

Quaternionic extension of R transform for non-Hermitian random matrix models

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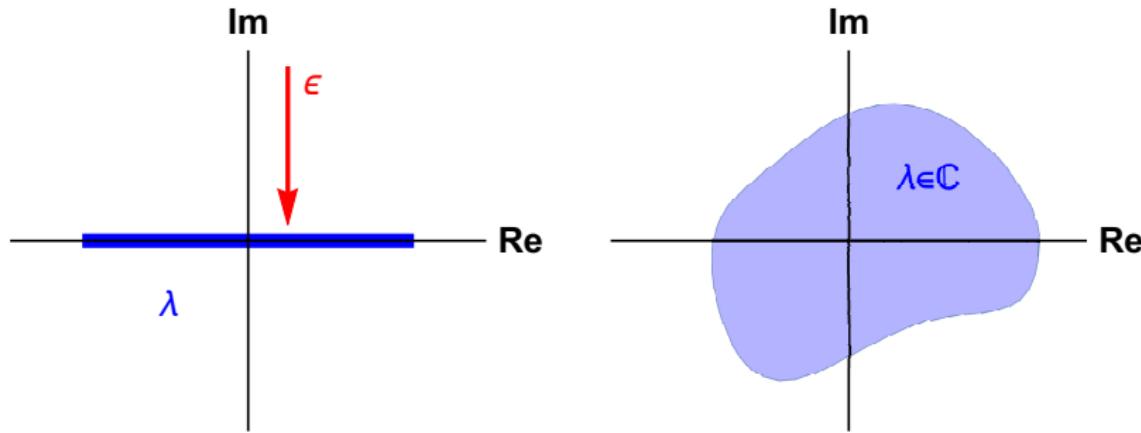
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Complex Green's function

Eigenvalue density is obtained from Green's function by approaching real axis from 'imaginary' direction:

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im} G(\lambda + i\epsilon) \quad (1)$$

$$G(z) = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} (z \mathbb{1} - H)^{-1} \right\} \quad (2)$$



Quaternions - Cayley-Dickson construction

Real → Complex

- $x \rightarrow z = x + iy$ extension
- $\mathbb{C} \rightarrow \mathbb{R}$ maps:
 - $\text{Re}(x + iy) = x$
 - $\text{Im}(x + iy) = y$

Complex → Quaternion (\mathbb{H})

- $z \rightarrow q = z + jw$ extension
- $\mathbb{H} \rightarrow \mathbb{C}$ maps:
 - $F(z + jw) = z$
 - $S(z + jw) = w$

2x2 matrix representation:

$$q = (z, w) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \quad (3)$$

Quaternionic matrix:

$$\mathcal{Q} = (Z, W) = \begin{pmatrix} Z & W \\ -W^\dagger & Z^\dagger \end{pmatrix}, \quad \text{Tr } \mathcal{Q} = \begin{pmatrix} \text{Tr } Z & \text{Tr } W \\ -\text{Tr } W^\dagger & \text{Tr } Z^\dagger \end{pmatrix} \quad (4)$$

Quaternionic resolvents

[Feinberg, Zee '97 and Jarosz, Nowak '03]

Dirac delta on the complex plane - quaternionic representation:

$$\delta(z) = \frac{1}{\pi} \lim_{w \rightarrow 0} \frac{\partial}{\partial \bar{z}} F((z, w)^{-1}) = \frac{1}{\pi} \lim_{w \rightarrow 0} \frac{|w|^2}{(|z|^2 + |w|^2)^2} \quad (5)$$

Leads to quaternionic extension of Green's function:

$$\mathcal{G}(q) = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} (q \mathbb{1} - X)^{-1} \right\} = \quad (6)$$

$$= \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr}_b \left(\begin{array}{cc} z \mathbb{1} - X & w \mathbb{1} \\ -\bar{w} \mathbb{1} & \bar{z} \mathbb{1} - X^\dagger \end{array} \right)^{-1} \right\} \quad (7)$$

S.t.

$$\rho(\lambda) = \frac{1}{\pi} \lim_{w \rightarrow 0} \frac{\partial}{\partial \bar{z}} F \mathcal{G}((z, w)) \quad (8)$$

Moments

$G(z)$ - generating function for moments of Hermitian matrix H

$$m_n = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} H^n \right\} \quad (9)$$

$$G(z) = \sum_{n=0}^{\infty} m_n z^{-n-1} \quad (10)$$

Planar diagrammatics: series expansion around infinity and clever resummation.

$$G(z) = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} (z^{-1} + z^{-1} Hz^{-1} + z^{-1} Hz^{-1} Hz^{-1} + \dots) \right\} \quad (11)$$

Non-Hermitian moments

Series expansion around infinity:

$$\mathcal{G}(q) = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} (q^{-1} + q^{-1} X q^{-1} + q^{-1} X q^{-1} X q^{-1} + \dots) \right\} \quad (12)$$

Non-Hermitian matrix X :

- $M_{\alpha_1 \dots \alpha_n}^{(n)} = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} X_{\alpha_1} \dots X_{\alpha_n} \right\}$
- $\mathcal{G}(q)_{\alpha\zeta} = q_{\alpha\zeta}^{-1} + \sum_{\beta} M_{\beta}^1 q_{\alpha\beta}^{-1} q_{\beta\zeta}^{-1} + \sum_{\beta\gamma} M_{\beta\gamma}^2 q_{\alpha\beta}^{-1} q_{\beta\gamma}^{-1} q_{\gamma\zeta}^{-1} + \dots$

indices = 1, 2, $X_1 = X$, $X_2 = X^\dagger$ etc.

Moments are given by arrays of numbers. E.g. element of fifth moment:

$$M_{21121}^5 = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} X^\dagger X^2 X^\dagger X \right\} = M_{12121}^5 \quad (13)$$

Cumulants

$$G(z) = \frac{1}{z - R(G(z))} \quad \mathcal{G}(q) = \frac{1}{q - \mathcal{R}(\mathcal{G}(q))}$$

R transform is a generating function of cumulants, which are now multidimensional arrays.

$$\mathcal{R}(q)_{\alpha\zeta} = K_{\alpha}^{(1)}\delta_{\alpha\zeta} + K_{\alpha\zeta}^{(2)}q_{\alpha\zeta} + \sum_{\beta} K_{\alpha\beta\zeta}^{(3)}q_{\alpha\beta}q_{\beta\zeta} + \dots \quad (14)$$

First few cumulants:

$$K_1^{(1)} = \bar{K}_2^{(1)} = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} X \right\}$$

$$K_{11}^{(2)} = \bar{K}_{22}^{(2)} = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} \left(X - K_1^{(1)} \mathbb{1} \right) \left(X - K_1^{(1)} \mathbb{1} \right) \right\}$$

$$K_{12}^{(2)} = K_{21}^{(2)} = \lim_{N \rightarrow \infty} \mathbf{E} \left\{ \frac{1}{N} \text{Tr} \left(X - K_1^{(1)} \mathbb{1} \right) \left(X - K_1^{(1)} \mathbb{1} \right)^{\dagger} \right\}$$

Elliptic ensembles

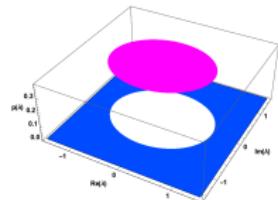
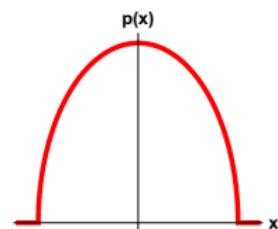
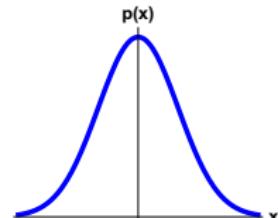
Only first two (free) cumulants nonzero:

- | | | |
|-------------------------|---|-----------------------|
| Commutative probability | - | Gaussian distribution |
| Hermitian RMT | - | Gaussian ensembles |
| Non-Hermitian RMT | - | Elliptic ensembles |

$$K^{(1)}, K_{11}^{(2)} \in \mathbb{C}, K_{12}^{(2)} \in \mathbb{R}$$

Convinient way to parametrize second cumulant for elliptic ensembles:

$$K^{(2)} = \sigma^2 \begin{pmatrix} \mu e^{2i\phi} & 1 \\ 1 & \mu e^{-2i\phi} \end{pmatrix} \quad (15)$$



Standardized elliptic ensemble

$$K^{(2)} = \sigma^2 \begin{pmatrix} \mu e^{2i\phi} & 1 \\ 1 & \mu e^{-2i\phi} \end{pmatrix}$$

$K^1 = 0, \sigma^2 = 1, \phi = 0$ - standardized ensemble.

$$R_\mu((z, w)) = (\mu z, w) = \begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix} \circ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \quad (16)$$

Linear transformation

$$X \rightarrow \frac{1}{\sigma} e^{-i\phi} (X - K^1)$$

takes care of other parameters.

Measure

What matrix model realizes this R transform? Take:

$$X = \frac{1}{\sigma_1 + \sigma_2} (\sigma_1 H_1 + i\sigma_2 H_2) , \quad \mu = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad (17)$$

with H_i - independent GUEs:

$$d\mu(H_1, H_2) \propto DH_1 DH_2 e^{-\frac{N}{2}\text{Tr}(H_1^2 + H_2^2)} \quad (18)$$

by simple calculation:

$$d\mu(X) \propto DX \exp \left(-\frac{N}{1-\mu^2} \text{Tr} \left(XX^\dagger - \frac{\mu}{2} (X^2 + X^{\dagger 2}) \right) \right) \quad (19)$$

Eigenvalue density flat on an ellipse around origin of \mathbb{C} [Girko '88].
 μ - eccentricity.

Most general Gaussian measure for non-Hermitian matrices

$$d\mu(X) \propto DX \exp \left(-\frac{N}{1-\mu^2} \text{Tr} \left(XX^\dagger - \frac{\mu}{2} (X^2 + X^{\dagger 2}) \right) \right) \quad (20)$$

Include other parameters by simple linear transformation:

$$X \rightarrow \frac{1}{\sigma} e^{-i\phi} (X - K^1) \quad (21)$$

Special cases:

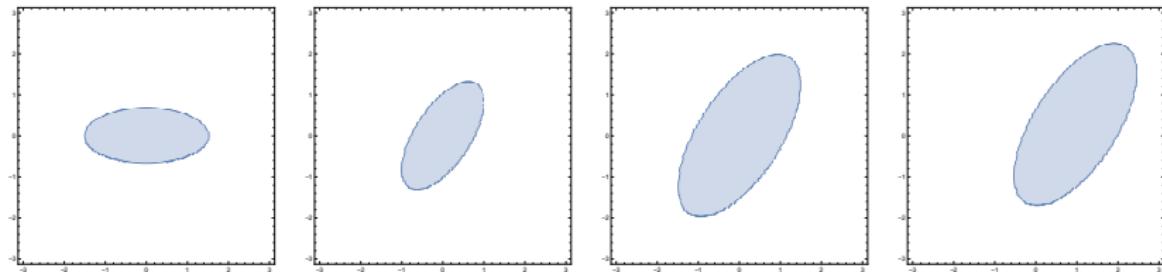
- $\mu = 0$ - Ginibre
- $\mu \rightarrow 1$ - GUE
- $\mu \rightarrow -1$ - antihermitian GUE, semicircle on imaginary axis

Elliptic ensemble - summary

Eigenvalue density of elliptic ensemble is uniform on an ellipse in complex plane.

- K^1 - parametrizes location of the center of ellipse
- $\sigma^2 = \sigma_1^2 + \sigma_2^2$ where σ_i - lengths of semi-axes
- ϕ - angle between ellipse axis and positive real semi-axis
- $\mu = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$ - eccentricity

$K^1 = 0, \sigma^2 = 1, \phi = 0$ - standardized ensemble. Any other can be obtained from it by translation(K^1), rotation(ϕ) and rescaling(σ^2).



Composite objects - Addition and multiplication laws

For invariant ensembles, $d\mu(X) \propto DX \exp(\text{Tr} V(X))$

Addition law [Janik et al. '97]:

$$\mathcal{R}_{A+B}(q) = \mathcal{R}_A(q) + \mathcal{R}_B(q) \quad (22)$$

Multiplication law [Burda, Janik, Nowak '11]:

$$\mathcal{R}_{AB}(\mathcal{G}_{AB}(z, \bar{z})) = [\mathcal{R}_A(q_b)]^L [\mathcal{R}_B(q_a)]^R \quad (23)$$

$$[q_a]^R = \mathcal{G}_{AB} [\mathcal{R}_A(q_b)]^L \quad (24)$$

$$[q_b]^L = [\mathcal{R}_B(q_a)]^R \mathcal{G}_{AB} \quad (25)$$

where, writing $z = re^{i\phi}$, $q = (\tilde{z}, w)$:

$$[q]^L = (\tilde{z}, w e^{i\phi/2}) \quad \text{and} \quad [q]^R = (\tilde{z}, w e^{-i\phi/2}) \quad (26)$$

Sketch of derivation

Linearization trick:

$$\mathcal{G}_{AB}^D(q) = \frac{1}{N} \text{Tr} \left(q \mathbb{1}_{2N} - \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right)^{-1} \quad (27)$$

- Expand as geometric series.
- Even moments agree with moments of the product.
- Average wrt. elliptic part of the measure through planar diagrammatic expansion.
- For A and B free, mixed diagrams vanish.

Similar idea proven rigorously in [Belinschi, Sniady, Speicher '15]
Done only in vicinity of \mathbb{C} .

Products of gaussian random matrices - examples

Notation:

$E(\mu)$ - standardized elliptic random matrix.

$E(0) = X$ - Ginibre matrix.

$E(1) = H$ - GUE.

- Example 1:

$$(a\mathbb{1} + bX_1)(c\mathbb{1} + dX_2) \quad (28)$$

- Example 2:

$$(\mathbb{1} + E_1(\mu))(\mathbb{1} + E_2(\mu)) \quad (29)$$

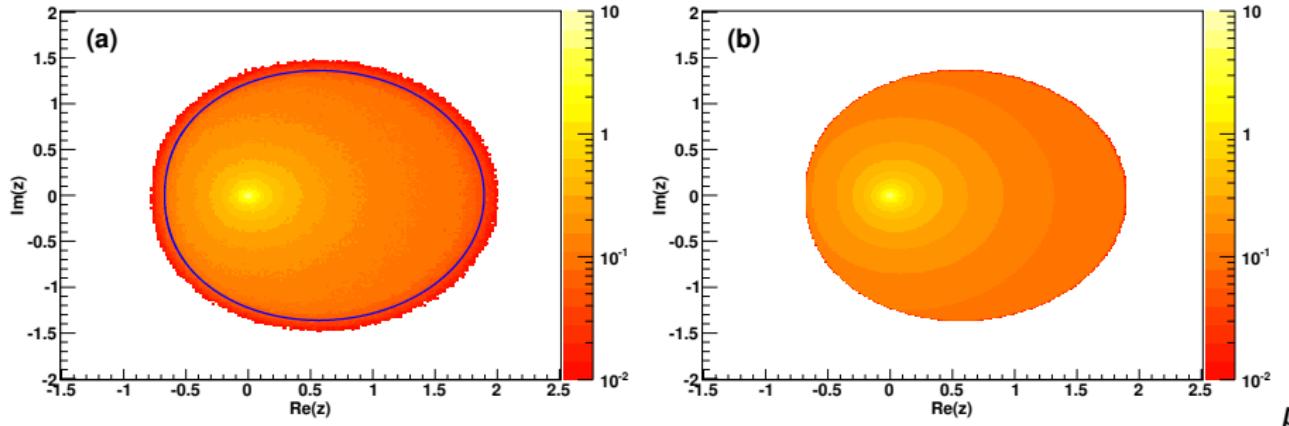
- Example 3:

$$(\mathbb{1} + H)(\mathbb{1} + X) \quad (30)$$

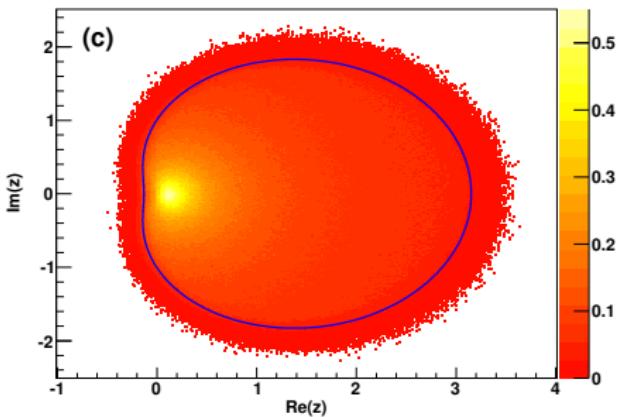
$$\text{Example 1: } (a\mathbb{1} + X_1)(c\mathbb{1} + dX_2)$$

Without rotations and rescaling:

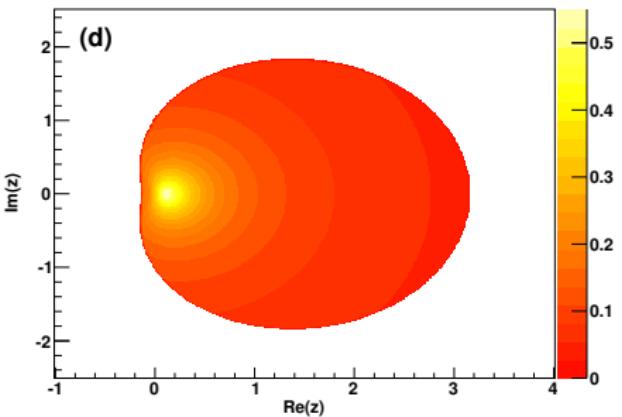
$$(p\mathbb{1} + X_1)(q\mathbb{1} + X_2), \quad p, q \in \mathbb{R}_+ \quad (31)$$



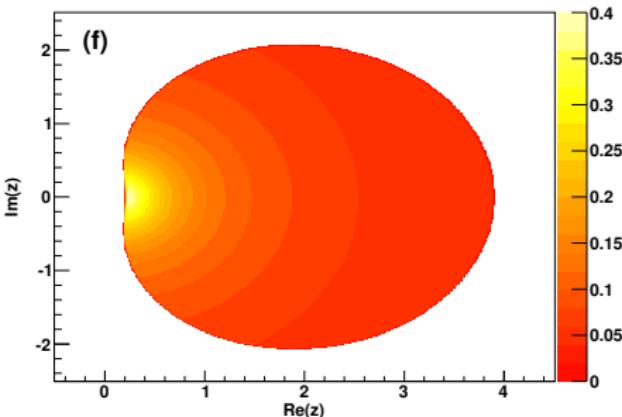
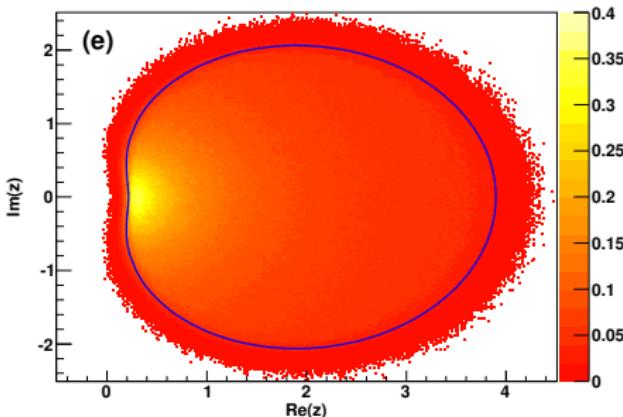
$$1/2, q = 3/4$$



$$p = 0.9, q = 1.2 \quad \uparrow$$

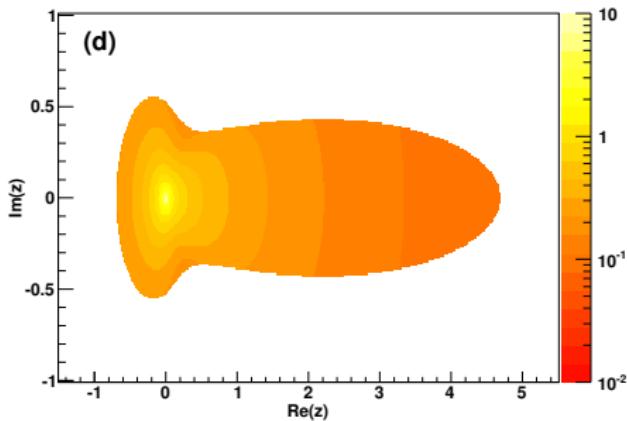
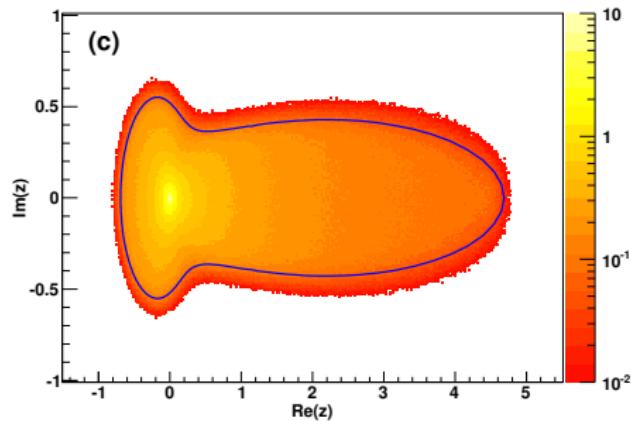


$$\downarrow \quad p = 1.2, q = 1.3$$



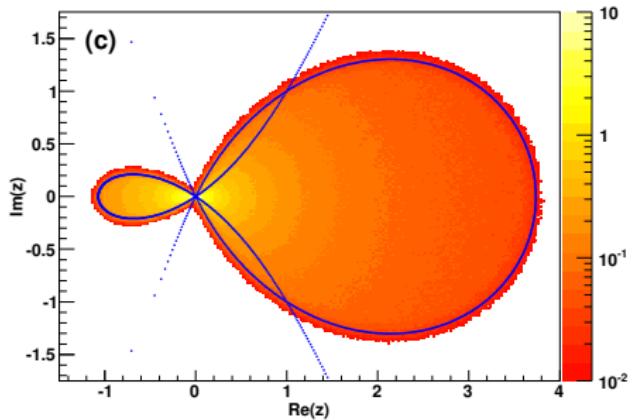
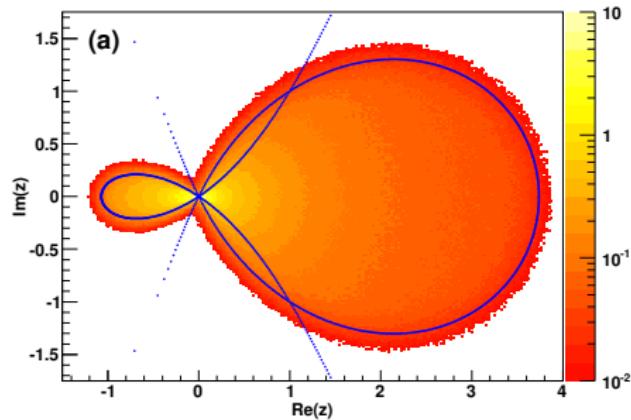
Example 2: $(\mathbb{1} + E(\mu))(\mathbb{1} + E(\mu))$

Symmetry reduces number of equations by half



$$\mu = 0.9$$

Example 3: $(\mathbb{1} + H)(\mathbb{1} + X)$



Two regions defined by closed loops of blue lines.

Why R-transform?

More efficient/stable/general way to calculate average spectra of polynomials developed in:

[Belinschi, Helton, Mai, Sniady, Speicher '13-15]

but with $\int e^{\text{Tr}KH} d\mu(H) = \Omega(K)$ - characteristic function

$$\omega(K) = N^{-1} \ln \Omega(K)$$

[Guionnet, Maida '04 and Mandt, Zirnbauer '09]

$$\mathbf{E} \left\{ \frac{\prod_b \text{Det}(p_{1,b} - H)}{\prod_a \text{Det}(p_{0,a} - H)} \right\} \propto \int DQ \text{SDet}^N(Q) \hat{\Omega}(Q) e^{-S \text{Tr} PQ} \quad (32)$$

$$\propto \int DQ e^{N(S \text{Tr} \ln(Q) + S \text{Tr} \hat{\omega}(Q) - S \text{Tr} PQ)} \quad (33)$$

Dominated by a saddle point:

$$Q^{-1} + \omega'(Q) = P \quad , \quad \omega'(Q) = R(Q) \quad (34)$$

Thank you for attention