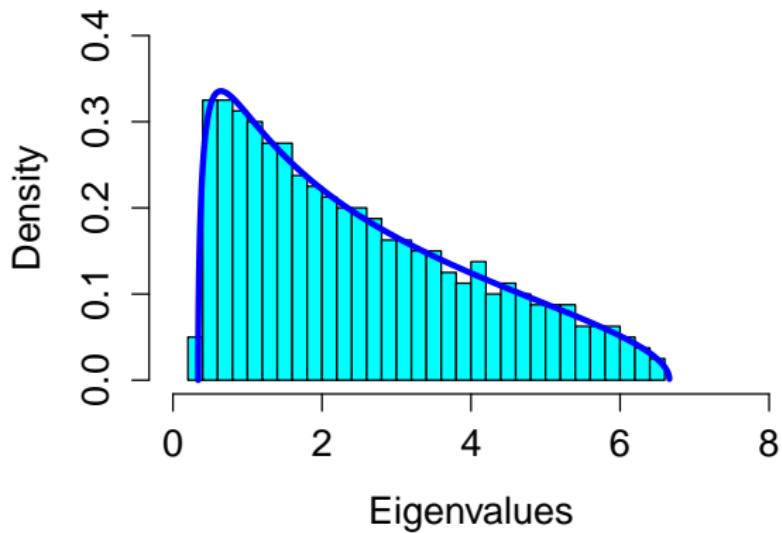
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# Marčenko-Pastur law

$$\rho(x) = \frac{1}{2\pi x} \sqrt{(\mathcal{B} - x)(x - \mathcal{A})}, \quad \mathcal{A} = (1 - \sqrt{\gamma})^2, \quad \mathcal{B} = (1 + \sqrt{\gamma})^2$$



## Non-correlated case: $\Sigma = I_n$

- [Johansson '00]

$$\text{max. eigenvalue} \simeq \mathcal{B} + \frac{\sigma}{N^{2/3}} \text{ Tracy-Widom}$$

- [Borodin-Forrester '03]

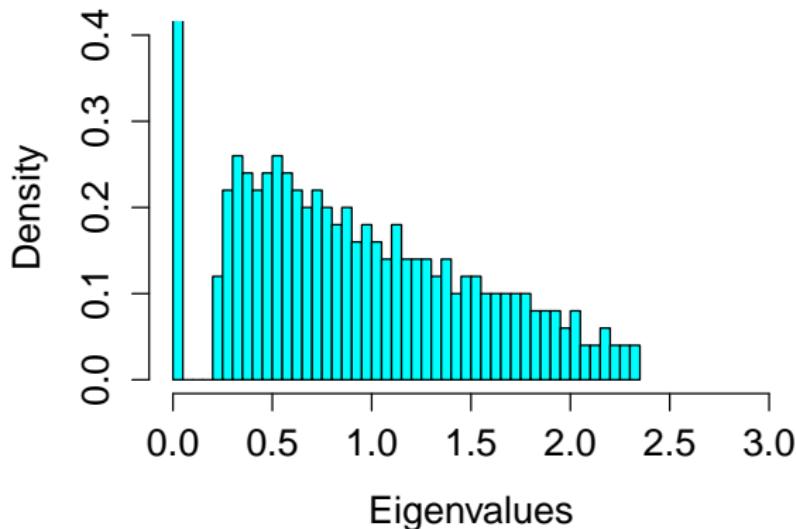
$$\text{min. eigenvalue} \simeq \mathcal{A} - \frac{\sigma}{N^{2/3}} \text{ Tracy-Widom}$$

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**Example:**  $N = 1000$ ,  $n = 300$  ( $\gamma < 1$ )

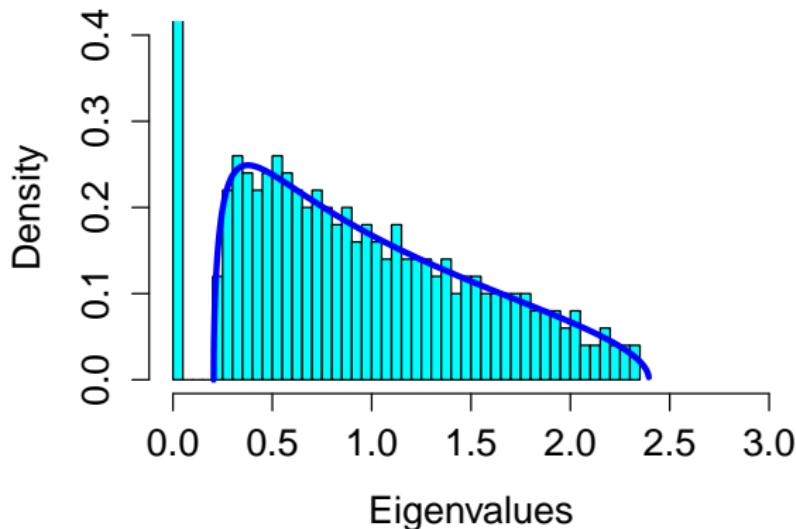


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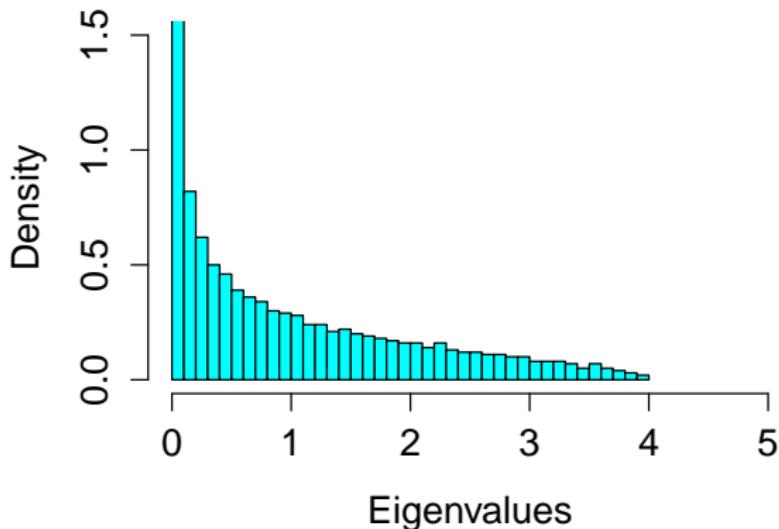


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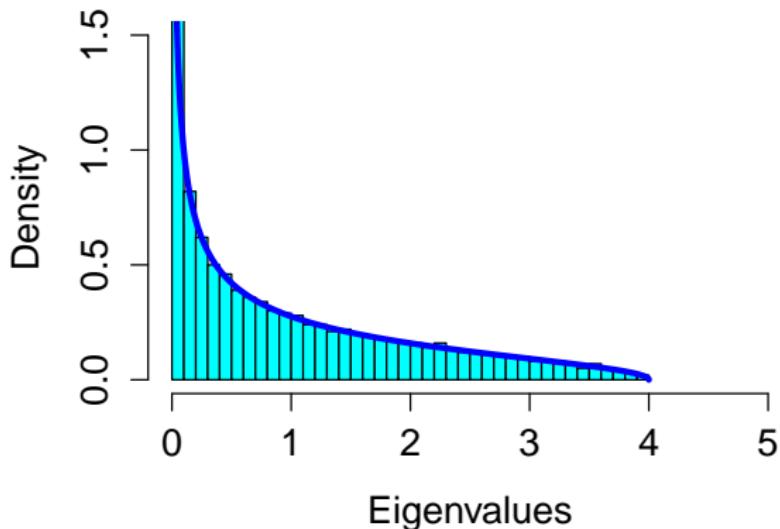


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## Non-correlated case: $\Sigma = I_n$

- When  $\gamma = 1$ , the Marčenko-Pastur law reads

$$\rho(x) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}}$$

- [Forrester '93] If  $n = N + \alpha$  with  $\alpha \in \mathbb{Z}$  fixed then

$$\text{min. eigenvalue} \simeq \frac{\sigma}{N^2} \text{Bessel}(\alpha)$$

# Bessel( $\alpha$ )

- Bessel( $\alpha$ ): law of the smallest particle of the determinantal point process on  $\mathbb{R}_+$  with kernel

$$K_{\text{Be}(\alpha)}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x - y)}$$

- If  $X \sim \text{Bessel}(\alpha)$ , then for any  $s > 0$ ,

$$\mathbb{P}(X \geq s) = \det(I - K_{\text{Be}(\alpha)})_{L^2(0, s)}$$

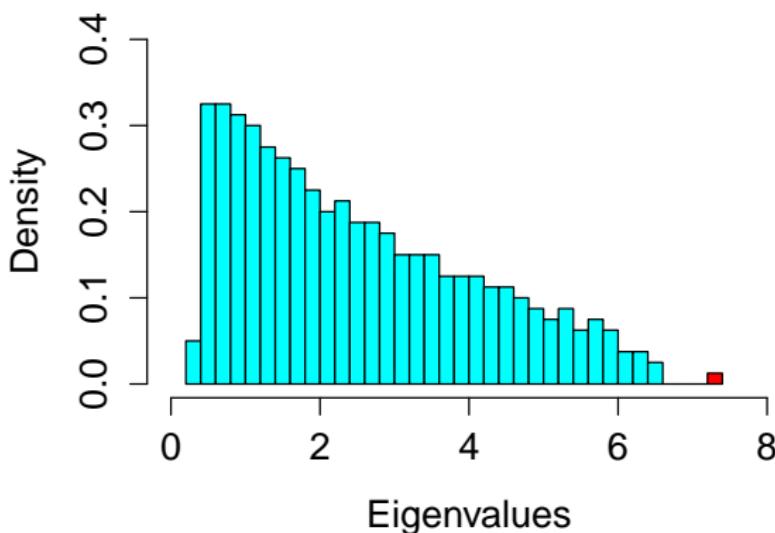
and it can be represented in terms of a solution of Painlevé III equation [Tracy-Widom '94]

$$\Sigma = I_n + \text{finite rank perturbation}$$

**Finite rank perturbation** [Baik-Ben Arous-Péché '05]:

$$\Sigma = \text{diag}(\underbrace{1, \dots, 1}_{n-1}, 1 + \varepsilon).$$

**Example:**  $N = 400$ ,  $n = 1000$ ,  $\varepsilon = 3$



$$\Sigma = I_n + \text{finite rank perturbation}$$

**Finite rank perturbation** [Baik-Ben Arous-Péché '05]:

$$\Sigma = \text{diag}(\underbrace{1, \dots, 1}_{n-1}, 1 + \varepsilon).$$

**Fact:** Same global behavior. But,

- When  $\varepsilon < \varepsilon_{crit}$ ,

$$\text{max. eigenvalue} \simeq \mathcal{B} + \frac{\sigma}{N^{2/3}} \text{ Tracy-Widom}$$

- When  $\varepsilon = \varepsilon_{crit}$ ,

$$\text{max. eigenvalue} \simeq \mathcal{B} + \frac{\sigma}{N^{2/3}} \text{ deformed TW}$$

- When  $\varepsilon > \varepsilon_{crit}$ ,

$$\text{max. eigenvalue} \simeq \mathcal{B}_{jump} + \frac{\sigma}{N^{1/2}} \text{ Gaussian}$$

where  $\mathcal{B}_{jump} > \mathcal{B} \Rightarrow \text{outlier}$

## General $\Sigma$

**General  $\Sigma$**  with (positive) eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$ . Assume

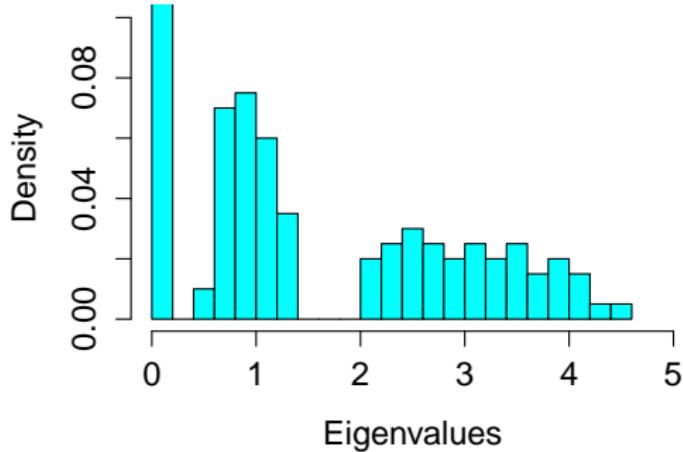
$$\frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j} \xrightarrow[N \rightarrow \infty]{*} \nu \quad \text{and} \quad 0 < \underline{\lim} \lambda_1 \leq \overline{\lim} \lambda_n < +\infty$$

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**Example:**  $N = 1000$ ,  $n = 100$ ,  $\nu = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_3$



## General $\Sigma$

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**Global asymptotics** [Marčenko-Pastur '67]:

- The eigenvalue distribution of  $\mathbf{W}_N$  has a limit  $\mu$ .
- The Stieltjes transform of  $\mu$  satisfies a fixed point equation
- $\mu$  only depends on  $\nu$  and  $\gamma$
- $\mu = (1 - \gamma)_+ \delta_0 + \rho(x) dx$
- $\text{Support}(\mu)$  is compact but may not be connected

## General $\Sigma$

[El Karoui '07, Onatski '08]

If the rightmost edge  $\mathcal{B}$  is regular and there is no outliers,

$$\text{max. eigenvalue} \simeq \mathcal{B} + \frac{\sigma}{N^{2/3}} \text{ Tracy-Widom}$$

**Remark:** [Silverstein-Choi '95]

$\mathcal{B}$  regular  $\Rightarrow \rho(x) \sim (\mathcal{B} - x)^{1/2}$  as  $x \rightarrow \mathcal{B}_-$

## General $\Sigma$

[Hachem-H.-Najim '16]  $(x_1 \leq \dots \leq x_N$  eigenvalues of  $\mathbf{W}_N)$

For any regular right edge  $\mathcal{E}$ ,

- There exists a deterministic sequence  $\Phi(N)$  such that a.s.

$$x_{\Phi(N)} \rightarrow \mathcal{E}, \quad \underline{\lim} x_{\Phi(N)+1} > \mathcal{E}$$

- Moreover,

$$x_{\Phi(N)} \simeq \mathcal{E} + \frac{\sigma}{N^{2/3}} \text{ Tracy-Widom}$$

Similar results for any regular positive left edge.

# General $\Sigma$

[Hachem-H.-Najim '16]

- The leftmost edge is the origin iff  $\gamma = 1$ , and in this case

$$\rho(x) \sim |x|^{-1/2} \quad \text{as } x \rightarrow 0_+$$

- If  $n = N + \alpha$  with  $\alpha \in \mathbb{Z}$  fixed then

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Moreover, if

$$F_\alpha(s) := \mathbb{P}(\text{Bessel}(\alpha) \geq s) = \det(I - K_{\text{Be}(\alpha)})_{L^2(0,s)},$$

then for any  $s > 0$ ,

$$\mathbb{P}(\text{min. eigenvalue} \geq s \frac{\sigma}{N^2}) = F_\alpha(s) + \frac{c_N}{N} \alpha s \frac{d}{ds} F_\alpha(s) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

where  $c_N$  is explicit and of order one.

## General $\Sigma$

[Hachem-H.-Najim '16]

- Given any finite family of positive regular edges, the associated Tracy-Widom fluctuations are asymptotically independent

# General $\Sigma$

[Hachem-H.-Najim '16]

- Given any finite family of **positive regular edges**, the associated Tracy-Widom fluctuations are asymptotically independent
- Convergence and fluctuations for the **condition number**:
  - When  $\gamma > 1$ , if the leftmost edge  $\mathcal{A}$  and the rightmost edge  $\mathcal{B}$  are regular, then

$$\kappa_N = \frac{x_{\max}}{x_{\min}} \simeq \frac{\mathcal{B}}{\mathcal{A}} + \frac{1}{N^{2/3}} \left( \frac{\sigma}{\mathcal{A}} \text{TW} + \frac{\sigma' \mathcal{B}}{\mathcal{A}^2} \text{TW}' \right)$$

- When  $n = N + \alpha$  with  $\alpha$  fixed,

$$\kappa_N = \frac{x_{\max}}{x_{\min}} \simeq \frac{N^2}{\sigma} \frac{1}{\text{Bessel}(\alpha)}$$

## General $\Sigma$

[Hachem-H.-Najim '16]

Let  $\mathcal{C}$  be an interior point such that  $\rho(\mathcal{C}) = 0$  which is **regular**

- The density behaves like a **cusp**:

$$\rho(x) \sim |x - \mathcal{C}|^{1/3} \quad \text{as } x \rightarrow \mathcal{C}$$

- Under an extra decay assumption,

$$N^{3/4} \left( \text{eigenvalues} - \mathcal{C} \right) \xrightarrow[N \rightarrow \infty]{\text{locally}} \text{Pearcey point process}$$

## General $\Sigma$

- Pearcey point process: determinantal point process on  $\mathbb{R}$  with kernel

$$K_{\text{Pe}(\tau)}(x, y) = \frac{\phi''(x)\psi(y) - \phi'(x)\psi'(y) + \phi(x)\psi''(y) - \tau\psi(x)\psi(y)}{x - y}$$

where  $\phi$  and  $\psi$  are the Pearcey functions satisfying the differential equations

$$\phi'''(x) - \tau\phi'(x) + x\phi(x) = 0, \quad \psi'''(y) - \tau\psi'(y) - y\psi(y) = 0$$

- The gap probabilities

$$\det(I - K_{\text{Pe}(\tau)})_{L^2(s, t)}$$

satisfy PDEs [Tracy-Widom '06, Bertola-Cafasso '12,  
Adler-Cafasso-van Moerbeke '12]

## General $\Sigma$

Beyond the Gaussian case?

Recall

$$\mathbf{W}_N = \frac{1}{N} \mathbf{G} \boldsymbol{\Sigma} \mathbf{G}^*$$

where

$\mathbf{G}$  :  $N \times n$  matrix with independent  $\mathcal{N}_{\mathbb{C}}(0, 1)$  entries

$\boldsymbol{\Sigma}$  :  $n \times n$  symmetric positive definite matrix

**Local law** [Knowles-Yin '14]: One can drop the Gaussian assumption and still have independent Tracy-Widom fluctuations

# A glimpse of the proof for Tracy-Widom

- Determinantal structure [Baik-Ben Arous-Péché '05] +  
Exact separation [Bai-Silverstein '98]

⇒ Repartition function  $\simeq$  Fredholm determinant,

$$\mathbb{P}\left(\frac{N^{2/3}}{\sigma}(x_{\Phi(N)} - \mathcal{E}) \leq s\right) = \det(I - K_N)_{L^2(s, \varepsilon N^{2/3})} + o(1)$$

⇒ Enough to prove the convergence

$$K_N \xrightarrow[N \rightarrow \infty]{} K_{Ai}$$

in trace class topology

# A glimpse of the proof for Tracy-Widom

**Contour integral representation for  $K_N$ 's kernel**

$$K_N(\textcolor{blue}{x}, \textcolor{blue}{y}) = \frac{1}{(2i\pi)^2} \oint_{\Gamma} dz \oint_{\Theta} dw F_N(\textcolor{blue}{x}, \textcolor{blue}{y}; z, w),$$

where

$$F_N(\textcolor{blue}{x}, \textcolor{blue}{y}; z, w) = e^{N(\mathbf{f}(z) - \mathbf{f}(w))} \times \text{Nice}(\textcolor{blue}{x}, \textcolor{blue}{y}; z, w)$$

with

$$\mathbf{f}(z) = \text{linear} + \log(z) - \frac{1}{N} \sum_{j=1}^n \log(1 - \lambda_j z).$$

**Steepest descent:** Look for [critical/saddle points](#)  $\mathfrak{d}_N$  satisfying

$$\mathbf{f}'(\mathfrak{d}_N) = \mathbf{f}''(\mathfrak{d}_N) = 0$$

# A glimpse of the proof for Tracy-Widom

**Asymptotic analysis as  $N \rightarrow \infty$  for**

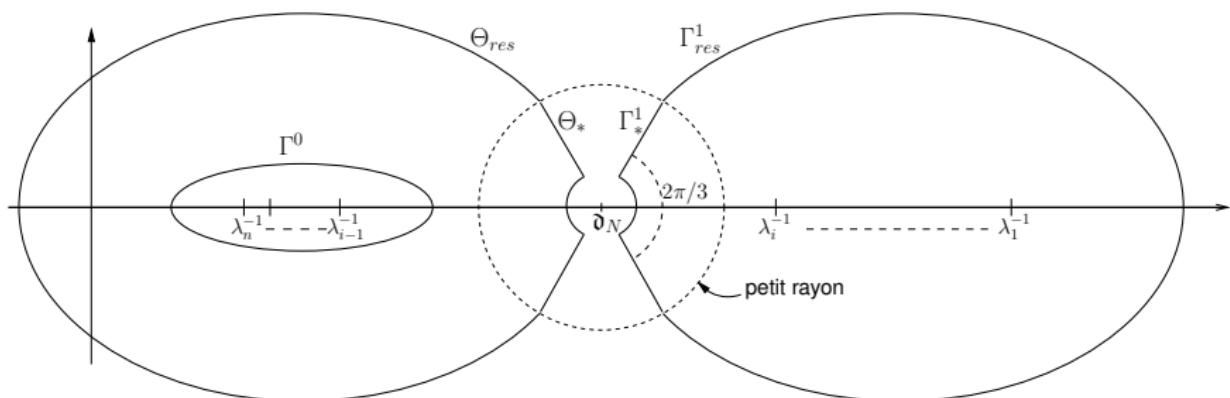
$$K_N(x, y) = \frac{1}{(2i\pi)^2} \oint_{\Gamma} dz \oint_{\Theta} dw F_N(x, y; z, w)$$

- **Local analysis around critical point  $\partial_N \Rightarrow$  Airy kernel**
  - Saddle point of order two, (almost) routine computation
- **The remaining of the integral is negligible**
  - Existence of appropriate analytic deformation of  $\Gamma$  and  $\Theta$

# A glimpse of the proof for Tracy-Widom

**Asymptotic analysis as  $N \rightarrow \infty$  for**

$$K_N(x, y) = \frac{1}{(2i\pi)^2} \oint_{\Gamma} dz \oint_{\Theta} dw F_N(x, y; z, w)$$



# A glimpse of the proof for Tracy-Widom

**Existence of the steepest descent contours?**

Non-constructive proof based on the  
maximum principle for subharmonic functions

# When universality breaks down?

What if an edge  $\mathcal{E}$  is **not regular**?

- **[Easy case]** A finite number of poles  $1/\lambda_j$ 's collapse on the critical point  $\vartheta_N \Rightarrow$  fluctuations are described by the **deformed Tracy-Widom distribution** of Baik-Ben Arous-Péché
- **[Mysterious case]** Infinite number of poles collapse onto  $\vartheta_N$ ?

**Similar situations:**

- Additive perturbations of GUE matrices **[Capitaine-Péché '16]**
- Random patterns on the Gelfand-Tsetlin cone **[Duse-Metcalfe '15]**

**Thank you for your attention!**

