On the number of real eigenvalues in a product of random matrices

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Random matrices: Products

Outline

- Motivation from an aspect of quantum entanglement and why number of real eigenvalues of a product?
- Three questions of measures, and some speculation.
- Nongaussian matrices.
- Summary, questions.

Background in brief

- The number of real **roots** of a random polynomial of degree N $\sim \log N$. (M. Kac 1943, Edelman, Kostlan 1995)

$$E_N = \frac{2}{\pi}\log(N) + 0.62573... + \frac{2}{N\pi} + \cdots$$

- "How many eigenvalues of a random matrix are real?" (Edelman, Kostlan, Shub, 1993).

$$E_N = \sqrt{\frac{2N}{\pi}}\left(1 - \frac{3}{8N} + \cdots\right) + \frac{1}{2}$$

- Fraction of real eigenvalues in a random matrix: $p_{k,n}$. (Kanzeiper, Akkeman, 2006).

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Quantum Entanglement

Bipartite Hilbert space: $\mathcal{H} = \mathcal{H}_A^N \otimes \mathcal{H}_B^N$

Pure unentangled states

$$|\chi_{AB}\rangle = |\psi_A\rangle \otimes |\phi_B\rangle$$

Entanglement in $|\psi_{AB}\rangle$ = von Neumann entropy of subsystems:

$$E(|\psi_{AB}\rangle) = -\mathrm{tr}_A(\rho_A \log \rho_A) = -\mathrm{tr}_B(\rho_B \log \rho_B)$$

$$\rho_A = \mathrm{tr}_B|\psi_{AB}\rangle\langle\psi_{AB}|)$$

Mixed separable states

$$\rho_{AB} = \sum_i q_i\, \rho_i^{(A)} \otimes \rho_i^{(B)}, \quad 0 \le q_i \le 1 \text{ and } \sum_i q_i = 1$$

Otherwise it is *entangled*

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Mixed state entanglement

Entanglement of formation

If $\rho_{AB} = \sum_i p_i \, |\Psi_i^{AB}\rangle\langle\Psi_i^{AB}|$ is one possible pure ensemble decomposition

Entanglement of formation is defined as

$E_f(\rho_{AB}) = \min_{p_i, \Psi_i^{AB}} \sum_i p_i \, E(|\Psi_i^{AB}\rangle)$.

2 qubit case ($N = 2$) is solved via **Concurrence** by *Hill* and *Wootters* (1998) and *Wootters* (1999).

2 qudit case ($N > 2$): open problem to evaluate the minimum in general.

Optimal entanglement ρ_{AB}: 2-qubit density matrix

Convexity: mixing reduces entanglement

$$C\left(\rho_{AB}=\sum_{i=1}^{k} p_i|\phi_i^{AB}\rangle\langle\phi_i^{AB}|\right)\leq\sum_{i=1}^{k} p_i C\left(|\phi_i^{AB}\rangle\langle\phi_i^{AB}|\right)$$

Optimal sets: Robust under mixing

Set $\{|\phi_i^{AB}\rangle, \, i=1,\cdots,k\}$ **optimal** if for **any** probability distribution $p_1\cdots p_k$

$$C\left(\rho_{AB}=\sum_{i=1}^{k} p_i|\phi_i^{AB}\rangle\langle\phi_i^{AB}|\right)=\sum_{i=1}^{k} p_i C\left(|\phi_i^{AB}\rangle\langle\phi_i^{AB}|\right)$$

Getting Real

Draw $|\phi_i\rangle$ from the set of **real** states $a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$ with $a_i \in \mathbb{R}$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. $\mathbf{a} \in S^3$.

Conditions for optimality of $\{|\phi_1\rangle, |\phi_2\rangle\}$ Iff

$$r_{11}r_{22} \geq 0, \quad \text{and} \quad -\det r = r_{12}^2 - r_{11}r_{22} \geq 0, \quad \text{where} \quad r_{ij} = \langle\phi_i|\sigma_y \otimes \sigma_y|\phi_j\rangle.$$

$$\begin{aligned} C\left(p|\phi_1\rangle\langle\phi_1| + (1-p)|\phi_2\rangle\langle\phi_2|\right) &= pC(|\phi_1\rangle\langle\phi_1|) + (1-p)C(|\phi_2\rangle\langle\phi_2|) \\ &= p\,|r_{11}| + (1-p)\,|r_{22}| \end{aligned}$$

(*Shuddhodhan, Ramkarthik, AL, J. Phys. A, 2011*)

Connection to products

Let $|\phi_1\rangle = a_{00}|01\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$,
and $|\phi_2\rangle = b_{00}|01\rangle + b_{01}|01\rangle + b_{10}|10\rangle + b_{11}|11\rangle$.
The $r_{ij} = \langle\phi_i|\sigma_y \otimes \sigma_y|\phi_j\rangle$ implies

$$r = \begin{pmatrix} -2\det M_1 & \mathrm{tr}(M_1 M_2) \\ \mathrm{tr}(M_1 M_2) & -2\det M_2 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \; M_2 = \begin{pmatrix} -b_{11} & b_{01} \\ b_{10} & -b_{00} \end{pmatrix} = -\det b \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}^{-1}$$

- Determinant = Discriminant

$$-\det r = (\mathrm{tr}(M_1 M_2))^2 - 4\det(M_1 M_2) \geq 0$$

$\implies M_1 M_2$ have **real** eigenvalues. Optimal if also $\det(M_1 M_2) \geq 0$

A first question of measure:

Let $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ **be a maximally entangled Bell state.**
$M_2 = -\mathbb{I}/\sqrt{2}$
If $|\phi_1\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$ **(** $\mathbf{a} \in S^3$**) and**

$$M_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

$\{|\phi_1\rangle, |\phi_2\rangle\}$ **is optimal** Iff $\det(M_1) \geq 0$ and M_1 has **real** eigenvalues.

Equivalent question in RMT

What is the probability, $p_{2,2}$, that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has real eigenvalues given that a, b, c, d are i.i.d. Gaussian numbers with zero mean?

$\det M > 0$ condition can be implemented as $p_{2,2} - \frac{1}{2}$.

Also $(a_1, a_2, a_3, a_4) = (a, b, c, d)/r$ is uniformly distributed (Haar) on S^3, where $r = \sqrt{a^2 + b^2 + c^2 + d^2}$. If M has real eigenvalues, so does M/r.

Answer to the equivalent question: $p_{2,2} = 1/\sqrt{2}$.

General answer known for probability of all eigenvalues of $n \times n$ real: $p_{n,n} = 2^{-n(n-1)/4}$ (*Edelman 1994*)

Answer to the first question of measure: $\frac{1}{\sqrt{2}} - \frac{1}{2} \approx 20\%$ of (real) states are co-optimal with the maximally entangled state.

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A second question of measure

Given an arbitrary, but fixed, state $|\phi_2\rangle$ what is the measure of states $|\phi_1\rangle$ such that $\{|\phi_1\rangle, |\phi_2\rangle\}$ is optimal?

Schmidt decomposition: The arbitary state can be taken as $|\phi_2\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ with $0 \le \theta \le \pi/4$. $C(|\phi_2\rangle) = \sin 2\theta$.

$|\phi_1\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$ ($\mathbf{a} \in S^3$) such that $\{|\phi_1\rangle, |\phi_2\rangle\}$ is optimal **iff**

$$M = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

is such that $\det(M) \ge 0$ and M **has real eigenvalues.**

Equivalent RMT:

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(a, b, c, d) i.i.d. $N(0, 1)$.

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The measure f_θ of states co-optimal with $|\phi_2\rangle$

$|\phi_2\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$

$$f_\theta = \frac{1}{2} - \frac{1}{2\pi}\int_0^\pi \sqrt{\frac{\sin\phi}{\sin\phi+\beta}}\, d\phi$$
$$= \frac{1}{2} - \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{k}{2}+\frac{3}{4})}{\Gamma(\frac{k}{2}+\frac{5}{4})}(\sin 2\theta)^{k+\frac{1}{2}}.$$

Decreases monotonically from $1/2$ at $\theta = 0$ to $1/\sqrt{2} - 1/2$ at $\theta = \pi/4$.

The fraction of states co-optimal with the maximally entangled state is the **smallest** and corresponds to the probability of a single random matrix having real eigenvalues.

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A third question of measure

What is the measure, f, of optimal pairs $\{|\phi_1\rangle, |\phi_2\rangle\}$?

Equivalent RMT: What is the probability, $p_{2,2}^{(2)}$ that the product of two 2×2 matrices have real eigenvalues?

Integrate out over θ. The appropriate invariant measure follows from the induced measure of singular values of random matrices and is known for $n \times m$ matrices. (*Zyczkowski, Sommers 2001*). For 2×2:

$$\mu(\theta) = 2\cos 2\theta$$

$$f = \int_0^{\pi/4} f_\theta \, \mu(\theta) d\theta = \frac{\pi}{4} - \frac{1}{2}$$

Probability of real eigenvalues of a product of 2 gaussian matrices:

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The probability that a product of two matrices have real eigenvalues

The fraction $\frac{\pi}{4} - \frac{1}{2} \approx 0.285$ of pairs of 2-qubit states are optimal.

$$p_{2,2} = \frac{1}{\sqrt{2}} = 0.70710678118654752440\cdots$$

$$< \; p_{2,2}^{(2)} = \frac{\pi}{4} = 0.78539816339744830962\cdots$$

Two are more real than one

Speculative general feature?

$$M = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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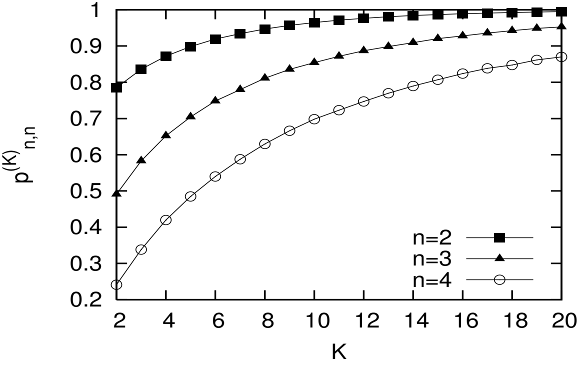
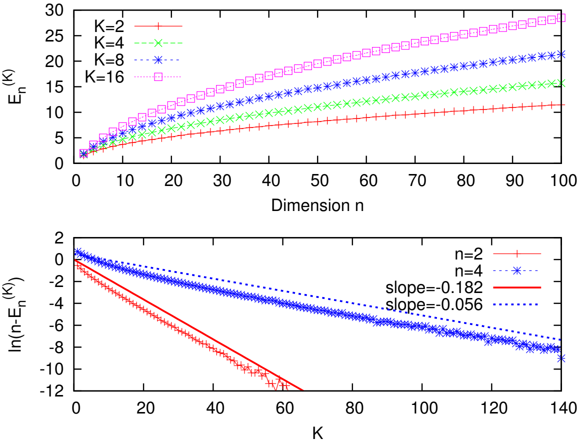
More matrices: Numerical results

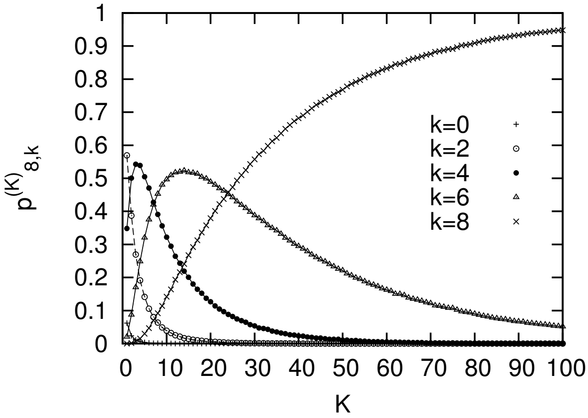
$p_{n,n}^{(K)}$ = Prob. that **all** eigenvalues of $A_1 \cdots A_K$ are real.
A_i: $n \times n$ random real matrix.



Expected number of real eigenvalues

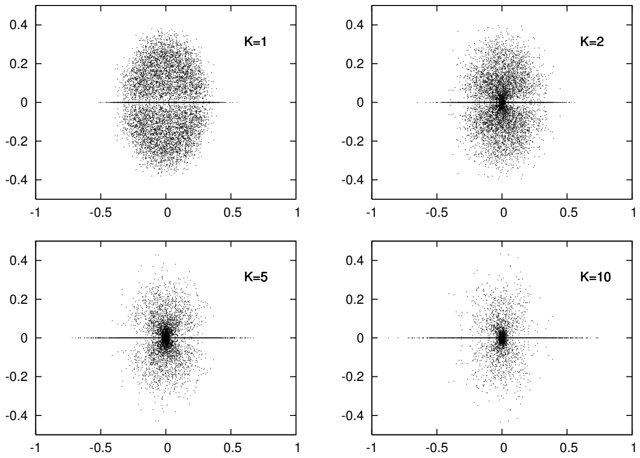


$$E_n^{(K)} \sim n - \exp(-\gamma_n K)$$



The probability that k eigenvalues of a product of K random 8 dimensional matrices are real, based on 100,000 realizations. The $k = 0$ case is barely seen in this scale.

AL: (J. Phys. A: Math. Theor. vol. 46 (2013)).



The eigenvalues of K products of 10 dimensional random matrices, after they have been divided by the corresponding Frobenius norms. The real and imaginary parts are plotted for 1000 realizations of such products.

Analytical results for $n > 2$, $K > 2$

P. J. FORRESTER, **"Probability of all eigenvalues real for products of standard Gaussian matrices"** *arXiv1309.7736, J. Phys. A. 2014*

Evaluates $p_{n,n}^{(2)}$ in terms of determinants whose entries are Meijer-G functions.

Conjectures: $p_{3,3}^{(2)} = \dfrac{5\pi}{32}, \cdots, p_{7,7}^{(2)} = \dfrac{31625532537\pi^3}{2^{47}}$

Proves: $p_{n,n}^{(K)} \longrightarrow 1$ as $K \longrightarrow \infty$.

Also SANTOSH KUMAR **"Exact evaluations of some Meijer G-functions and probability of all eigenvalues real for the product of two Gaussian matrices"** *J. Phys. A. 2015*

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General nonatomic distributions

i.i.d. (but not necessarily gaussian), symmetric zero mean and continuous

Under rather general conditions for $n = 2$, the probability of real eigenvalues $\geq \mathbf{5/8}$ and *seems* to be $\leq \mathbf{7/8}$.

1. Uniform on $[-1, 1]$: $\dfrac{49}{72} = 0.680556.$
2. Gaussian: $1/\sqrt{2} = 0.707\cdots.$
3. Laplace $\exp(-|x|)$: $\dfrac{11}{15} = 0.733\cdots.$
4. Cauchy: $\dfrac{1}{\pi(1+x^2)}$: $\dfrac{3}{4} = 0.75.$

Probability of real eigenvalues

Symmetric Beta distribution: $|x|^{\nu}\Theta(1-|x|)$

ν	Probability
$-4095/4096$	0.874959
$-7/8$	0.849868
$-1/2$	0.759836
0	$49/72 = 0.680556$
1	0.63709
$3/2$	0.632888
2	0.631023
3	0.62928
4	0.628361
200	0.625078
400	0.625039

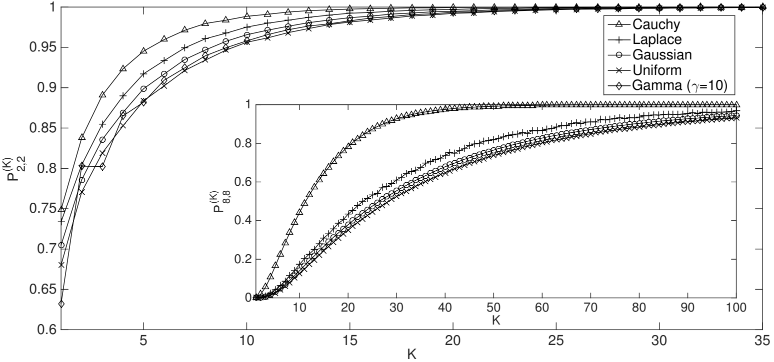
$$\nu = -1/2: \ \frac{1}{48}(41 - \pi - 2\ln 2)$$

$$\nu = 1: \ \frac{3653}{5760} + \frac{\ln 2}{240}$$

$$\nu = 2: \ \frac{8905}{14112}$$

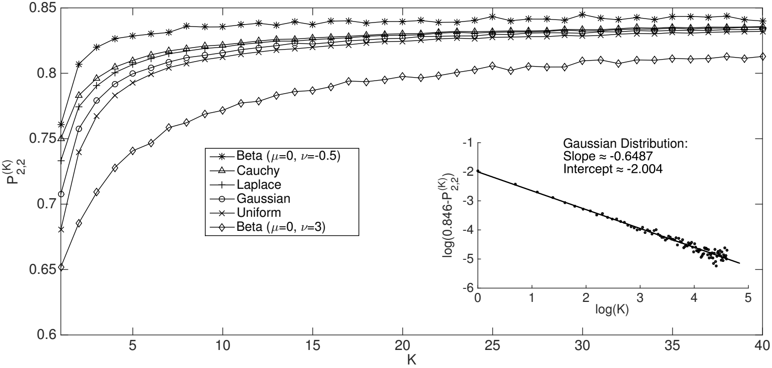
$$\nu = 4: \ \frac{45332489}{72144072}$$

Products follow the same ordering



Comparison of probability that all eigenvalues are real for a product of K random matrices with different symmetric distributions and the dimensionality $n = 2$ (main) and 8 (inset). The plot is based on 10^5 independent realizations.

So do Hadamard products ...



Comparison of probability that all eigenvalues are real for *Hadamard* products of K 2×2 random matrices for some symmetric distributions based on 10^5 realizations.The inset shows the power law approach of the probability of all real eigenvalues to the asymptotic value which is less than unity, for the Gaussian case.

Summary and questions

- A question about measure of **Concurrence-optimal** states led to the question about the fraction of product of two 2×2 matrices that have real eigenvalues.
- fraction of real eigenvalues increases from $1/\sqrt{2}$ for $k = 1$ to $\pi/4$ for $K = 2$ and with further products tends to 1.
- For a triple of optimal states of 2 qubits, the fraction is not more than the probability that $\{AB, AC, BC\}$ *all* have real eigenvalues for triples $\{A, B, C\}$. How much is this?
- What is the probability $p_{k,n}^{(K)}$ that $k < n$ eigenvalues are real in a product of K random matrices? Find $E_n^K = \sum_{k=0}^{n} k p_{k,n}^{(K)}$, does it approach n exponentially?
- Universality: eigenvalues tends to become real with more terms in the products for nongaussian matrices. Hierarchy at $K = 1$ seems to be maintained. Hadamard products also increase number of real eigenvalues but not to full fraction.

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Vielen Danke

Based on the collaborations in

Optimality: K. V. Shuddhodhan, TIFR Math. Mumbai; K. Ramkarthik, VJNIT, Nagpur.(J. Phys. A: Math. Theor. 44, 345301 (2011))

Nongaussian matrices: Sajna Hameed, Michigan; Kavita Jain, JNCSAR Bangalore. (J. Phys. A : Math. Theor. 48, 385204 (2015)) and

AL: (J. Phys. A: Math. Theor. vol. 46 (2013)).