

Transformation results for sums and products of random matrices

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Random Product Matrices,

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Product of Ginibre matrices

- **Complex Ginibre matrices G_j of sizes**

$$(n + \nu_j) \times (n + \nu_{j-1}) \quad \text{all } \nu_j \geq 0 \text{ and } \nu_0 = 0$$

- **We make the product matrix $Y = G_r \cdots G_2 G_1$**

Theorem (Akemann-Kieburg-Wei, Akemann-Ipsen-Kieburg)

Eigenvalues of $Y^* Y$ have joint density

$$\frac{1}{Z_n} \Delta_n(x) \det [w_k(x_j)]_{j,k=1}^n, \quad \text{all } x_j > 0,$$

where $\Delta_n(x) = \prod_{j < k} (x_k - x_j)$ and w_k has Mellin transform

$$\int_0^\infty x^{s-1} w_k(x) dx = s^{k-1} \prod_{j=1}^r \Gamma(s + \nu_j)$$



Polynomial ensemble

Polynomial ensemble is density of the form

$$\frac{1}{Z_n} \Delta_n(x) \det [w_k(x_j)]_{j,k=1}^n,$$

Examples are **eigenvalues** of unitary ensembles, squared singular values of **products** of Ginibre matrices, **Muttalib-Borodin biorthogonal ensembles**, ...

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- Determinantal point process with kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y)$$

- P_k is polynomial of degree k , and Q_k is in the span of w_1, \dots, w_n such that

$$\int P_j(x) Q_k(x) dx = \delta_{j,k}, \quad \text{for } j, k = 0, 1, \dots, n-1.$$

Transformation of polynomial ensemble

G is $(n + \nu) \times n$ complex Ginibre matrix

Theorem (K-Stivigny (2014))

If squared singular values of X have joint density

$$\frac{1}{Z_n} \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n$$

then squared singular values of $Y = GX$ have density

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det [g_k(y_k)]_{j,k=1}^n$$

where g_k is the **Mellin convolution** of $x^\nu e^{-x}$ with $f_k(x)$

$$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k \left(\frac{y}{x} \right) \frac{dx}{x}$$

Transformation of kernel

There are transformation results for

- Polynomials P_j
- Dual functions Q_k .
- Correlation kernel $K_n^X \mapsto K_n^Y$

$$K_n^Y(x, y) = \frac{1}{2\pi i} \oint_{\Sigma} \frac{ds}{s} \int_0^{\infty} \frac{dt}{t} \left(\frac{t}{s}\right)^{\nu} e^{s-t} K_n^X\left(\frac{x}{s}, \frac{y}{t}\right)$$

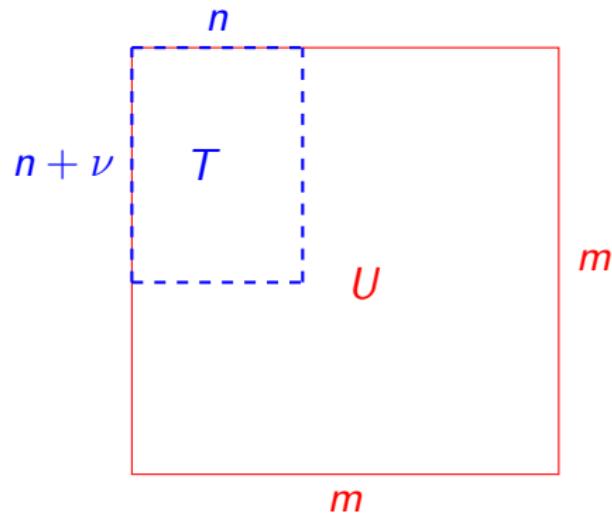
Claeys-K-Wang (2015)

Other products

- Products with inverses of complex Ginibre matrices
[Forrester \(2014\)](#)
- Products with truncations of unitary matrices
[Kieburg-K-Stivigny \(2016\)](#)

Truncated unitary matrix

- **Unitary matrix U has size $m \times m$**
- **Truncation T has size $(n+\nu) \times n$ with $n \leq n+\nu \leq m$**



Transformation of polynomial ensemble

T is $(n + \nu) \times n$ truncation of Haar distributed $m \times m$ unitary matrix

Theorem (Kieburg-K-Stivigny (2016))

If squared singular values of X have joint density

$$\frac{1}{Z_n} \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n \quad \text{all } x_j > 0$$

then squared singular values of $Y = TX$ have density

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det [g_k(y_j)]_{j,k=1}^n \quad \text{all } y_j > 0$$

with
$$g_k(y) = \int_0^1 x^\nu (1-x)^{m-n-\nu-1} f_k \left(\frac{y}{x} \right) \frac{dx}{x}$$

Mellin convolution with Beta density.

Four proofs

There are **four proofs** of this result

- 1) Change of variables $T \mapsto Y = TX$ and **new integral over the unitary group.**
- 2) Other changes of variable and matrix integrals.
Kieburg-K-Stivigny (2016)
- 3) Transformations based on removal of rows and columns.
K (2016)
- 4) **Spherical transform** proof for Gelfand pair $(\mathrm{GL}(n, \mathbb{C}), \mathrm{U}(n))$.
Kieburg-Kösters (arXiv 2016)

Ingredient

- Andreief identity (generalized Cauchy-Binet)

$$\int \det[f_k(x_j)]_{j,k=1}^n \cdot \det[g_k(x_j)]_{j,k=1}^n dx_1 \cdots dx_n \\ = n! \det \left[\int f_j(x) g_k(x) dx \right]_{j,k=1}^n$$

Proof 1, integral over unitary group

- Integral over subset of unitary group

$$\int_{U \in \mathrm{U}(n) : UBU^* \leq A} \det(A - UBU^*)^p dU = c_{n,p} \frac{\det \left[(a_j - b_k)_+^{p+n-1} \right]_{j,k=1}^n}{\Delta_n(a)\Delta_n(b)}$$

- A and B are Hermitian matrices with eigenvalues a_1, \dots, a_n , and b_1, \dots, b_n ,

$$(a_j - b_k)_+ = \begin{cases} a_j - b_k & \text{if } a_j \geq b_k \\ 0 & \text{otherwise.} \end{cases}$$

$$c_{n,p} = \prod_{j=0}^{n-1} \binom{p+n-1}{j}^{-1}$$

Recall the theorem

T is $(n + \nu) \times n$ truncation of Haar distributed $m \times m$ unitary matrix

Theorem (Kieburg-K-Stivigny (2016))

If squared singular values of X have joint density

$$\frac{1}{Z_n} \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n \quad \text{all } x_j > 0$$

then squared singular values of $Y = TX$ have density

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det [g_k(y_j)]_{j,k=1}^n \quad \text{all } y_j > 0$$

with
$$g_k(y) = \int_0^1 x^\nu (1-x)^{m-n-\nu-1} f_k \left(\frac{y}{x} \right) \frac{dx}{x}$$

Mellin convolution with Beta density.

Proof 1, step 1

First consider fixed X

Step 1: Truncation T has distribution

$$\propto \det(I - T^* T)^{m-2n-\nu} dT \quad \text{subject to } T^* T \leq I.$$

$T \mapsto Y = TX$ is change of variables

Proof 1, step 1

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$T \mapsto Y = TX$ is change of variables

- **Jacobian** $\det(X^* X)^{-n-\nu} = \prod_j x_j^{-n-\nu}$
- **Then** $T = YX^{-1}$ and $Y^* Y \leq X^* X$

$$\begin{aligned} & \propto \left(\prod_{j=1}^n x_j^{-n-\nu} \right) \det(I - X^{-*} Y^* Y X^{-1})^{m-2n-\nu} \mathbb{1}_{Y^* Y \leq X^* X} dY \\ & = \left(\prod_{j=1}^n x_j^{-m+n} \right) \det(X^* X - Y^* Y)^{m-2n-\nu} \mathbb{1}_{Y^* Y \leq X^* X} dY \end{aligned}$$

Proof 1, steps 2 and 3

Step 2: Singular value decomposition $Y = V\Sigma U$

- Then density

$$\propto \left(\prod_{j=1}^n x_j^{-m+n} \right) \det(X^*X - U^*\Sigma^2 U)^{m-2n-\nu} \mathbb{1}_{U^*\Sigma^2 U \leq X^*X}$$
$$\underbrace{\left(\prod_{k=1}^n y_k^\nu \right)}_{\text{Jacobian of SVD}} \Delta_n(y)^2$$

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$$\underbrace{\left(\prod_{k=1}^n y_k^\nu \right)}_{\text{Jacobian of SVD}} \Delta_n(y)^2$$

Step 3: Integrate out U and V

$$\propto \left(\prod_{j=1}^n x_j^{-m+n} \right) \left(\prod_{k=1}^n y_k^\nu \right) \Delta_n(y)^2 \underbrace{\frac{\det [(x_j - y_k)_+^{m-n-\nu-1}]_{j,k=1}^n}{\Delta_n(y)\Delta_n(x)}}_{\text{result of integral over unitary group}}$$

Proof 1, step 4

Step 4: Simplify and bring factors into the determinant:

$$\propto \frac{\Delta_n(y)}{\Delta_n(x)} \det \left[\frac{y_k^\nu}{x_j^{m-n}} (x_j - y_k)_+^{m-n-\nu-1} \right]_{j,k=1}^n$$

Proof 1, step 4

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Step 5: Average over $\frac{1}{Z_n} \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n$ and use
Andreief identity

$$\propto \Delta_n(y) \det [g_k(y_j)]_{j,k=1}^n$$

with

$$\begin{aligned} g_k(y) &= \int_0^\infty \frac{y^\nu}{x^{m-n}} (x - y)_+^{m-n-\nu-1} f_k(x) dx \\ &= \int_0^\infty x^\nu (1 - x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x} \end{aligned}$$

Proof 3

Observation: suppose $U = \begin{pmatrix} T & * \\ * & * \end{pmatrix}$

then

$$U \begin{pmatrix} XX^* & 0 \\ 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} TXX^*T^* & * \\ * & * \end{pmatrix}$$

- Squared singular values of TX are non-zero eigenvalues of **leading principal submatrix** of UAU^* where

$$A = \begin{pmatrix} XX^* & 0 \\ 0 & 0 \end{pmatrix}$$

- A is a semi positive-definite matrix whose non-zero eigenvalues are the squared singular values of X .

Proof 3, Key result

Baryshnikov (2001)

Suppose A is $(n+1) \times (n+1)$ Hermitian matrix with eigenvalues

$$a_0 < a_1 < \cdots < a_n$$

B is the $n \times n$ principal submatrix of UAU^* where U is Haar distributed unitary matrix

- Eigenvalues of B almost surely interlace

$$a_0 < b_1 < a_1 < b_2 < \cdots < a_{n-1} < b_n < a_n$$

- Joint density of eigenvalues of B

$$n! \frac{\Delta_n(b)}{\Delta_{n+1}(a)}$$

subject to the interlacing condition.

Proof 3, Variation on this theme

Forrester-Rains (2005)

Suppose A is $(n+p) \times (n+p)$ positive semidefinite with eigenvalues

$$0 < a_1 < \cdots < a_n, \quad 0 \text{ has multiplicity } p \geq 1.$$

B is $(n+p-1) \times (n+p-1)$ principal submatrix of UAU^*

- B has $p-1$ eigenvalues at 0 and remaining n eigenvalues almost surely interlace

$$0 < b_1 < a_1 < b_2 < \cdots < a_{n-1} < b_n < a_n$$

- Joint density of non-zero eigenvalues

$$\propto \frac{\prod_{k=1}^n b_k^{p-1}}{\prod_{k=1}^n a_k^p} \frac{\Delta_n(b)}{\Delta_n(a)}$$

subject to the interlacing condition.

Proof 3, Polynomial ensemble

A is $(n+p) \times (n+p)$ positive semidefinite with eigenvalues

$$0 < a_1 < \cdots < a_n, \quad 0 \text{ has multiplicity } p \geq 1,$$

B is $(n+p-1) \times (n+p-1)$ principal submatrix of UAU^*

- Suppose A is random and non-zero eigenvalues are polynomial ensemble

$$\propto \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n$$

- Then non-zero eigenvalues of B have density

$$\propto \Delta_n(y) \det [g_k(y_j)]_{j,k=1}^n \quad g_k(y) = \int_0^1 x^{p-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

Proof 3, Repeat the transformation

A is $(n+p) \times (n+p)$ is positive semidefinite

$0 < a_1 < \dots < a_n, \quad 0 \text{ has multiplicity } p \geq 1,$

**B is $(n+q) \times (n+q)$ principal submatrix of UAU^* with
 $0 \leq q < p$**

- Suppose A is random and non-zero eigenvalues are polynomial ensemble

$$\propto \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n$$

- Then non-zero eigenvalues of B have density

$$\propto \Delta_n(y) \det [g_k(y_j)]_{j,k=1}^n, \quad g_k(y) = \int_0^1 x^q (1-x)^{p-q-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

Sum with GUE matrix

GUE matrix H has density $\frac{1}{Z_n} e^{-\frac{1}{2} \text{Tr } H^2} dH$.

Theorem (Claeys-K-Wang (2015))

If eigenvalues of $n \times n$ random Hermitian matrix X have density

$$\frac{1}{Z_n} \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n$$

then eigenvalues of $Y = H + X$ have density

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det [F_k(y_j)]_{j,k=1}^n$$

where

$$F_k(y) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} f_k(y - x) dx$$

Sum with LUE matrix

LUE matrix L has density $\propto (\det L)^\alpha e^{-\text{Tr } L}$ on positive definite Hermitian matrices.

Theorem

If eigenvalues of $n \times n$ random Hermitian matrix X have density

$$\frac{1}{Z_n} \Delta_n(x) \det [f_k(x_j)]_{j,k=1}^n$$

then eigenvalues of $Y = L + X$ have density

$$\frac{1}{\tilde{Z}_n} \Delta_n(y) \det [F_k(y_j)]_{j,k=1}^n$$

where

$$F_k(y) = \int_0^\infty x^{\alpha+n-1} e^{-x} f_k(y-x) dx$$

Proof 1: matrix integral

$Y = L + X$ **has density**

$$\begin{aligned} &\propto \det(Y - X)^\alpha e^{-\text{Tr}(Y - X)} \mathbb{1}_{Y \geq X} \\ &= \prod_{j=1}^n e^{-y_j} \prod_{k=1}^n e^{x_k} \det(Y - X)^\alpha \mathbb{1}_{Y \geq X} \end{aligned}$$

Proof 1: matrix integral

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Eigenvalue decomposition $Y = U^* D U$

- Average over U is integral over unitary group

$$\int_{U \in \mathrm{U}(n)} \det(U^* D U - X)^\alpha \mathbb{1}_{U^* D U \geq X} dU = c_{n,\alpha} \frac{\det [(y_j - x_k)_+^{\alpha+n-1}]}{\Delta_n(y) \Delta_n(x)}$$

- Average over X from polynomial ensemble and use Andreieff identity as before.

Proof 2: spherical functions

Kieburg-Kösters (2016):

Group theoretic interpretation for matrix products in terms of **spherical functions** for the Gelfand pair

$$(G, K) \quad G = \mathrm{GL}(n, \mathbb{C}), \quad K = \mathrm{U}(n)$$

- Spherical functions are labeled by

$$s = (s_1, \dots, s_n) \in \mathbb{C}^n,$$

$$\varphi_s(A) = \left(\prod_{j=0}^{n-1} j! \right) \frac{\det [x_j^{s_k}]_{j,k=1}^n}{\Delta_n(s) \Delta_n(x)}$$

where x_1, \dots, x_n are the eigenvalues of A^*A

Gelfand-Naimark (1950)

Gelfand pair: general theory 1

Gelfand pair (G, K)

- G is Lie group
- K is compact subgroup
- $f : G \rightarrow \mathbb{C}$ is **K -biinvariant** if

$$f(xk) = f(kx) = f(x), \quad \text{for } x \in G, k \in K$$

Gelfand pair: general theory 1

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- $f : G \rightarrow \mathbb{C}$ is **K -biinvariant** if

$$f(xk) = f(kx) = f(x), \quad \text{for } x \in G, k \in K$$

- Convolution product

$$(f * g)(x) = \int_G f(y)g(xy^{-1})dy$$

- $L^1(K \backslash G / K)$ is **commutative** (= Gelfand pair)

Gelfand pair: general theory 2

Gelfand pair (G, K)

- $\varphi : G \rightarrow \mathbb{C}$ is a **spherical function** if $\varphi(e) = 1$ and

$$\int_K \varphi(xky) dk = \varphi(x)\varphi(y) \quad \text{for } x, y \in G.$$

- **Spherical transform**

$$f \mapsto \widehat{f}, \quad \widehat{f}(\varphi) = \int_G f(x)\varphi(x) dx$$

has property

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}$$

- There is **inverse spherical transform** $\widehat{f} \mapsto f$.

Gelfand pair for sums 1

Gelfand pair that is relevant for **sums of random matrices**

$$(G, K) \quad G = \mathrm{U}(n) \ltimes \mathrm{Herm}(n), \quad K = \mathrm{U}(n)$$

Group operation

$$(U, A) \cdot (V, B) = (UV, A + UBU^*)$$

Gelfand pair for sums 1

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Group operation

$$(U, A) \cdot (V, B) = (UV, A + UBU^*)$$

- $f : G \rightarrow \mathbb{C}$ is **K -biinvariant** if $f(U, A)$ depends on the **eigenvalues of A only**.
- For **K -biinvariant functions**

$$(f * g)(A) = \int_{\mathrm{Herm}(n)} f(B)g(A - B)dB$$

Gelfand pair for sums 2

- For K -biinvariant functions

$$(f * g)(A) = \int_{\text{Herm}(n)} f(B)g(A - B)dB$$

If f and g are probability densities for random matrices X and Y then $f * g$ is probability density for $X + Y$.

Gelfand pair for sums 3

Gelfand pair $G = \mathrm{U}(n) \ltimes \mathrm{Herm}(n)$, $K = \mathrm{U}(n)$

Spherical functions are

$$\varphi_s(A) = \int_{\mathrm{U}(n)} e^{\mathrm{Tr} SUAU^*} dU = \left(\prod_{j=0}^{n-1} j! \right) \frac{\det [e^{x_j s_k}]_{j,k=1}^n}{\Delta_n(s) \Delta_n(x)}$$

if (x_1, \dots, x_n) are eigenvalues of A .

Harish-Chandra / Itzykson-Zuber

Gelfand pair for sums 3

Gelfand pair $G = \mathrm{U}(n) \ltimes \mathrm{Herm}(n)$, $K = \mathrm{U}(n)$

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$$\varphi_s(A) = \int_{\mathrm{U}(n)} e^{\mathrm{Tr} SUAU^*} dU = \left(\prod_{j=0}^{n-1} j! \right) \frac{\det [e^{x_j s_k}]_{j,k=1}^n}{\Delta_n(s) \Delta_n(x)}$$

if (x_1, \dots, x_n) are eigenvalues of A .

Harish-Chandra / Itzykson-Zuber

- **Spherical transform** $\widehat{f}(s) = \int_{\mathrm{Herm}(n)} f(A) \varphi_s(A) dA$
can be calculated in certain cases.

Gelfand pair for sums 4

- **GUE:** If $f_{GUE}(A) = \frac{1}{Z_n} e^{-\frac{1}{2} \operatorname{Tr} A^2}$ then

$$\widehat{f}_{GUE}(s) = \prod_{k=1}^n e^{\frac{1}{2}s_k^2}.$$

- **LUE:** If $f_{LUE}(A) = \frac{1}{Z_n} (\det A)^\alpha e^{-\operatorname{Tr} A} \mathbb{1}_{A \geq 0}$ then

$$\widehat{f}_{LUE}(s) = \prod_{k=1}^n \frac{1}{(1 - s_k)^{\alpha+n}}, \quad \text{all } \operatorname{Re} s_k < 1$$

- **Polynomial ensemble:** If $f_{PE}(A)$ has eigenvalue density $\propto \Delta_n(x) \det [f_k(x_j)]$ then

$$\widehat{f}_{PE}(s) \propto \frac{1}{\Delta_n(s)} \det \left[\int_{-\infty}^{\infty} f_k(x) e^{s_j x} dx \right]_{j,k=1}^n$$

Gelfand pair for sums 5

$$\widehat{f}_{LUE}(s) = \prod_{k=1}^n \frac{1}{(1-s_k)^{\alpha+n}}, \quad \text{all } \operatorname{Re} s_k < 1$$

$$\widehat{f}_{PE}(s) \propto \frac{1}{\Delta_n(s)} \det \left[\int_{-\infty}^{\infty} f_k(x) e^{s_j x} dx \right]_{j,k=1}^n$$

- Verify that $\widehat{f}_{LUE} \cdot \widehat{f}_{PE}$ is the spherical transform of a polynomial ensemble with functions

$$F_k(y) = \int_0^{\infty} x^{\alpha+n-1} e^{-x} f_k(y-x) dx$$