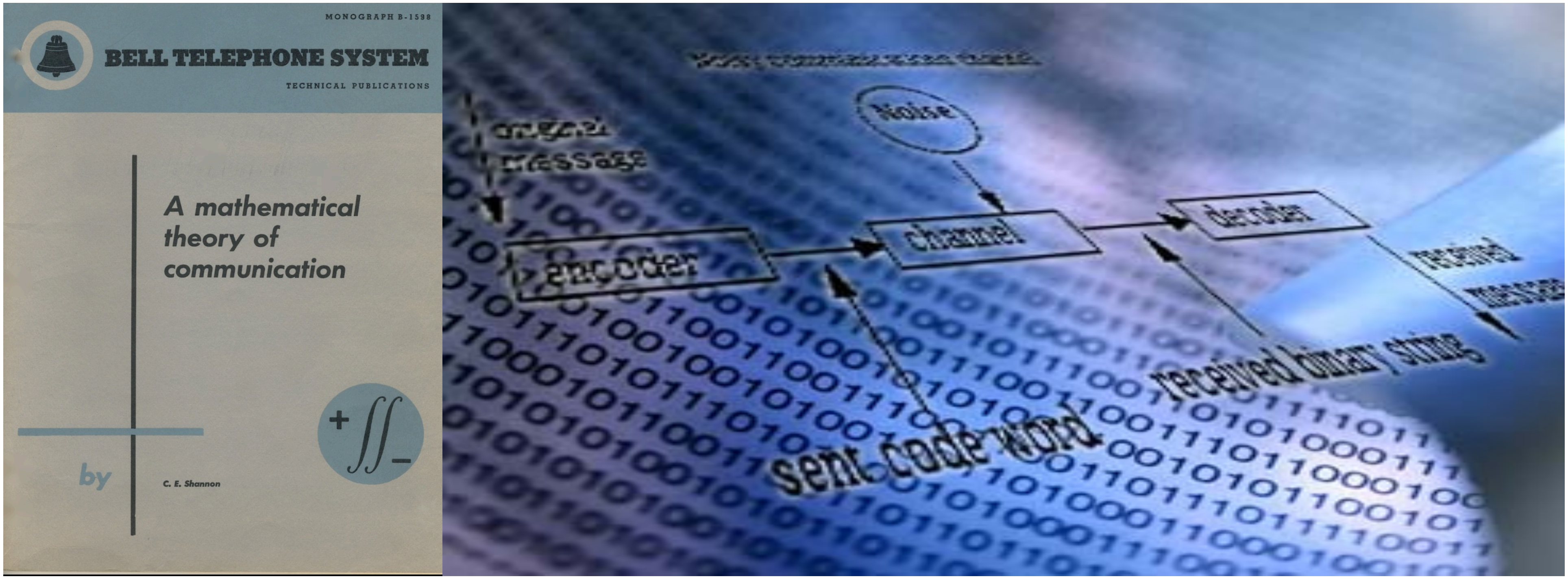

BROADCAST APPROACH FOR THE SPARSE-INPUT RANDOM-SAMPLED MIMO GAUSSIAN CHANNEL

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THE PROBLEM: MIMO BROADCAST

Random matrix with distribution discussed later on

$$\mathbf{y}_{p_i} = \underbrace{\mathbf{A}_{p_i}\mathbf{U}\mathbf{B}}_{\text{N X N}}\mathbf{x} + \mathbf{z} \qquad \mathbb{E}[||\mathbf{x}||^2] \leq P$$

where:

- $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$
- $\mathbf{A} = \begin{pmatrix} A_{1,1} & 0 & \dots & 0 \\ 0 & A_{2,2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & A_{N,N} \end{pmatrix}, \quad A_{j,j} \in \{0,1\} \qquad \mathbb{P}[A_{j,j} = 1] = p_i \qquad p_i \in (0, 1]$

Random with pdf $w(p)$

- $\mathbf{B} = \begin{pmatrix} B_{1,1} & 0 & \dots & 0 \\ 0 & B_{2,2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & B_{N,N} \end{pmatrix}, \quad B_{j,j} \in \{0,1\} \qquad \mathbb{P}[B_{j,j} = 1] = q$

GIVEN AN INPUT SPARSITY, MAXIMIZE TOTAL MULTICAST RATE WITH CHANNEL PARAMETER DISTRIBUTED AS $p \in (0,1] \sim W(p)$

STOCHASTICALLY DEGRADED MIMO BROADCAST CHANNEL

$$\mathbf{y}_{p_1} = \mathbf{A}_{p_1}\mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z}$$

$$\mathbf{y}_{p_2} = \mathbf{A}_{p_2}\mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z}$$

The channel with sampling rate p_1 is a stochastically degraded version of a channel with sampling rate p_2 if:

$$p_1 \leq p_2$$

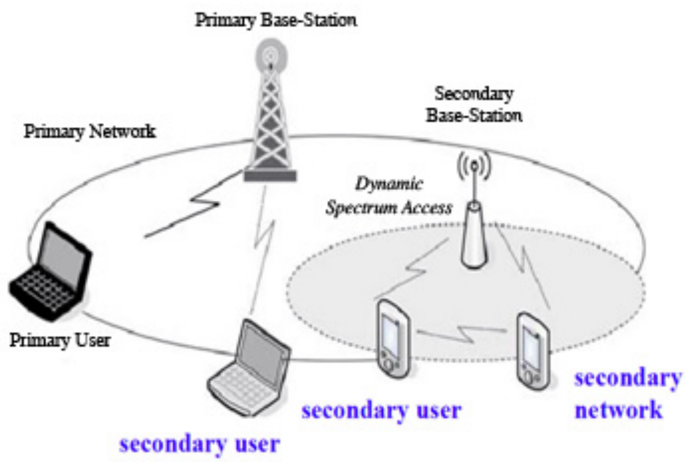
Our objective will be to maximize the total multicast average rate.

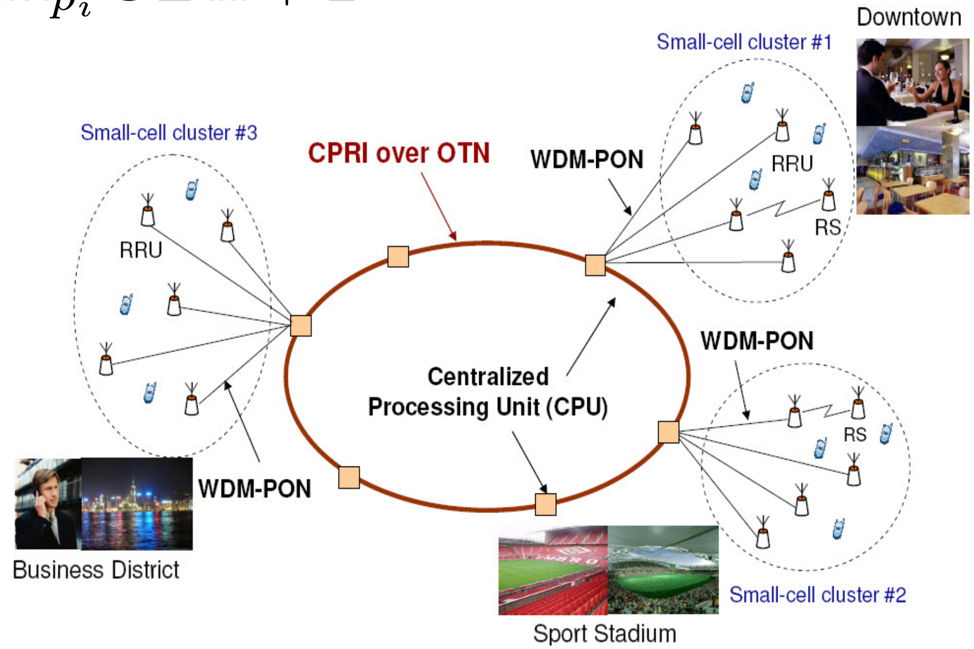
To do this we will resort to the *broadcast approach*

S. SHAMAI AND A. STEINER, "A BROADCAST APPROACH FOR A SINGLE-USER SLOWLY FADING MIMO CHANNEL," IEEE TRANS. ON INFORM. THEORY.

MOTIVATIONS

$$\mathbf{y}_{p_i} = \mathbf{A}_{p_i}\mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z}$$





OFDM-BASED COGNITIVE RADIO

OFDM with randomly sparse subcarrier aggregation: the non-zeros of B are the subcarriers used for transmission; U is a unitary DFT matrix,

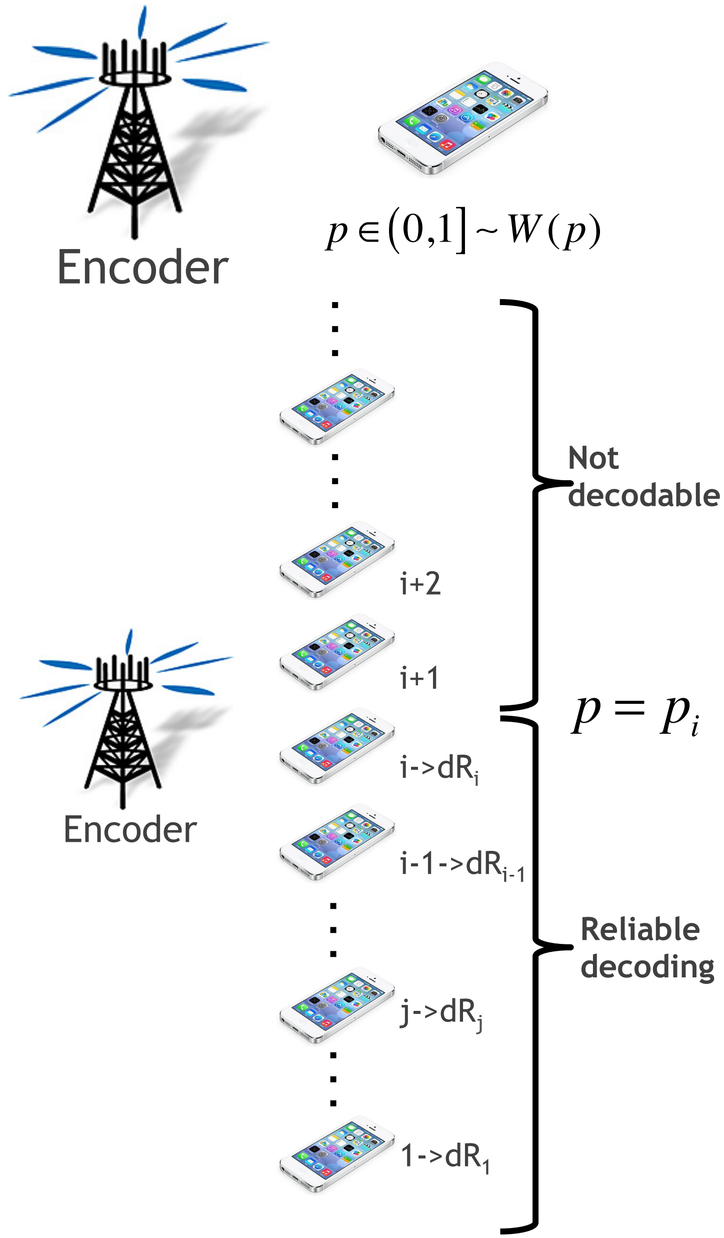
CLOUD RAN ARCHITECTURE

The CPU has no feedback on the status of the RRU: the non-zeros of B are the RRU active; U is a Guassian matrix (iid for example.)

TRANSMITTER AWARE ONLY OF THE STATISTICS OF THE MATRIX AUB

BROADCAST APPROACH

$$\mathbf{y}_p = \mathbf{A}_p \mathbf{U} \mathbf{B} \mathbf{x} + \mathbf{z};$$

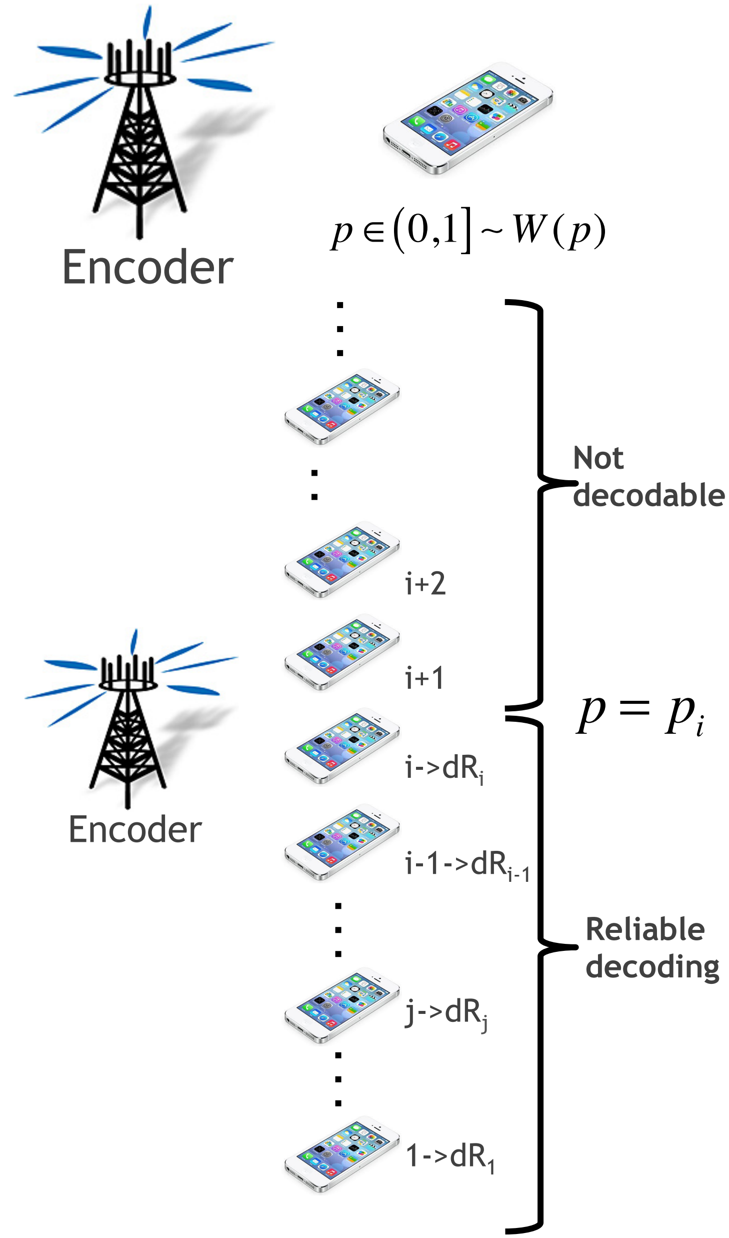


BROADCAST APPROACH

$$\mathbf{y}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z};$$

$$dR_j$$

$$R_i = \sum_{j \le i} dR_j$$



BROADCAST APPROACH

$$\sqrt{\mathbf{R}_p} \triangleq \mathbf{A}_p\mathbf{U}\mathbf{B}$$

$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \frac{1}{N}\mathbb{E}\left[\log\left|\mathbf{I}+\rho\mathbf{R}_p\right|\right]$$

$$\mathbf{y}_p = \mathbf{A}_p\mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z};$$

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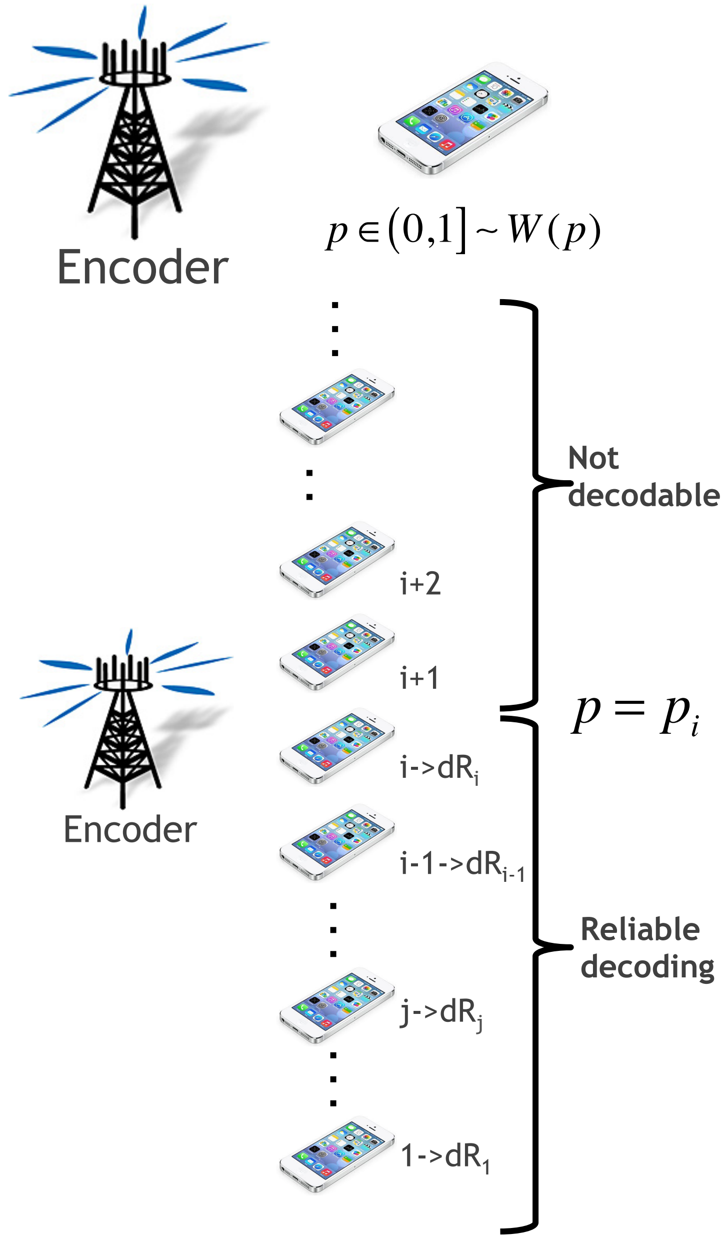

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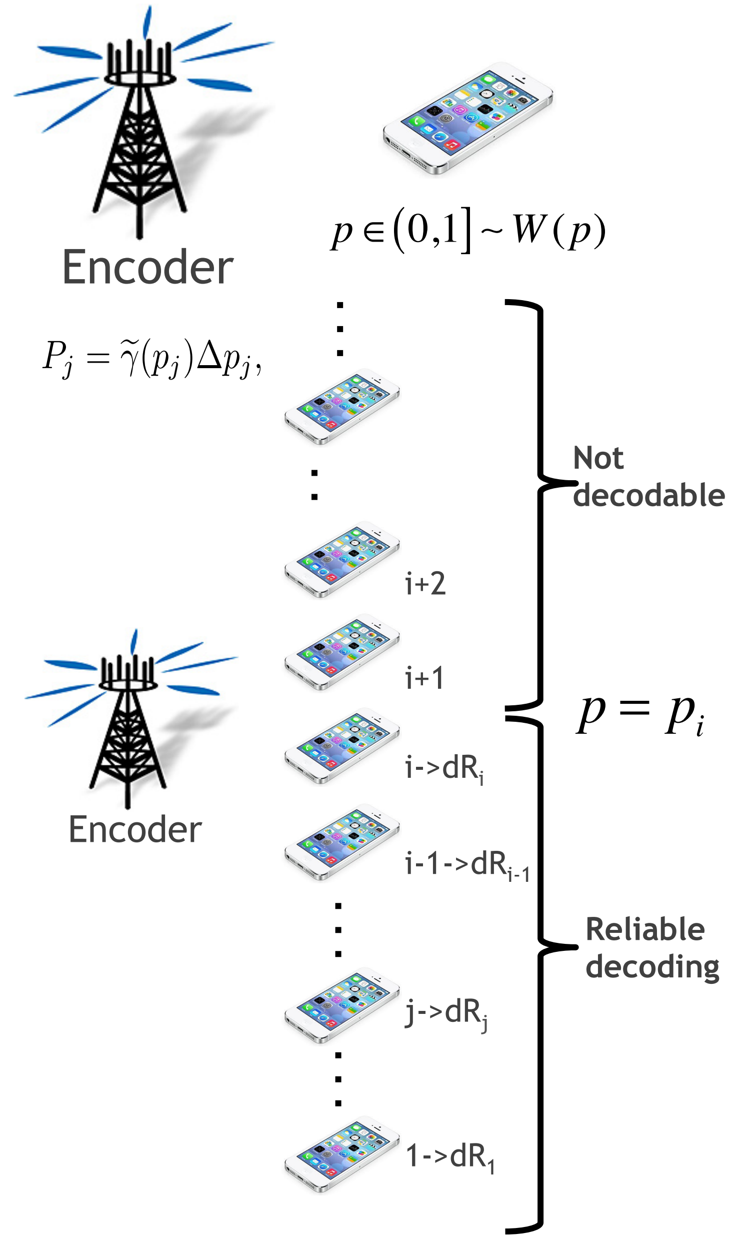
BROADCAST APPROACH

$$\mathbf{y}_p = \mathbf{A}_p \mathbf{U} \mathbf{B} \mathbf{x} + \mathbf{z};$$

$$\sqrt{\mathbf{R}_p} \triangleq \mathbf{A}_p \mathbf{U} \mathbf{B}$$
$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \frac{1}{N} \mathbb{E}\left[\log \left| \mathbf{I} + \rho \mathbf{R}_p \right| \right]$$

$$dR_j = \mathcal{V}_{\mathbf{R}_{p_j}} \left(\sum_{k \geq j} \widetilde{\gamma}(p_k) \Delta s_k \right) - \mathcal{V}_{\mathbf{R}_{p_j}} \left(\sum_{k > j} \widetilde{\gamma}(p_k) \Delta s_k \right)$$

$$R_i = \sum_{j \leq i} dR_j$$



BROADCAST APPROACH

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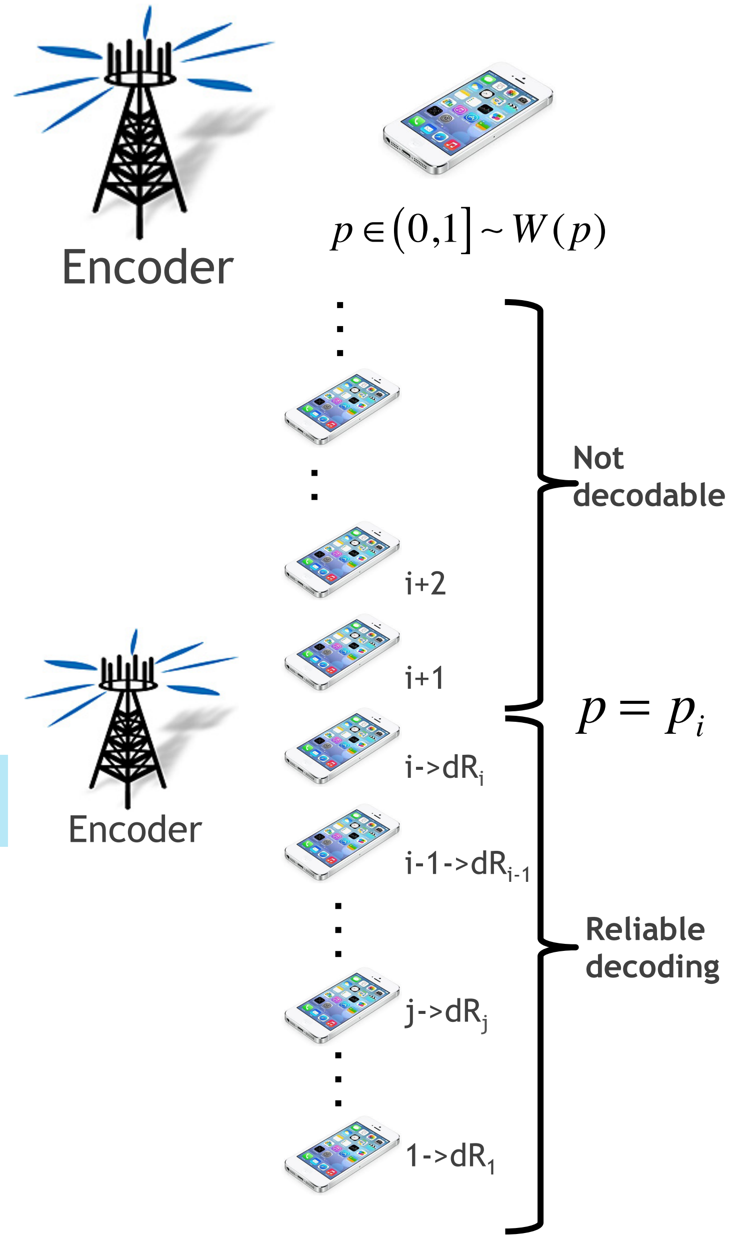
$$dR_j = \mathcal{V}_{\mathbf{R}_{p_j}}\left(\sum_{k \geq j} \widetilde{\gamma}(p_k)\Delta s_k\right) - \mathcal{V}_{\mathbf{R}_{p_j}}\left(\sum_{k > j} \widetilde{\gamma}(p_k)\Delta s_k\right)$$

$$\approx \frac{\partial}{\partial \rho}\mathcal{V}_{\mathbf{R}_{p_j}}(\rho)\bigg|_{\rho = \sum_{k>j}\widetilde{\gamma}(p_k)\Delta p_k} \times \widetilde{\gamma}(p_j)\Delta p_j.$$

$$P_j = \widetilde{\gamma}(p_j)\Delta p_j, \quad \Delta p_j = p_j - p_{j-1},$$

$$R_i = \sum_{j \leq i} dR_j$$



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BROADCAST APPROACH

$$\mathbf{y}_p = \mathbf{A}_p\mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z};$$

$$\sqrt{\mathbf{R}_p} \triangleq \mathbf{A}_p\mathbf{U}\mathbf{B}$$

$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \frac{1}{N}\mathbb{E}\left[\log\left|\mathbf{I} + \rho\mathbf{R}_p\right|\right]$$

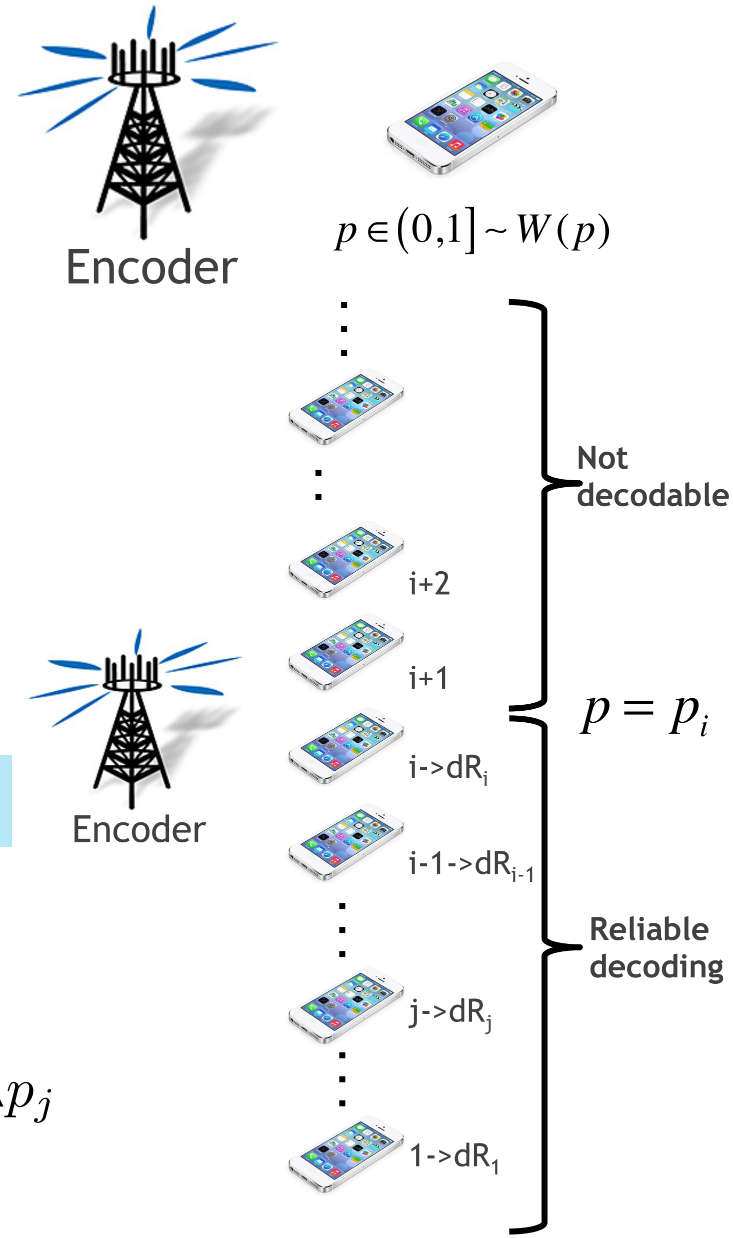
$$dR_j = \mathcal{V}_{\mathbf{R}_{p_j}}\left(\sum_{k\geq j}\widetilde{\gamma}(p_k)\Delta s_k\right) - \mathcal{V}_{\mathbf{R}_{p_j}}\left(\sum_{k>j}\widetilde{\gamma}(p_k)\Delta s_k\right)$$

$$\approx \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_{p_j}}(\rho)\Big|_{\rho=\sum_{k>j}\widetilde{\gamma}(p_k)\Delta p_k} \times \widetilde{\gamma}(p_j)\Delta p_j.$$

$$P_j = \widetilde{\gamma}(p_j)\Delta p_j,$$
$$\Delta p_j = p_j - p_{j-1},$$

$$R_i = \sum_{j\leq i} dR_j$$

$$= \sum_{j\leq i}\frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_{p_j}}(\rho)\Big|_{\rho=\sum_{k>j}\widetilde{\gamma}(p_k)\Delta p_k} \times \widetilde{\gamma}(p_j)\Delta p_j$$



BROADCAST APPROACH

$$\mathbf{y}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z};$$

$$\sqrt{\mathbf{R}_p} \triangleq \mathbf{A}_p \mathbf{U}\mathbf{B}$$

$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \frac{1}{N}\mathbb{E}\left[\log\left|\mathbf{I} + \rho\mathbf{R}_p\right|\right]$$

$$dR_j = \mathcal{V}_{\mathbf{R}_{p_j}}\left(\sum_{k\geq j}\widetilde{\gamma}(p_k)\Delta s_k\right) - \mathcal{V}_{\mathbf{R}_{p_j}}\left(\sum_{k>j}\widetilde{\gamma}(p_k)\Delta s_k\right)$$

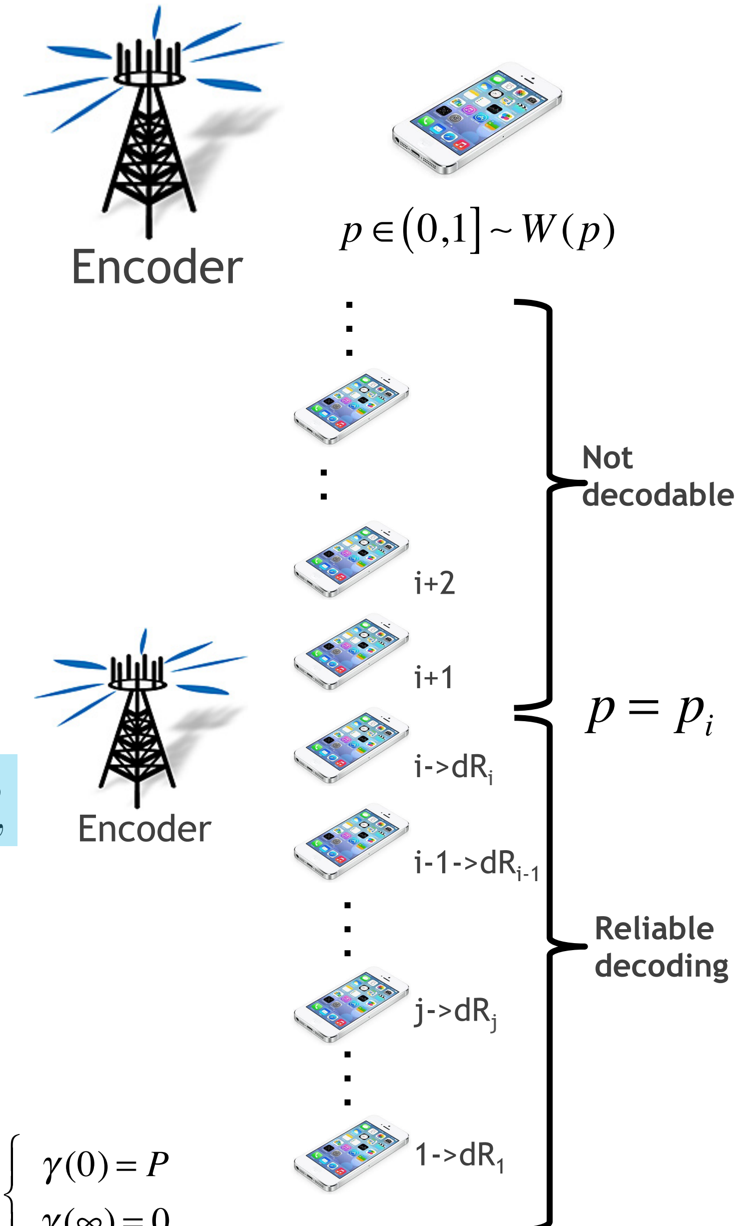
$$\approx \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_{p_j}}(\rho)\Big|_{\rho=\sum_{k>j}\widetilde{\gamma}(p_k)\Delta p_k} \times \boxed{\widetilde{\gamma}(p_j)\Delta p_j}.$$

$$P_j = \widetilde{\gamma}(p_j)\Delta p_j,$$
$$\Delta p_j = p_j - p_{j-1},$$

$$R_i = \sum_{j\leq i} dR_j$$

$$\xrightarrow{\Delta p_j \to 0} \int_0^{p_i} \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_a}(\rho)\Big|_{\rho=\boxed{\int_a^\infty \widetilde{\gamma}(u)du}} \widetilde{\gamma}(a)da,$$

$$\gamma(a) = \begin{cases} \gamma(0) = P \\ \gamma(\infty) = 0 \end{cases}$$



TOTAL MULTICAST RATE

$$\mathbf{R}_p \triangleq \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^{\mathsf{H}}\mathbf{A}_p.$$

The average rate achievable by the broadcast approach for a user with $p \in (0,1] \sim W(p)$

$$\frac{1}{N}R = \frac{1}{N}\int_0^{\infty} w(p)R(p)dp = \int_0^{\infty} w(p)\int_0^{p} \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_p}(\rho)\Big|_{\rho=\gamma(p)} dp$$

Optimizing R with respect to $\gamma(p)$ consists of maximizing

$$\frac{1}{N}R = \int_0^{\infty} (1 - W(p))\, \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_p}(\rho)\Big|_{\rho=\gamma(p)} \widetilde{\gamma}(p)dp,$$

subject to:

- $\gamma(0) = E\left[\|\mathbf{x}\|^2\right] = P$ and $\gamma(\infty) = 0$
- $\gamma(p)$ is a monotonically non-increasing and non-negative function of p

WE NEED CLOSED-FORM EXPRESSION FOR SHANNON-TRANSFORM

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TOTAL MULTICAST RATE

$$\mathbf{R}_p \triangleq \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^{\mathsf{H}} \mathbf{A}_p.$$

The average rate achievable by the broadcast approach for a user with $p \in (0,1] \sim W(p)$

$$\frac{1}{N}R = \frac{1}{N}\int_0^\infty w(p)R(p)dp = \int_0^\infty w(p)\int_0^p \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_p}(\rho)\Big|_{\rho=\gamma(p)} dp$$

Optimizing R with respect to $\gamma(p)$ consists of maximizing

$$\frac{1}{N}R = \int_0^\infty (1 - W(p)) \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_p}(\rho)\Big|_{\rho=\gamma(p)} \tilde{\gamma}(p)dp,$$

subject to:

- $\gamma(0) = E\left[\|\mathbf{x}\|^2\right] = P$ and $\gamma(\infty) = 0$
- $\gamma(p)$ is a monotonically non-increasing and non-negative function of p

$$(1 - W(p)) \frac{\partial}{\partial\rho}\dot{\mathcal{V}}_{\mathbf{R}_p}(\rho)\Big|_{\rho=\gamma(p)} - w(p) \frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_p}(\rho)\Big|_{\rho=\gamma(p)} = 0$$

WE NEED CLOSED-FORM EXPRESSION FOR SHANNON-TRANSFORM

BROADCAST APPROACH

$$\mathbf{y}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{x} + \mathbf{z};$$

$$\frac{1}{N} R = \int_0^\infty (1 - W(p)) \left. \frac{\partial}{\partial \rho} \mathcal{V}_{\mathbf{R}_p}(\rho) \right|_{\rho = \gamma(p)} \widetilde{\gamma}(p) dp,$$

$$\mathbf{R}_p \triangleq \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^{\mathsf{H}} \mathbf{A}_p$$

$$(1 - W(p)) \left. \frac{\partial}{\partial \rho} \dot{\mathcal{V}}_{\mathbf{R}_p}(\rho) \right|_{\rho = \gamma(p)} - w(p) \left. \frac{\partial}{\partial \rho} \mathcal{V}_{\mathbf{R}_p}(\rho) \right|_{\rho = \gamma(p)} = 0$$

Solving the Euler-Lagrange equation we need expressions:

$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[\log |\mathbf{I} + \rho \mathbf{R}_p|\right], \qquad \frac{\partial}{\partial \rho} \mathcal{V}_{\mathbf{R}_p}(\rho) = \frac{1}{\rho} \left(1 - \eta_{\mathbf{R}_p}(\rho)\right)$$

ASYMP. RANDOM MATRIX THEORY: CLOSE-FORM EXPRESSION FOR SHANNON AND η−TRANSFORM

OBJECTIVE

Evaluate Shannon and η-transform of the following matrix

$$\mathbf{R}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^{\dagger}\mathbf{A}_p$$

$\mathbf{U}$	Arbitrary Matrix		
$\mathbf{A}_p$	N x N diag matrix	$(\mathbf{A}_p)_i \in \{0,1\}$	$P\{(\mathbf{A}_p)_i = 1\} = p$
$\mathbf{B}$	N x N diag matrix	$(\mathbf{B})_i \in \{0,1\}$	$P\{(\mathbf{B})_i = 1\} = q$

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Interlude:

RANDOM MATRIX THEORY

THE STIELTJES TRANSFORM

The Stieltjes transform (also called the Cauchy transform) of an arbitrary random variable X is defined as:

$$S_X(z) = \mathbb{E}\left[\frac{1}{X - z}\right]$$

Inversion formula was obtained by Stieltjes in 1894:

$$\mathsf{f}_X(\lambda) = \lim_{\omega \to 0^+} \frac{1}{\pi} \mathsf{Im}\left[\mathcal{S}_X(\lambda + \jmath\omega)\right]$$

Rationale for Stieltjes: Description of Asymptotic Distribution of Singular Values (Marčenko-Pastur (1967))

THE η-TRANSFORM

The η-transform of a nonnegative random variable X is given by:

$$\eta_X(\rho) \triangleq \mathbb{E}\left[\frac{1}{1+\rho X}\right]$$

where ρ is a nonnegative real number, and thus, $0 < \eta_X(\rho) \leq 1$

Rationale for η: Description of Asymptotic Distribution of Singular Values + Signal Processing Insight

THE SHANNON TRANSFORM

The Shannon transform of a nonnegative random variable X is defined as:

$$\mathcal{V}_X(\rho) \triangleq \mathbb{E}\left[\log\left(1+\rho X\right)\right],$$

where ρ is a nonnegative real number.

Rationale for Shannon: Description of Asymptotic Distribution of Singular Values + Information Theory Insight

RELATIONSHIP BETWEEN TRANSFORMS

- Relationship η-Shannon

$$\gamma \frac{d}{d\gamma} \mathcal{V}_X(\gamma) \quad = \quad 1 - \eta_X(\gamma)$$

- Relationship η-Stieltjes

$$\eta_X(\gamma) = \frac{\mathcal{S}_X(-\frac{1}{\gamma})}{\gamma}$$

- Relationship Shannon-Stieltjes

$$\gamma \frac{d}{d\gamma} \mathcal{V}_X(\gamma) \quad = \quad 1 - \frac{1}{\gamma} \mathcal{S}_X \left(-\frac{1}{\gamma} \right)$$

η-TRANSFORM OF A RANDOM MATRIX

Given a N × N Hermitian matrix **R:**

The η-transform of its asymptotic ESD us given by

$$\eta_{\mathbf{R}}(\rho) \triangleq \mathbb{E}\left[\frac{1}{1+\rho\lambda(\mathbf{R})}\right] = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathbf{Tr}\left(\mathbf{I}+\rho\mathbf{R}\right)^{-1}\right]$$

SHANNON TRANSFORM OF A RANDOM MATRIX

Given a N × N Hermitian matrix **R:**

The Shannon transform of its asymptotic ESD is given by

$$\mathcal{V}_{\mathbf{R}}(\rho) \triangleq \mathbb{E}\left[\log\left(1+\rho\lambda(\mathbf{R})\right)\right] = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log\left|\mathbf{I}+\rho\mathbf{R}\right|\right]$$

$$\frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}}(\rho) = \frac{1}{\rho}\left(1-\eta_{\mathbf{R}}(\rho)\right).$$

RANDOM MATRIX THEORY ($N \to \infty$)

$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log\left|\mathbf{I}+\rho\mathbf{R}_p\right|\right], \qquad \eta_{\mathbf{R}_p}(\rho) \triangleq \mathbb{E}\left[\frac{1}{1+\rho\lambda(\mathbf{R}_p)}\right]$$

$$\frac{\partial}{\partial\rho}\mathcal{V}_{\mathbf{R}_p}(\rho) = \frac{1}{\rho}\left(1-\eta_{\mathbf{R}_p}(\rho)\right)$$

ASYMP. RM THEORY: CLOSE-FORM EXPRESSION FOR SHANNON AND η−TRANSFORM

$\mathbf{R} = \mathbf{H}\mathbf{H}^\dagger$ WITH $\mathbf{H}$ $pN \times qN$ IID MATRIX

- Shannon Transform:

$$\begin{aligned}\mathcal{V}(\gamma) &= \log\left(1+\gamma-\frac{1}{4}\mathcal{F}(\gamma,\beta)\right) \\ &+ \frac{1}{\beta}\log\left(1+\gamma\beta-\frac{1}{4}\mathcal{F}(\gamma,\beta)\right)-\frac{\log e}{4\,\beta\,\gamma}\mathcal{F}(\gamma,\beta)\end{aligned}$$

- η-Transform:

$$\eta_{\mathbf{R}}(\gamma) = \left(1-\frac{\mathcal{F}(\gamma,\beta)}{4\,\gamma}\right)$$

$$\beta = \frac{p}{q}$$

$$\mathcal{F}(x,z) = \left(\sqrt{x(1+\sqrt{z})^2+1}-\sqrt{x(1-\sqrt{z})^2+1}\right)^2$$

S. SHAMAI AND S. VERDU, "THE EFFECT OF FREQUENCY-FLAT FADING ON THE SPECTRAL EFFICIENCY OF CDMA", IEEE TRANS. INFORMATION THEORY, 2001.

OBJECTIVE

Evaluate Shannon and η-transform of the following matrix

$$\mathbf{R}_p = \mathbf{A}_p \mathbf{U} \mathbf{B} \mathbf{U}^\dagger \mathbf{A}_p$$

$\mathbf{U}$	Arbitrary Matrix		
$\mathbf{A}_p$	N x N diag matrix	$(\mathbf{A}_p)_i \in \{0,1\}$	$P\{(\mathbf{A}_p)_i = 1\} = p$
$\mathbf{B}$	N x N diag matrix	$(\mathbf{B})_i \in \{0,1\}$	$P\{(\mathbf{B})_i = 1\} = q$

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$\mathbf{R}=\mathbf{H}\mathbf{H}^{\dagger}$ WITH $\mathbf{H}$ $pN \times qN$ IID MATRIX

- Shannon Transform:

$$\mathcal{V}(\gamma) = \log\left(1+\gamma-\frac{1}{4}\mathcal{F}(\gamma,\beta)\right) + \frac{1}{\beta}\log\left(1+\gamma\beta-\frac{1}{4}\mathcal{F}(\gamma,\beta)\right) - \frac{\log e}{4\,\beta\,\gamma}\mathcal{F}(\gamma,\beta)$$

$$\beta = \frac{p}{q}$$

$$\mathbf{R}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^{\dagger}\mathbf{A}_p$$

$\mathbf{U}$ N x N IID matrix $\quad \mathbf{A}_p$ N x N diag matrix $\quad \mathbf{B}$ N x N diag matrix

$$(\mathbf{A}_p)_i \in \{0,1\} \quad P\left\{(\mathbf{A}_p)_i = 1\right\} = p \qquad (\mathbf{B})_i \in \{0,1\} \quad P\left\{(\mathbf{B})_i = 1\right\} = q$$

S. SHAMAI AND S. VERDU, "THE EFFECT OF FREQUENCY-FLAT FADING ON THE SPECTRAL EFFICIENCY OF CDMA", IEEE TRANS. INFORMATION THEORY, 2001.

$\mathbf{R}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^{\dagger}\mathbf{A}_p$ WITH U $N \times N$ HAAR MATRIX

- Shannon Transform:

$$\mathcal{V}(\gamma) \quad = \quad q \log\left(1 + \gamma(1 - \hat{e})\right) + d(e \parallel \hat{e})$$

$$\hat{e} = \frac{e}{\eta_{\mathbf{R}_p}(\gamma)}$$

- η-Transform:

$$\eta_{\mathbf{R}_p}(\gamma) = \frac{2\gamma(p-1)(q-1)}{1 - \gamma(p+q-2) - \sqrt{\mathcal{G}(\gamma, p, q)}}$$

$$\mathcal{G}(z, x, y) \;=\; z^2(x - y)^2 + 2z(x + y - 2xy) + 1.$$

TULINO, CAIRE, SHAMAI, VERDU, "CAPACITY OF CHANNELS WITH FREQUENCY-SELECTIVE AND TIME-SELECTIVE FADING," IEEE TRANS. INFORMATION THEORY, 2010.

$\mathbf{R}_p = \mathbf{A}_p \mathbf{U}\mathbf{B}\mathbf{U}^\dagger \mathbf{A}_p$ WITH $\mathbf{U}$ $N \times N$ FOURIER MATRIX

Theorem:

$\mathbf{U}\mathbf{B}\mathbf{U}^\dagger$ **and** $\mathbf{A}_p$ **are asymptotically free.**

- Shannon Transform:

$$\mathcal{V}(\gamma) = q \log\left(1 + \gamma(1 - \hat{e})\right) + d(e \,\|\, \hat{e})$$

$$\hat{e} = \frac{e}{\eta_{\mathbf{R}_p}(\gamma)}$$

- η-Transform:

$$\eta_{\mathbf{R}_p}(\gamma) = \frac{2\gamma(p-1)(q-1)}{1 - \gamma(p+q-2) - \sqrt{\mathcal{G}(\gamma,p,q)}}$$

TULINO, CAIRE, SHAMAI, VERDU, "CAPACITY OF CHANNELS WITH FREQUENCY-SELECTIVE AND TIME-SELECTIVE FADING," IEEE TRANS. INFORMATION THEORY, 2010.

BACK TO OUR PROBLEM

SOLVING THE EULER-LAGRANGE EQUATION

$$(1 - W(p)) \left. \frac{\partial}{\partial \rho} \dot{\mathcal{V}}_{\mathbf{R}_p}(\rho) \right|_{\rho=\gamma(p)} - w(p) \left. \frac{\partial}{\partial \rho} \mathcal{V}_{\mathbf{R}_p}(\rho) \right|_{\rho=\gamma(p)} = 0$$

We need expressions for the mutual information and its derivatives, for the channel model:

$$\mathbf{y}_p = \mathbf{A}_p \mathbf{U} \mathbf{B} \mathbf{x} + \mathbf{z},$$

$$\mathcal{V}_{\mathbf{R}_p}(\rho) \triangleq \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[\log |\mathbf{I} + \rho \mathbf{R}_p| \right], \qquad \frac{\partial}{\partial \rho} \mathcal{V}_{\mathbf{R}_p}(\rho) = \frac{1}{\rho} \left(1 - \eta_{\mathbf{R}_p}(\rho) \right)$$

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CHANGE OF VARIABLE

We can rewrite the Euler-Lagrange equation in terms of the eta-transform and its derivative with respect to p:

$$-(1-W(p))\frac{\dot{\eta}_{\mathbf{R}_p}(\gamma(p))}{\gamma(p)} - \frac{w(p)}{\gamma(p)}\left(1-\eta_{\mathbf{R}_p}(\gamma(p))\right) = 0$$

With a change of variable $\gamma = \gamma(p)$, we can re-write the rate:

$$R = \int_0^1 (W(p)-1)\left.\frac{\partial}{\partial \rho}\mathcal{V}_{\mathbf{R}_p}(\rho)\right|_{\rho=\gamma(p)} \dot{\gamma}(p)dp$$

as:

$$R = \int_{\gamma(\mathcal{A})} (1-W(g(\gamma)))\frac{1-\eta_{\mathbf{R}_{g(\gamma)}}(\gamma)}{\gamma}d\gamma.$$

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RANDOM MATRIX THEORY ($N \rightarrow \infty$)

$$-(1-W(p))\frac{\dot{\eta}_{\mathbf{R}_p}(\gamma(p))}{\gamma(p)} - \frac{w(p)}{\gamma(p)}\left(1-\eta_{\mathbf{R}_p}(\gamma(p))\right) = 0$$

$\mathbf{U}$ IID	$\mathbf{U}$ HAAR / FOURIER
$\eta_{\mathbf{R}_p}(\gamma) = 1 - \frac{\mathcal{F}(\gamma p, \frac{q}{p})}{4\gamma},$ $\mathcal{F}(x,z) = \left(\sqrt{1+x\left(1+\sqrt{z}\right)^2} - \sqrt{1+x\left(1-\sqrt{z}\right)^2}\right)^2$	$\eta_{\mathbf{R}_p}(\gamma) = \frac{2\gamma(p-1)(q-1)}{1-\gamma(p+q-2)-\sqrt{\mathcal{G}(\gamma,p,q)}}$ $\mathcal{G}(z,x,y) = z^2(x-y)^2 + 2z(x+y-2xy) + 1.$

ASYMP. RM THEORY: CLOSE-FORM EXPRESSION FOR SHANNON AND η−TRANSFORM

FINDING THE RIGHT SOLUTION

- Having expressions for both $\eta_{\mathbf{R}_p}(\gamma)$ and $\dot{\eta}_{\mathbf{R}_p}(\gamma)$, we can replace them into the Euler-Lagrange and solve for $\gamma(p)$ for all values of $p \in [0, 1]$.
- **Such solution must be discussed carefully.** In general, $\gamma(p)$ is equal to the constant P (total power) for some interval $[0, p_0]$ and it is equal to the constant 0 for some interval $[p_1, 1]$ with $p_0 \leq p_1$.
- In the range $[p_0, p_1]$, $\gamma(p) = \hat{\gamma}(p)$ where the latter is a monotonically non-increasing function of p.
- In order to find p_0 and p_1 we replace the boundary conditions $\gamma = P$ and $\gamma = 0$ into the Euler-Lagrange equation and solve for p. Hence, we verify that in the interval $[p_0, p_1]$ the solution is indeed unique and monotonic.

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$\mathbf{U}$: AN IID RANDOM MATRIX

Fix:

$$P = 10, \qquad q = 0.2, \qquad w(p) = \begin{cases} 1 & 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

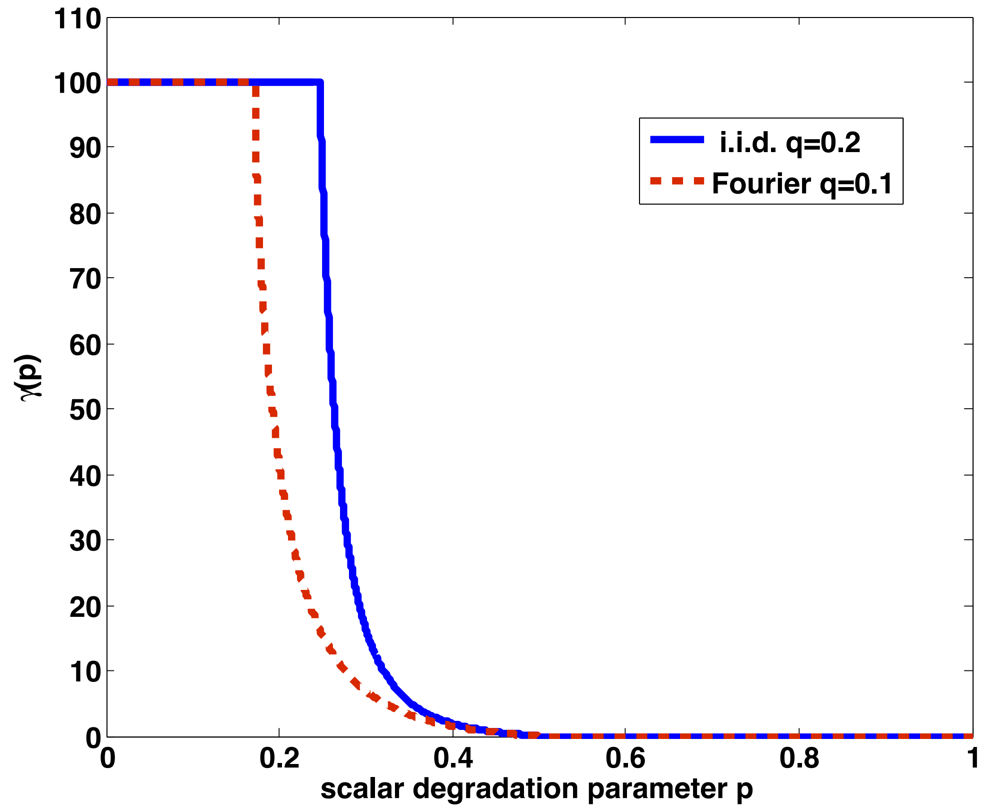
Then:

$$\gamma(p) = \begin{cases} P & 0 \le p \le \underbrace{0.32}_{p_0} \\ \hat{\gamma}(p) & 0.32 \le p \le 0.5 \\ 0 & \underbrace{0.5}_{p_1} \le p \le 1 \end{cases}$$

where $\hat{\gamma}(p)$ is the solution for $p \in [0.32, 0.5]$ $\overset{\gamma \to 0}{\to} q(1-2p) = 0,$

$$\frac{1-p}{2\gamma}\left[1 - \frac{1+\gamma(p-q)}{\sqrt{1+2\gamma(p+q)+\gamma^2(p-q)^2}}\right] - \frac{\mathcal{F}(\gamma p, \frac{q}{p})}{4\gamma^2} = 0.$$

$\gamma(p)$ FOR $\mathbf{U}$ IID AND HAAR MATRIX



CONCLUSIONS

- MIMO (linear Gaussian) channel where:
  - Inputs turned on and off at random,
  - Outputs sampled at random

- Given:
  - input sparsity probability,
  - statistics of the MIMO channel
  - broadcast approach as coding technique

  method for calculating the power allocation across the layers, in order to maximize the system weighted sum rate for arbitrary non-negative weighting function .

- Analytical solutions both for iid and Haar distributed MIMO channel matrices are provided.

- The Haar case accounts also for DFT matrices, with application to sparse spectrum signals with random sub-Nyquist sampling.